Tutorial Classification

January 23, 2017

1 Tutorial: Classification

Agenda: 1. Classification running example: Iris Flowers 2. Weight space & feature space intuition 3. Perceptron convergence proof 4. Gradient Descent for Multiclass Logisitc Regression

```
In [1]: import matplotlib
        import numpy as np
        import matplotlib.pyplot as plt
        %matplotlib inline
```

1.1 Classification with Iris

We're going to use the Iris dataset.

We will only work with the first 2 flower classes (Setosa and Versicolour), and with just the first two features: length and width of the sepal

If you don't know what the sepal is, see this diagram: https://www.math.umd.edu/~petersd/666/html/iris_with_labels.jpg

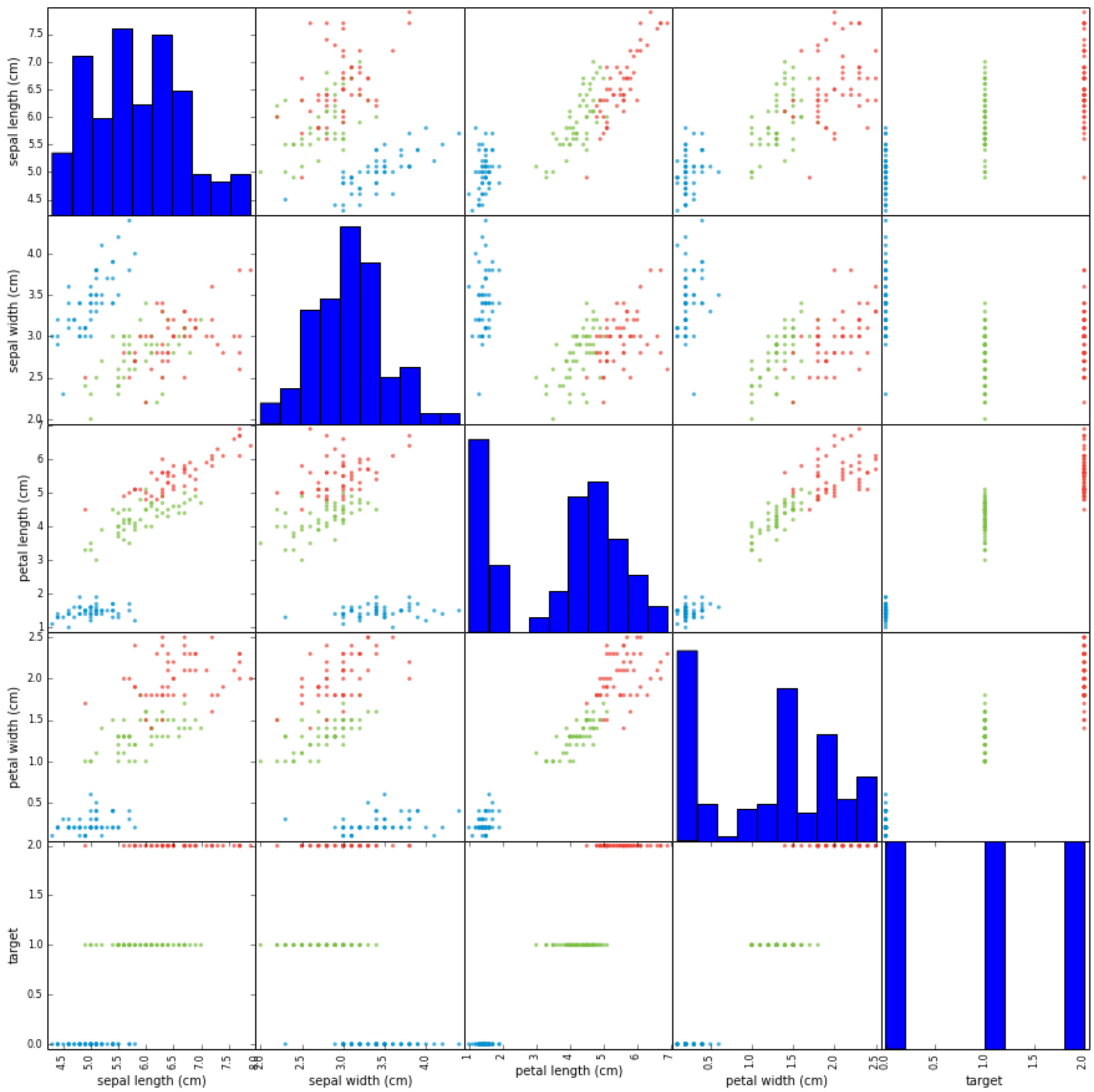
```
In [2]: from sklearn.datasets import load_iris
        iris = load_iris()
        print iris['DESCR']
```

```
Iris Plants Database

Notes
-----
Data Set Characteristics:
    :Number of Instances: 150 (50 in each of three classes)
    :Number of Attributes: 4 numeric, predictive attributes and the class
    :Attribute Information:
        - sepal length in cm
        - sepal width in cm
        - petal length in cm
        - petal width in cm
        - class:
                - Iris-Setosa
                - Iris-Versicolour
                - Iris-Virginica
```

```
    :Summary Statistics:
    ============== ==== ==== ======= ===== ====================
                    Min  Max   Mean    SD   Class Correlation
    ============== ==== ==== ======= ===== ====================
    sepal length:   4.3  7.9   5.84   0.83    0.7826
    sepal width:    2.0  4.4   3.05   0.43   -0.4194
    petal length:   1.0  6.9   3.76   1.76    0.9490  (high!)
    petal width:    0.1  2.5   1.20  0.76     0.9565  (high!)
    ============== ==== ==== ======= ===== ====================
    :Missing Attribute Values: None
    :Class Distribution: 33.3% for each of 3 classes.
    :Creator: R.A. Fisher
    :Donor: Michael Marshall (MARSHALL%PLU@io.arc.nasa.gov)
    :Date: July, 1988

This is a copy of UCI ML iris datasets.
http://archive.ics.uci.edu/ml/datasets/Iris

The famous Iris database, first used by Sir R.A Fisher

This is perhaps the best known database to be found in the
pattern recognition literature.  Fisher's paper is a classic in the field and
is referenced frequently to this day.  (See Duda & Hart, for example.)  The
data set contains 3 classes of 50 instances each, where each class refers to a
type of iris plant.  One class is linearly separable from the other 2; the
latter are NOT linearly separable from each other.

References
----------
   - Fisher,R.A. "The use of multiple measurements in taxonomic problems"
     Annual Eugenics, 7, Part II, 179-188 (1936); also in "Contributions to
     Mathematical Statistics" (John Wiley, NY, 1950).
   - Duda,R.O., & Hart,P.E. (1973) Pattern Classification and Scene Analysis.
     (Q327.D83) John Wiley & Sons.  ISBN 0-471-22361-1.  See page 218.
   - Dasarathy, B.V. (1980) "Nosing Around the Neighborhood: A New System
     Structure and Classification Rule for Recognition in Partially Exposed
     Environments".  IEEE Transactions on Pattern Analysis and Machine
     Intelligence, Vol. PAMI-2, No. 1, 67-71.
   - Gates, G.W. (1972) "The Reduced Nearest Neighbor Rule".  IEEE Transactions
     on Information Theory, May 1972, 431-433.
   - See also: 1988 MLC Proceedings, 54-64.  Cheeseman et al"s AUTOCLASS II
     conceptual clustering system finds 3 classes in the data.
   - Many, many more ...
```

In [4]:
```
# code from
# http://stackoverflow.com/questions/21131707/multiple-data-in-scatter-matr
```

```
from pandas.tools.plotting import scatter_matrix
import pandas as pd

iris_data = pd.DataFrame(data=iris['data'],columns=iris['feature_names'])
iris_data["target"] = iris['target']
color_wheel = {1: "#0392cf",
               2: "#7bc043",
               3: "#ee4035"}
colors = iris_data["target"].map(lambda x: color_wheel.get(x + 1))
ax = scatter_matrix(iris_data, color=colors, alpha=0.6, figsize=(15, 15), c
```

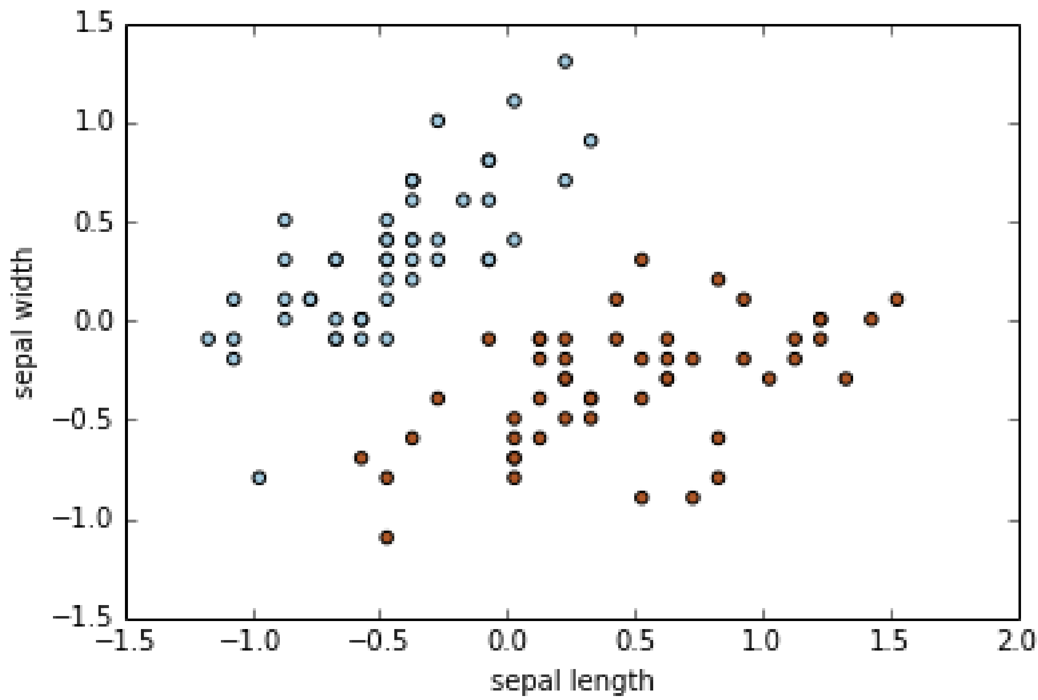



```
In [5]: # Select first 2 flower classes (~100 rows)
        # And first 2 features
```

```
sepal_len = iris['data'][:100,0]
sepal_wid = iris['data'][:100,1]
labels = iris['target'][:100]

# We will also center the data
# This is done to make numbers nice, so that we have no
# need for biases in our classification. (You might not
# be able to remove biases this way in general.)

sepal_len -= np.mean(sepal_len)
sepal_wid -= np.mean(sepal_wid)
```

In [6]:
```
# Plot Iris

plt.scatter(sepal_len,
            sepal_wid,
            c=labels,
            cmap=plt.cm.Paired)
plt.xlabel("sepal length")
plt.ylabel("sepal width")
```

Out[6]: <matplotlib.text.Text at 0x10ec88f50>



1.1.1 Plotting Decision Boundary

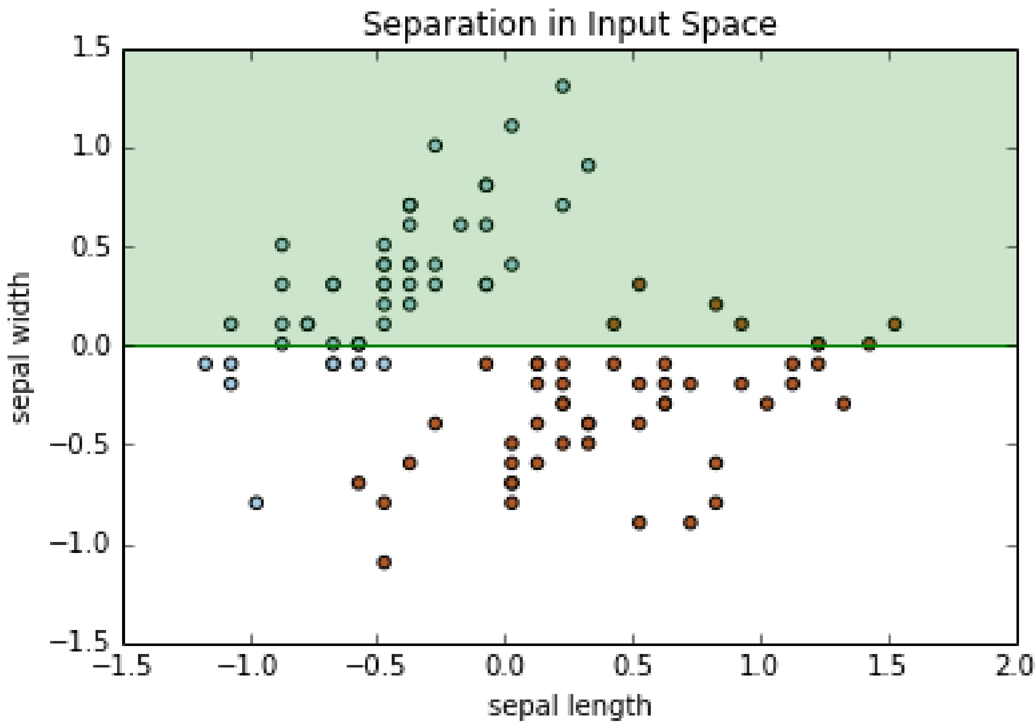
Plot decision boundary hypothese

$$w_1 x_1 + w_2 x_2 \geq 0$$

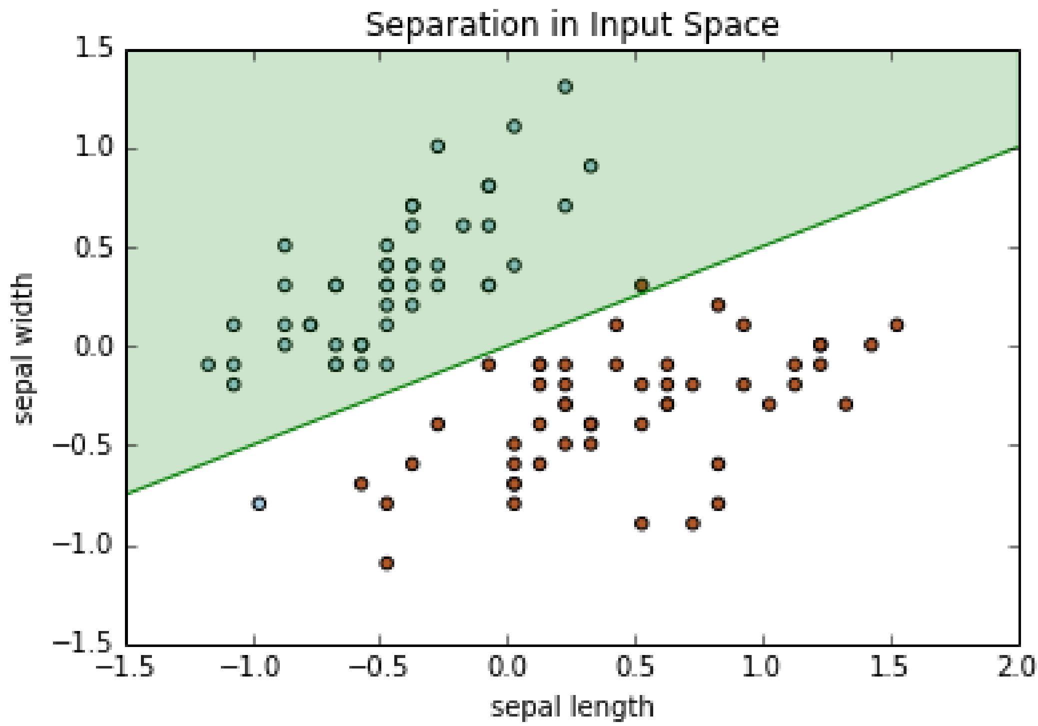
for classification as Setosa.

```
In [7]: def plot_sep(w1, w2, color='green'):
            '''
            Plot decision boundary hypothesis
              w1 * sepal_len + w2 * sepal_wid = 0
            in input space, highlighting the hyperplane
            '''
            plt.scatter(sepal_len,
                        sepal_wid,
                        c=labels,
                        cmap=plt.cm.Paired)
            plt.title("Separation in Input Space")
            plt.ylim([-1.5,1.5])
            plt.xlim([-1.5,2])
            plt.xlabel("sepal length")
            plt.ylabel("sepal width")
            if w2 != 0:
                m = -w1/w2
                t = 1 if w2 > 0 else -1
                plt.plot(
                    [-1.5,2.0],
                    [-1.5*m, 2.0*m],
                    '-y',
                    color=color)
                plt.fill_between(
                    [-1.5, 2.0],
                    [m*-1.5, m*2.0],
                    [t*1.5, t*1.5],
                    alpha=0.2,
                    color=color)
            if w2 == 0: # decision boundary is vertical
                t = 1 if w1 > 0 else -1
                plt.plot([0, 0],
                         [-1.5, 2.0],
                         '-y',
                        color=color)
                plt.fill_between(
                    [0, 2.0*t],
                    [-1.5, -2.0],
                    [1.5, 2],
                    alpha=0.2,
                    color=color)
```

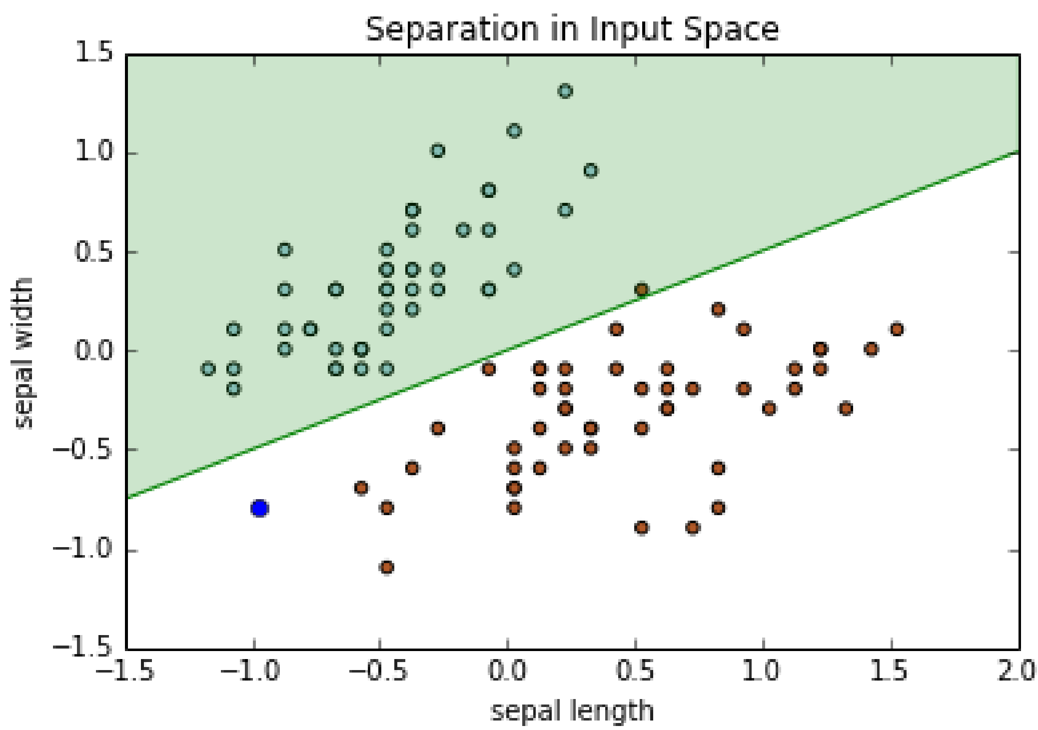
```
In [8]: # Example hypothesis
        #   sepal_wid >= 0

        plot_sep(0, 1)
```




```
In [9]: # Another example hypothesis:
        #   -0.5*sepal_len + 1*sepal_wid >= 0

        plot_sep(-0.5, 1)
```




```
In [10]: # We're going to hand pick one point and
         # analyze that point:

         a1 = sepal_len[41]
         a2 = sepal_wid[41]
         print (a1, a2) # (-0.97, -0.79)

         plot_sep(-0.5, 1)
         plt.plot(a1, a2, 'ob') # highlight the point

(-0.9710000000000097, -0.7940000000000004)


Out[10]: [<matplotlib.lines.Line2D at 0x10cee6cd0>]
```



1.1.2 Plot Constraints in Weight Space

We'll plot the constraints for some of the points that we chose earlier.

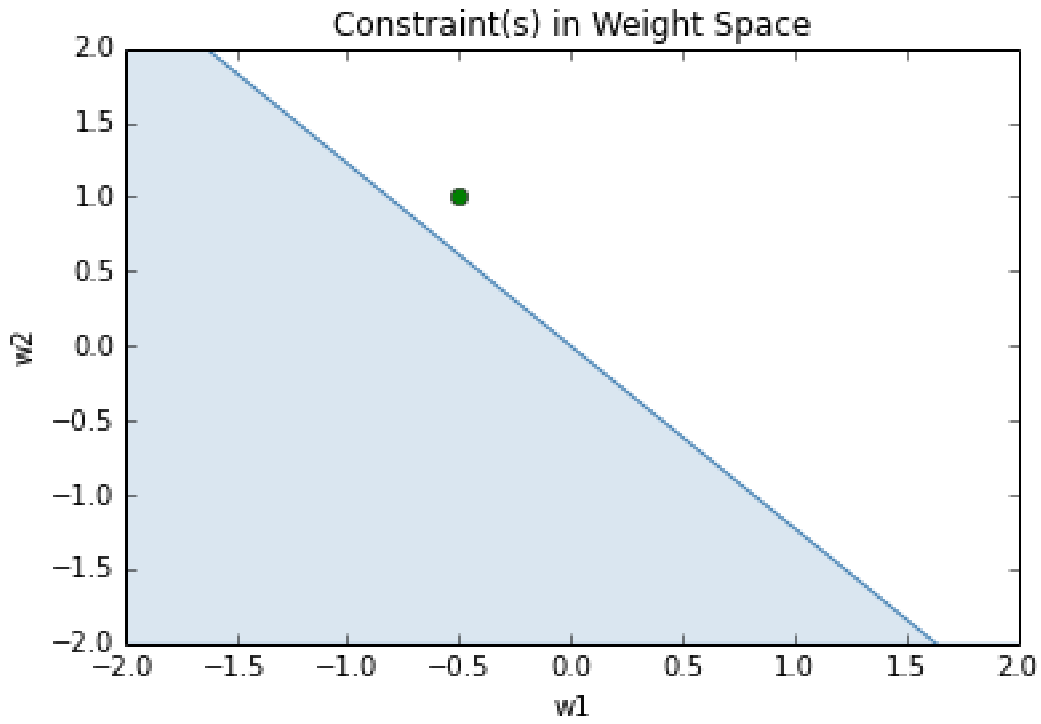
```
In [11]: def plot_weight_space(sepal_len, sepal_wid, lab=1,
                               color='steelblue',
                               maxlim=2.0):
             plt.title("Constraint(s) in Weight Space")
             plt.ylim([-maxlim,maxlim])
             plt.xlim([-maxlim,maxlim])
             plt.xlabel("w1")
             plt.ylabel("w2")

             if sepal_wid != 0:
                 m = -sepal_len/sepal_wid
                 t = 1*lab if sepal_wid > 0 else -1*lab
                 plt.plot([-maxlim, maxlim],
                          [-maxlim*m, maxlim*m],
                          '-y',
                          color=color)
                 plt.fill_between(
                     [-maxlim, maxlim],      # x
                     [m*-maxlim, m*maxlim], # y-min
```

```
                [t*maxlim, t*maxlim],       # y-max
                alpha=0.2,
                color=color)
        if sepal_wid == 0: # decision boundary is vertical
            t = 1*lab if sepal_len > 0 else -1*lab
            plt.plot([0, 0],
                     [-maxlim, maxlim],
                     '-y',
                     color=color)
            plt.fill_between(
                [0, 2.0*t],
                [-maxlim, -maxlim],
                [maxlim, maxlim],
                alpha=0.2,
                color=color)
```

In [12]:
```
# Plot the constraint for the point identified earlier:

a1 = sepal_len[41]
a2 = sepal_wid[41]
print (a1, a2)

# Do this on the board first by hand

plot_weight_space(a1, a2, lab=1)

# Below is the hypothesis we plotted earlier
# Notice it falls outside the range.
plt.plot(-0.5, 1, 'og')
```

```
(-0.9710000000000097, -0.7940000000000004)
```

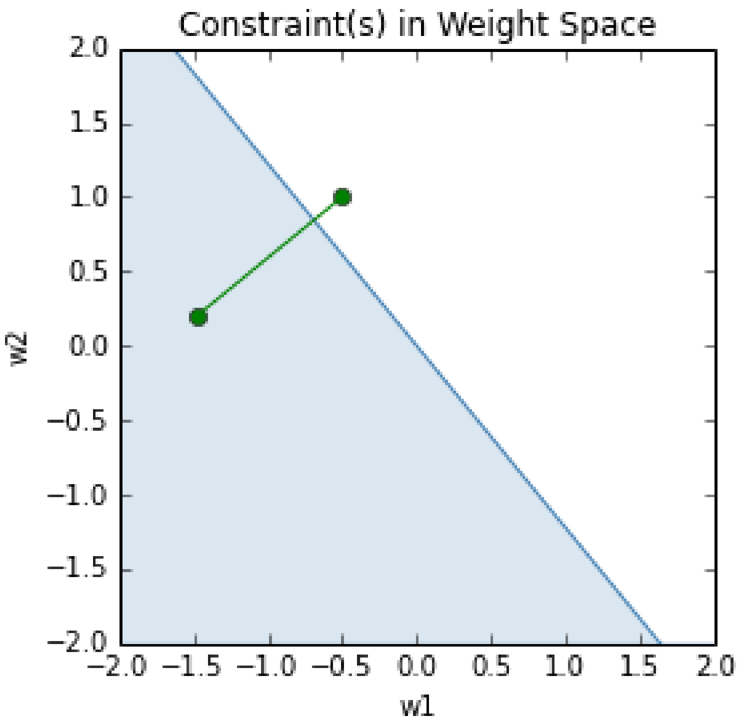
Out[12]:
```
[<matplotlib.lines.Line2D at 0x10e928fd0>]
```



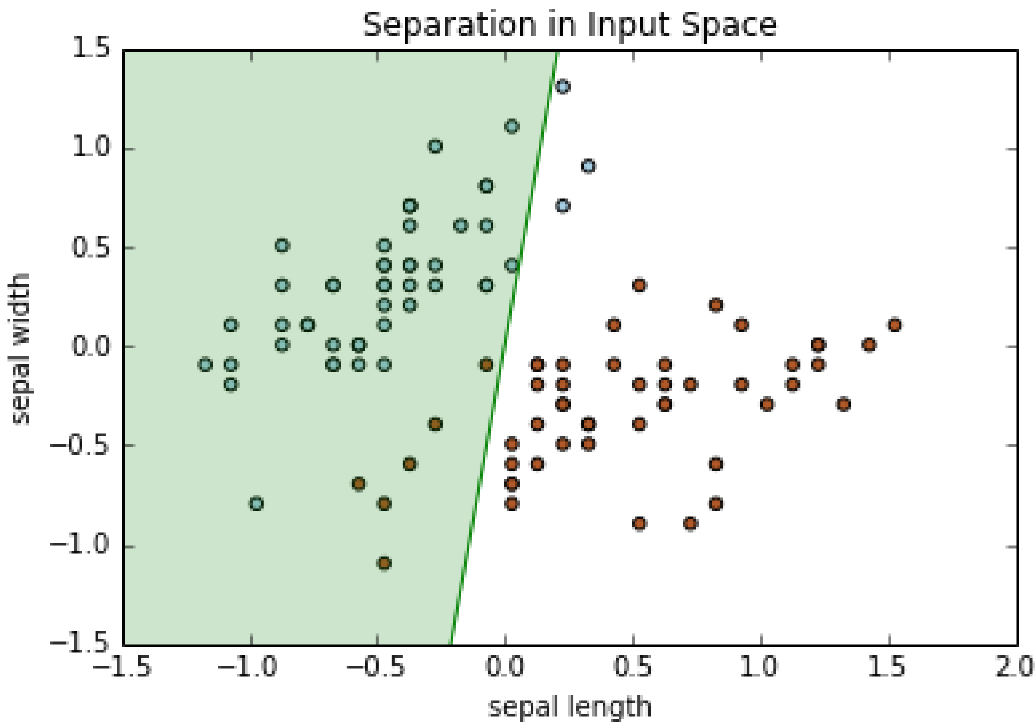
1.1.3 Perceptron Learning Rule Example

We'll take one step using the perceptron learning rule

```
In [20]: # Using the perceptron learning rule
         # TODO: Fill in

         w1 = -0.5 # + ...
         w2 =  1   # + ...

In [21]: # This should bring the point closer to the boundary
         # In this case, the step brought the point into the
         # condition boundary
         plot_weight_space(a1, a2, lab=1)
         plt.plot(-0.5+a1, 1+a2, 'og')
         # old hypothesis
         plt.plot(-0.5, 1, 'og')
         plt.plot([-0.5, -0.5+a1], [1, 1+a2], '-g')

         plt.axes().set_aspect('equal', 'box')
```



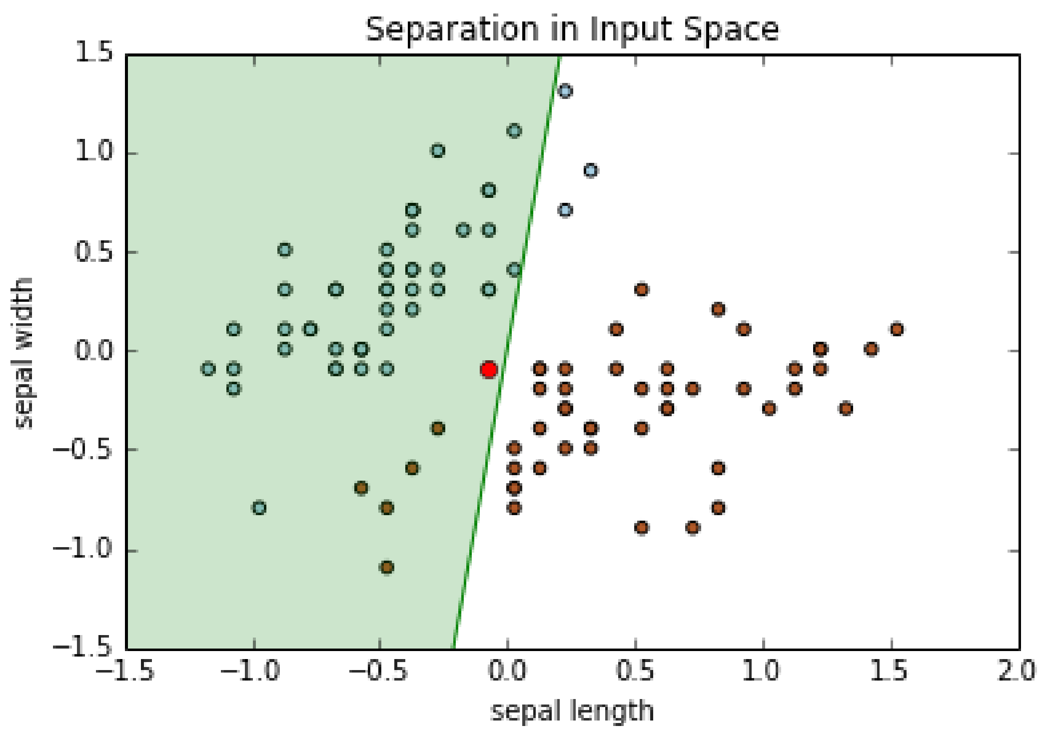

```
In [22]: # Which means that the point (a1, a2) in input
         # space is correctly classified.

         plot_sep(-0.5+a1, 1+a2)
```



1.1.4 Visualizing Multiple Constraints

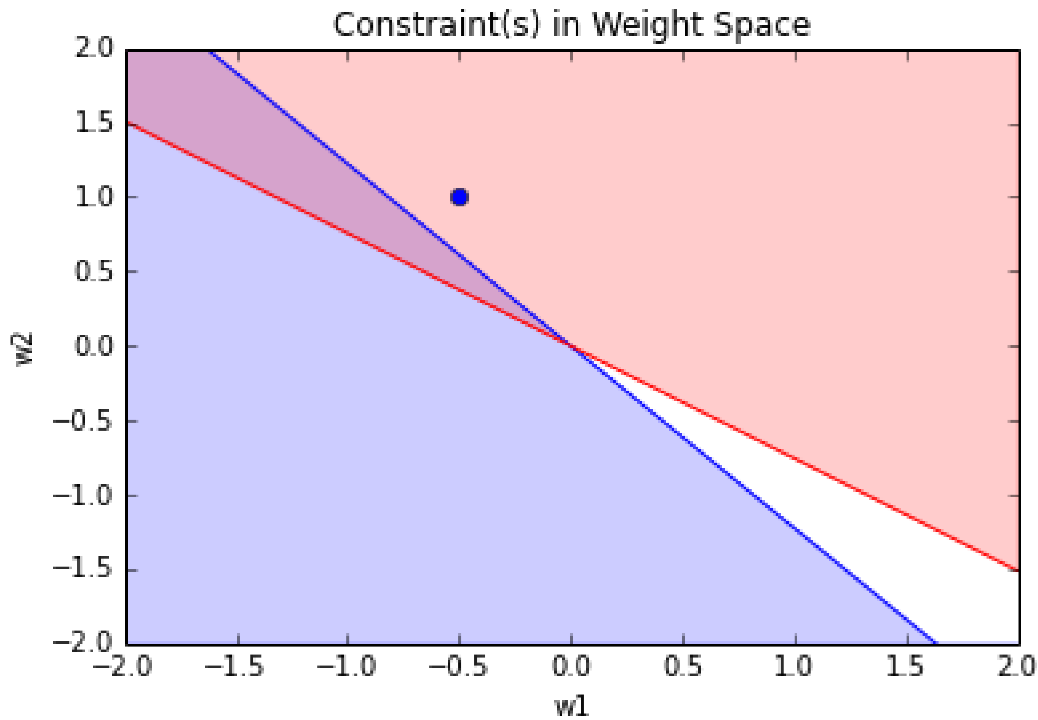
We'll visualize multiple constraints in weight space.

```
In [23]: # Pick a second point
         b1 = sepal_len[84]
         b2 = sepal_wid[84]

         plot_sep(-0.5+a1, 1+a2)
         plt.plot(b1, b2, 'or') # plot the circle in red

Out[23]: [<matplotlib.lines.Line2D at 0x10cc68ed0>]
```

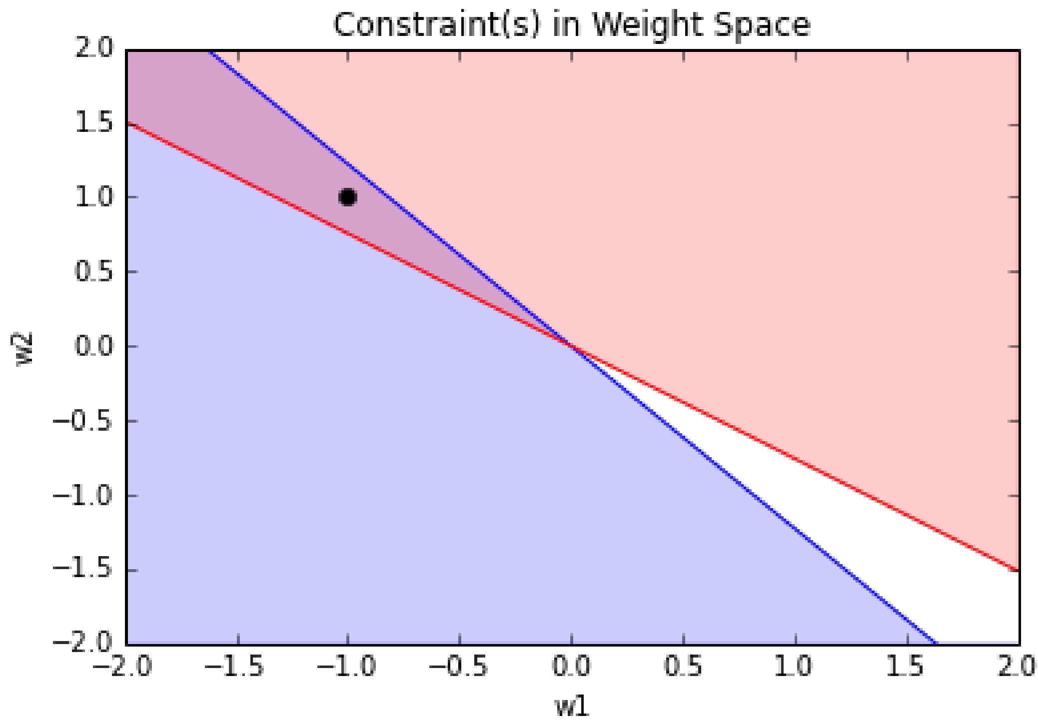


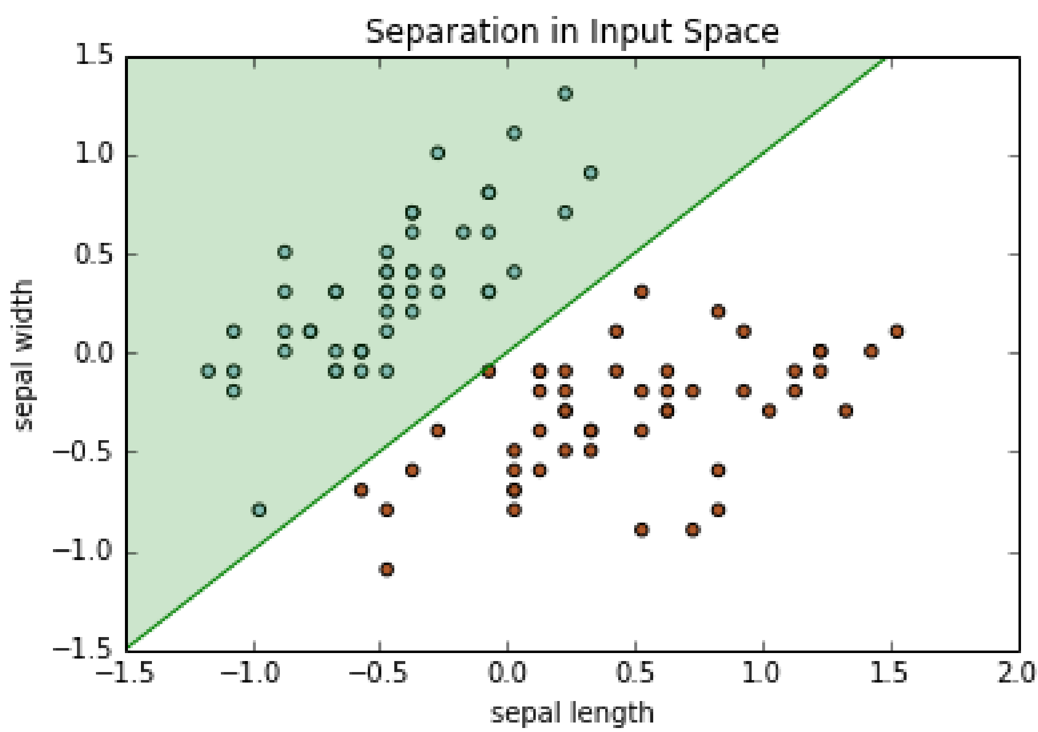
In [24]:
```
# our weights fall outside constraint of second pt.

plot_weight_space(a1, a2, lab=1, color='blue')
plot_weight_space(b1, b2, lab=-1, color='red')
plt.plot(w1, w2, 'ob')
```

Out[24]: [<matplotlib.lines.Line2D at 0x10dc8a4d0>]




```
In [25]: # Example of a separating hyperplane
         plot_weight_space(a1, a2, lab=1, color='blue')
         plot_weight_space(b1, b2, lab=-1, color='red')
         plt.plot(-1, 1, 'ok')
         plt.show()
         plot_sep(-1, 1)
         plt.show()
```


Constraint(s) in Weight Space
w1
w2
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0


Separation in Input Space
sepal length
sepal width
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0

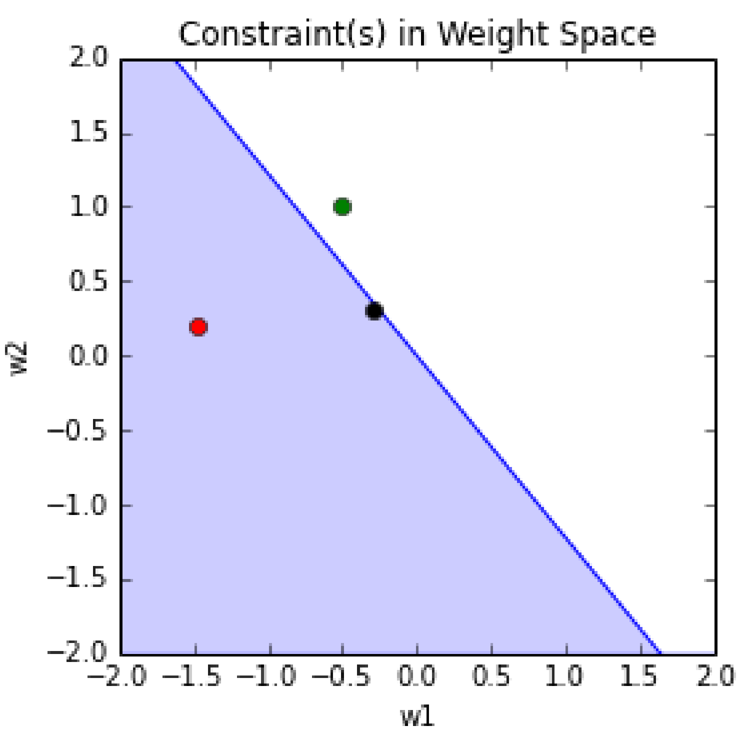
1.2 Perceptron Convergence Proof:

(From Geoffrey Hinton's slides 2d)

Hopeful claim: Every time the perceptron makes a mistake, the learning algo moves the current weight vector closer to all feasible weight vectors

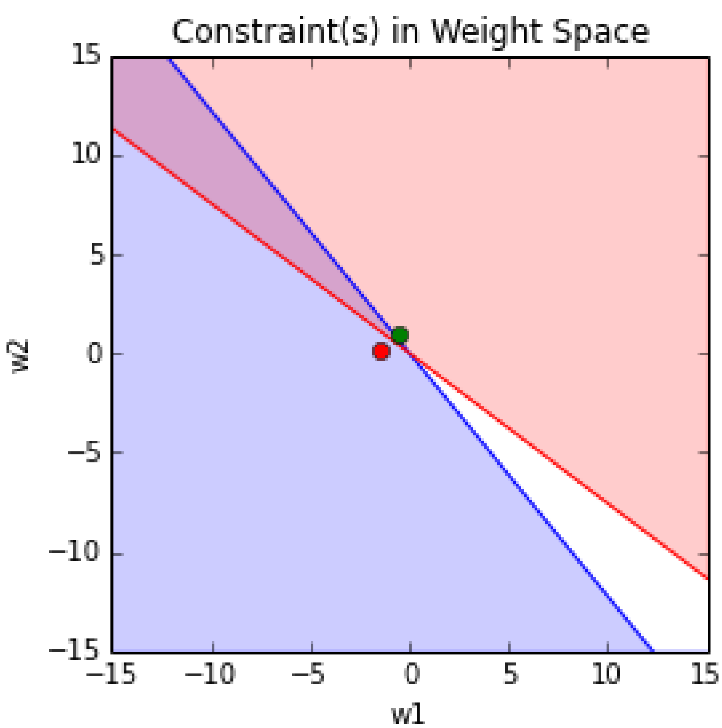
BUT: weight vector may not get close to feasible vector in the boundary

In [26]:
```
# The feasible region is inside the intersection of these two regions:
plot_weight_space(a1, a2, lab=1, color='blue')
#plot_weight_space(b1, b2, lab=-1, color='red')

# This is a vector in the feasible region.
plt.plot(-0.3, 0.3, 'ok')

# We started with this point
plt.plot(-0.5, 1, 'og')

# And ended up here
plt.plot(-0.5+a1, 1+a2, 'or')

# Notice that red point is further away to black than the green

plt.axes().set_aspect('equal', 'box')
```



- So consider "generously feasible" weight vectors that lie within the feasible region by a margin at least as great as the length of the input vector that defines each constraint plane.
- Every time the perceptron makes a mistake, the squared distance to all of these generously feasible weight vectors is always decreased by at least the squared length of the update vector.

In [27]:
```
plot_weight_space(a1, a2, lab=1, color='blue' ,maxlim=15)
plot_weight_space(b1, b2, lab=-1, color='red', maxlim=15)

# We started with this point
plt.plot(-0.5, 1, 'og')
plt.plot(-0.5+a1, 1+a2, 'or')
plt.axes().set_aspect('equal', 'box')

# red is closer to "generously feasible" vectors on the top left
```



1.2.1 Inform Sketch of Proof of Convergence

- Each time the perceptron makes a mistake, the current weight vector moves to decrease its squared distance from every weight vector in the "generously feasible" region.
- The squared distance decreases by at least the squared length of the input vector.
- So after a finite number of mistakes, the weight vector must lie in the feasible region if this region exists.

1.3 Gradient Descent for Multiclass Logisitc Regression

Multiclass logistic regression:

$$\mathbf{z} = \mathbf{W}\mathbf{x} + \mathbf{b} \tag{1}$$
$$\mathbf{y} = \text{softmax}(\mathbf{z}) \tag{2}$$
$$\mathcal{L}_{\text{CE}} = -\mathbf{t}^T(\log \mathbf{y}) \tag{3}$$

Draw out the shapes on the board before continuing.

```
In [28]: # Aside: lots of functions work on vectors

         print np.log([1.5,2,3])
         print np.exp([1.5,2,3])
```

```
[ 0.40546511  0.69314718  1.09861229]
[  4.48168907   7.3890561   20.08553692]
```

Start by expanding the cross entropy loss so that we can work with it

$$\mathcal{L}_{\text{CE}} = -\sum_l t_l \log(y_l)$$

1.3.1 Main setup

We'll take the derivative with respect to the loss:

$$\frac{\partial \mathcal{L}_{\text{CE}}}{\partial w_{kj}} = \frac{\partial}{\partial w_{kj}}(-\sum_l t_l \log(y_l)) \tag{4}$$
$$= -\sum_l \frac{t_l}{y_l}\frac{\partial y_l}{\partial w_{kj}} \tag{5}$$

Normally in calculus we have the rule:

$$\frac{\partial y_l}{\partial w_{kj}} = \sum_m \frac{\partial y_l}{\partial z_m}\frac{\partial z_m}{\partial w_{kj}} \tag{6}$$

But w_{kj} is independent of z_m for $m \neq k$, so

$$\frac{\partial y_l}{\partial w_{kj}} = \frac{\partial y_l}{\partial z_k}\frac{\partial z_k}{\partial w_{kj}} \tag{7}$$

AND

$$\frac{\partial z_k}{\partial w_{kj}} = x_j$$

Thus

$$\frac{\partial \mathcal{L}_{\text{CE}}}{\partial w_{kj}} = -\sum_l \frac{t_l}{y_l} \frac{\partial y_l}{\partial z_k} \frac{\partial z_k}{\partial w_{kj}} \tag{8}$$

$$= -\sum_l \frac{t_l}{y_l} \frac{\partial y_l}{\partial z_k} x_j \tag{9}$$

$$= x_j \Big(-\sum_l \frac{t_l}{y_l} \frac{\partial y_l}{\partial z_k}\Big) \tag{10}$$

$$= x_j \frac{\partial \mathcal{L}_{\text{CE}}}{\partial z_k} \tag{11}$$

1.3.2 Derivative with respect to z_k

But we can show (on board) that

$$\frac{\partial y_l}{\partial z_k} = y_k(I_{k,l} - y_l)$$

Where $I_{k,l} = 1$ if $k = l$ and 0 otherwise.
Therefore

$$\frac{\partial \mathcal{L}_{\text{CE}}}{\partial z_k} = -\sum_l \frac{t_l}{y_l}(y_k(I_{k,l} - y_l)) \tag{12}$$

$$= -\frac{t_k}{y_k} y_k(1 - y_k) - \sum_{l \neq k} \frac{t_l}{y_l}(-y_k y_l) \tag{13}$$

$$= -t_k(1 - y_k) + \sum_{l \neq k} t_l y_k \tag{14}$$

$$= -t_k + t_k y_k + \sum_{l \neq k} t_l y_k \tag{15}$$

$$= -t_k + \sum_l t_l y_k \tag{16}$$

$$= -t_k + y_k \sum_l t_l \tag{17}$$

$$= -t_k + y_k \tag{18}$$

$$= y_k - t_k \tag{19}$$

1.3.3 Putting it all together

$$\frac{\partial \mathcal{L}_{\text{CE}}}{\partial w_{kj}} = x_j(y_k - t_k) \tag{20}$$

1.3.4 Vectorization

Outer product.

$$\frac{\partial \mathcal{L}_{\text{CE}}}{\partial \mathbf{W}} = (\mathbf{y} - \mathbf{t})\mathbf{x}^{\mathbf{T}} \tag{21}$$

$$\frac{\partial \mathcal{L}_{\text{CE}}}{\partial \mathbf{b}} = (\mathbf{y} - \mathbf{t}) \tag{22}$$

```
In [29]: def softmax(x):
             #return np.exp(x) / np.sum(np.exp(x))
             return np.exp(x - max(x)) / np.sum(np.exp(x - max(x)))
```

```
In [30]: x1 = np.array([1,3,3])
         softmax(x1)
```

```
Out[30]: array([ 0.06337894,  0.46831053,  0.46831053])
```

```
In [31]: x2 = np.array([1000,3000,3000])
         softmax(x2)
```

```
Out[31]: array([ 0. ,  0.5,  0.5])
```

```
In [32]: def gradient(W, b, x, t):
             '''
             Gradient update for a single data point.
                 returns dW and db
             This is meant to show how to implement the
             obtained equation in code. (not tested)
             '''
             z = np.matmul(W, x) + b
             y = softmax(z)
             dW = np.matmul(x, (y-t).T)
             db = (y-t)
             return dW, db
```

```
In [ ]:
```