CSC 2541: Neural Net Training Dynamics

Lecture 3 - Metrics

Roger Grosse

University of Toronto, Winter 2022

Today

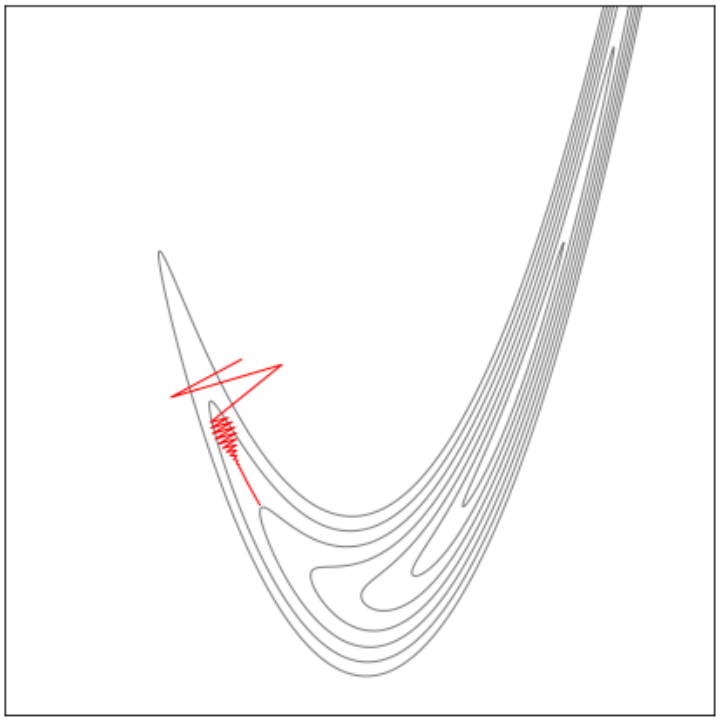
- Today's question: how to measure the "distance" between neural networks?
  - Euclidean distance between weights isn't very meaningful, since neural nets are complicated nonlinear functions
  - Instead, we'll look at distance in the space of functions represented by the networks
  - Taking the second-order Taylor approximation of function space distance gives a metric, which we can build interesting algorithms out of
  - If the ouptuts represent probabilities, this yields the Fisher-Rao metric
- Main motivating example: optimization (natural gradient descent)

Motivating Example

- Here's the Rosenbrock function, a famous toy problem from optimization:

$$h(x_1, x_2) = (a - x_1)^2 + b(x_2 - x_1^2)^2$$

- Gradient descent bounces across the valley and gets stuck:

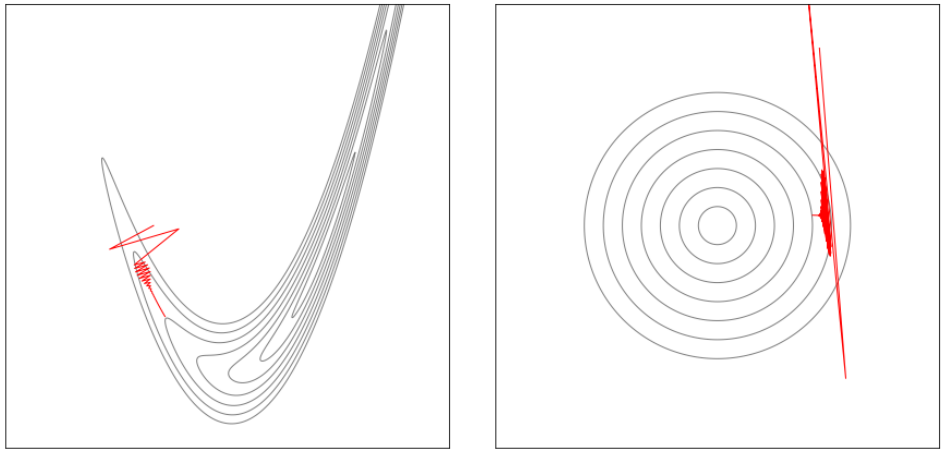


Motivating Example

- We can understand the Rosenbrock function as a composition of two functions, analogously to cost functions in ML. This is known as composite optimization.

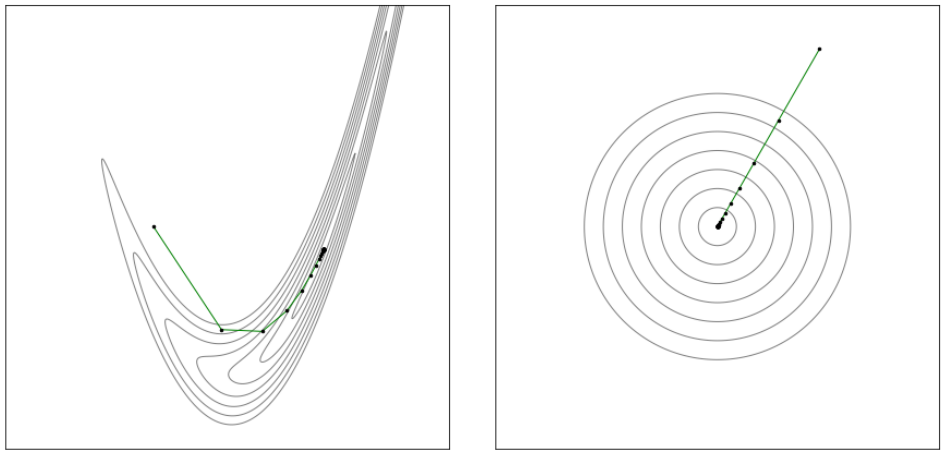
$$\mathcal{J}(x_1, x_2) = \mathcal{L}(f(x_1, x_2))$$
$$f(x_1, x_2) = (a - x_1, \sqrt{b}(x_2 - x_1^2))$$
$$\mathcal{L}(z_1, z_2) = z_1^2 + z_2^2.$$

- Here's the gradient descent trajectory in parameter space and output space:



Motivating Example

- If we could do gradient descent on the outputs, then it would converge instantly. Of course, this is cheating.



- This solution doesn't directly carry over to neural nets
  - Mapping from parameters to weights may not be invertible, or not easily invertible
  - Will the solution generalize?
- But we can achieve a similar effect using proximal optimization

Proximal Optimization

Proximal Optimization

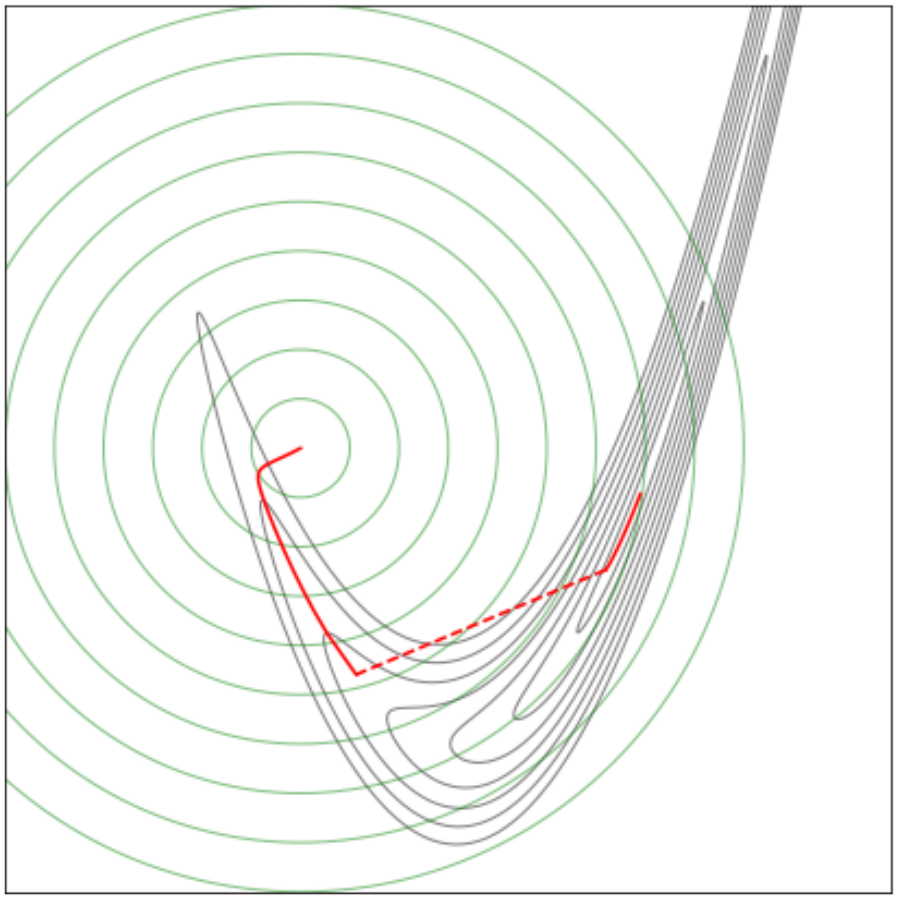
- Suppose we want to minimize $\mathcal{J}(\mathbf{w})$. The proximal point method trades off the cost and the distance from the current iterate:

$$\mathbf{w}^{(k+1)} = \text{prox}_{\mathcal{J},\lambda}(\mathbf{w}^{(k)}) = \arg\min_{\mathbf{u}} \left[\mathcal{J}(\mathbf{u}) + \lambda\rho(\mathbf{u}, \mathbf{w}^{(k)})\right]$$

- ρ is a dissimilarity function (like a distance, but doesn't need to satisfy all the axioms)
- $\lambda\rho(\mathbf{u}, \mathbf{w}^{(k)})$ is the proximity term, and $\text{prox}_{f,\lambda}$ is the proximal operator

Proximal Optimization

- **Example:** $\rho(\mathbf{w}, \mathbf{w}') = \frac{1}{2}\|\mathbf{w} - \mathbf{w}'\|^2$



- **Note:** this is an idealized update rule, not something we can efficiently implement. E.g., if $\lambda = 0$, it simply optimizes $\mathcal{J}$ directly.

Proximal Optimization

$$\mathbf{w}^{(k+1)} = \text{prox}_{\mathcal{J},\lambda}(\mathbf{w}^{(k)}) = \arg\min_{\mathbf{u}} \left[\mathcal{J}(\mathbf{u}) + \lambda \rho(\mathbf{u}, \mathbf{w}^{(k)}) \right]$$

- With squared Euclidean distance:

$$\text{prox}_{\mathcal{J},\lambda}(\mathbf{w}^{(k)}) = \arg\min_{\mathbf{u}} \left[\mathcal{J}(\mathbf{u}) + \frac{\lambda}{2} \|\mathbf{u} - \mathbf{w}^{(k)}\|^2 \right]$$

- Setting the gradient equal to $\mathbf{0}$ and rearranging:

$$\text{prox}_{\mathcal{J},\lambda}(\mathbf{w}^{(k)}) = \mathbf{u}_\star = \mathbf{w}^{(k)} - \lambda^{-1} \nabla \mathcal{J}(\mathbf{u}_\star)$$

- This is implicit gradient descent: it's almost a gradient descent update, except that the gradient is evaluated at $\mathbf{u}_\star$.
  - Not a practical algorithm, but often convenient to analyze.

Proximal Optimization

- **Approximation 1:** linearize the cost around the current weights

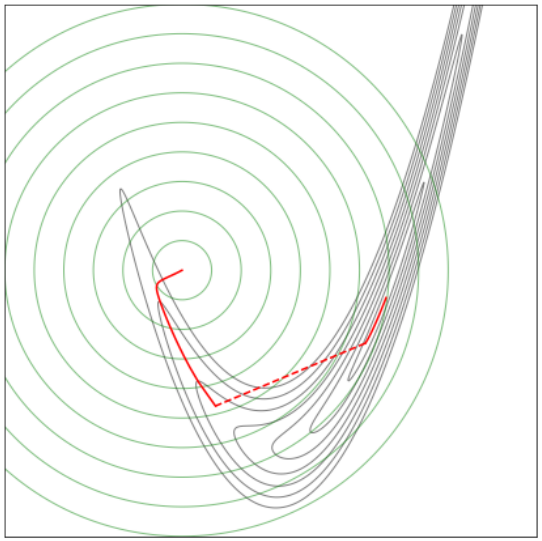
$$\begin{aligned}\text{prox}_{\mathcal{J},\lambda}(\mathbf{w}^{(k)}) &= \arg\min_{\mathbf{u}} \left[\mathcal{J}(\mathbf{w}^{(k)}) + \nabla\mathcal{J}(\mathbf{w}^{(k)})^\top(\mathbf{u} - \mathbf{w}^{(k)}) + \lambda\rho(\mathbf{u}, \mathbf{w}^{(k)})\right] \\ &= \arg\min_{\mathbf{u}} \left[\nabla\mathcal{J}(\mathbf{w}^{(k)})^\top\mathbf{u} + \lambda\rho(\mathbf{u}, \mathbf{w}^{(k)})\right]\end{aligned}$$

- Setting the gradient to 0:

$$\nabla_{\mathbf{u}}\rho(\mathbf{u}_\star, \mathbf{w}^{(k)}) = -\lambda^{-1}\nabla\mathcal{J}(\mathbf{w}^{(k)}).$$

- This is called mirror descent, for reasons explained in the readings
  - A foundational technique in online learning
  - Closed form solutions for many ρ of interest (e.g. KL divergence)
- The case of squared error reduces to ordinary gradient descent

Proximal Optimization



- **Approximation 2:** Take the infinitesimal limit, $\lambda \to \infty$.
- Then we take the first-order Taylor approximation to $\mathcal{J}$ and second-order Taylor approximation to ρ:

$$\rho(\mathbf{u}, \mathbf{w}^{(k)}) = \tfrac{1}{2}(\mathbf{u} - \mathbf{w}^{(k)})^\top \mathbf{G}(\mathbf{u} - \mathbf{w}^{(k)}) + \mathcal{O}(\|\mathbf{u} - \mathbf{w}^{(k)}\|^3)$$

$$\mathbf{G} = \nabla^2_{\mathbf{u}} \rho(\mathbf{u}, \mathbf{w}^{(k)})\big|_{\mathbf{u}=\mathbf{w}^{(k)}}$$

- $\mathbf{G}$ is the metric matrix

Proximal Optimization

- Plugging in both approximations:

$$\text{prox}_{\mathcal{J},\lambda}(\mathbf{w}^{(k)}) = \arg\min_{\mathbf{u}} \left[\nabla \mathcal{J}(\mathbf{w}^{(k)})^\top \mathbf{u} + \frac{\lambda}{2} (\mathbf{u} - \mathbf{w}^{(k)})^\top \mathbf{G} (\mathbf{u} - \mathbf{w}^{(k)}) \right]$$

- Optimal solution:

$$\mathbf{u}_\star = \mathbf{w}^{(k)} - \lambda^{-1} \mathbf{G}^{-1} \nabla \mathcal{J}(\mathbf{w}^{(k)})$$

- For certain ρ, $\mathbf{G}^{-1} \nabla \mathcal{J}(\mathbf{w}^{(k)})$ is called the natural gradient
- If ρ is squared Euclidean distance, this is just ordinary gradient descent

Proximal Optimization

- **Approximation 3:** Second-order Taylor approximations to both $\mathcal{J}$ and ρ
- Update rule:

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - (\mathbf{H} + \lambda \mathbf{G})^{-1} \nabla \mathcal{J}(\mathbf{w}^{(k)})$$

- If ρ is squared error, then $\mathbf{G} = \mathbf{I}$, and this gives a damped Newton update:

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - (\mathbf{H} + \lambda \mathbf{I})^{-1} \nabla \mathcal{J}(\mathbf{w}^{(k)})$$

- Relationship to trust-region methods (see readings for details)

Fisher Information

Fisher Information

- So far we've only considered squared error proximity terms. If we're optimizing over probability distributions, a more meaningful dissimilarity function is KL divergence:

$$\mathrm{D}_{\mathrm{KL}}(q \,\|\, p) = \mathbb{E}_{\mathbf{x}\sim q}[\log q(\mathbf{x}) - \log p(\mathbf{x})]$$

- Doesn't satisfy some of the axioms of distance metrics (symmetry, triangle inequality), but that's OK
- Nonnegative, equals 0 iff $p = q$
- Information theoretic interpretation as relative entropy: the number of bits wasted if you encode data from q using a code designed for p
- This formula doesn't mention the parameters, so it's an intrinsic dissimilarity function
  - But we'll typically optimize over a parametric family $\{p_{\boldsymbol{\theta}}\}$ parameterized by $\boldsymbol{\theta}$

Fisher Information

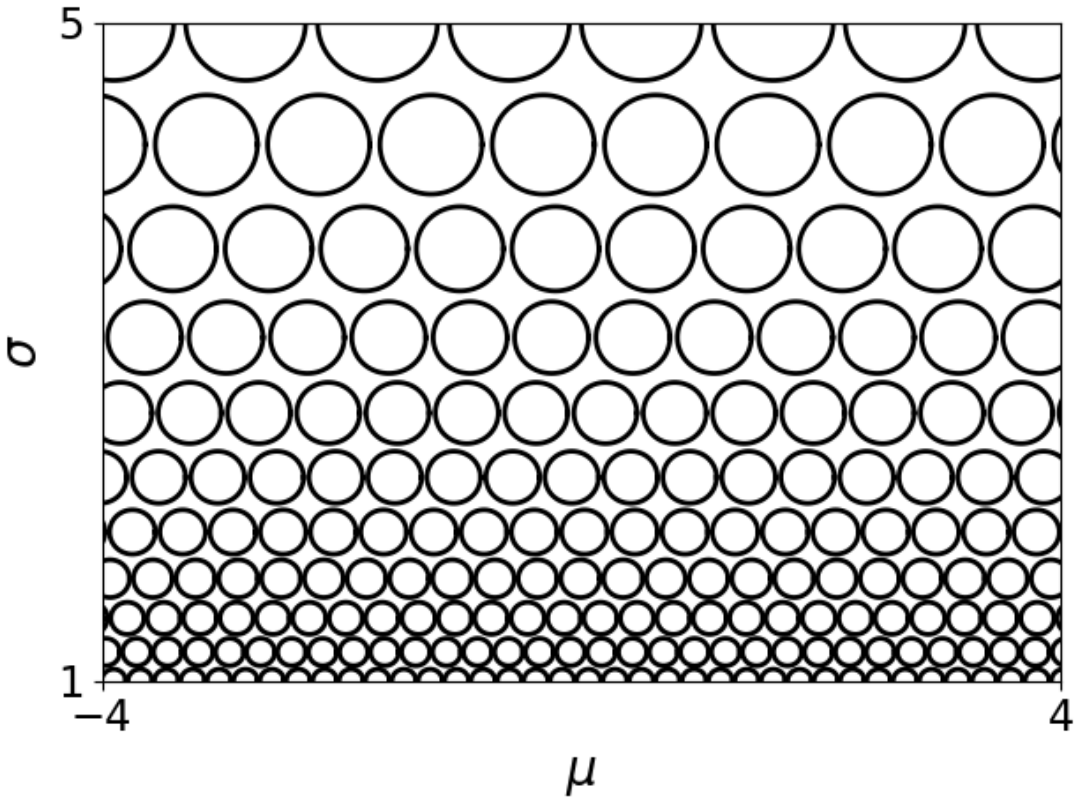
- The metric matrix is the Fisher information matrix (derivation in the readings):

$$\begin{aligned}\nabla^2_{\mathbf{u}} \mathrm{D}_{\mathrm{KL}}(p_{\mathbf{u}} \,\|\, p_{\boldsymbol{\theta}})\big|_{\mathbf{u}=\boldsymbol{\theta}} &= \mathbf{F}_{\boldsymbol{\theta}} \\ &= \mathrm{Cov}_{\mathbf{x}\sim p_{\boldsymbol{\theta}}}(\nabla_{\boldsymbol{\theta}} \log p_{\boldsymbol{\theta}}(\mathbf{x})) \\ &= \mathbb{E}_{\mathbf{x}\sim p_{\boldsymbol{\theta}}}\left[(\nabla_{\boldsymbol{\theta}} \log p_{\boldsymbol{\theta}}(\mathbf{x}))(\nabla_{\boldsymbol{\theta}} \log p_{\boldsymbol{\theta}}(\mathbf{x}))^\top\right]\end{aligned}$$

- The vectors $\nabla_{\boldsymbol{\theta}} \log p_{\boldsymbol{\theta}}(\mathbf{x})$ are called Fisher scores
- Intuitively, $\Delta\boldsymbol{\theta}^\top \mathbf{F}_{\boldsymbol{\theta}} \Delta\boldsymbol{\theta}$ tells you how much the distribution changes if you adjust the parameters by $\Delta\boldsymbol{\theta}$

Fisher Information

Fisher metric for univariate Gaussians:



Fisher Information

- Proximal operator:

$$\text{prox}_{\mathcal{J},\lambda}(\boldsymbol{\theta}) = \arg\min_{\mathbf{u}} \left[\mathcal{J}(\mathbf{u}) + \lambda \text{D}_{\text{KL}}(p_{\mathbf{u}} \,\|\, p_{\boldsymbol{\theta}})\right]$$

- Infinitesimal limit:

$$\begin{aligned}\boldsymbol{\theta}^{(k+1)} &= \boldsymbol{\theta}^{(k)} - \alpha \mathbf{F}_{\boldsymbol{\theta}}^{-1} \nabla \mathcal{J}(\boldsymbol{\theta}^{(k)}) \\ &= \boldsymbol{\theta}^{(k)} - \alpha \tilde{\nabla} \mathcal{J}(\boldsymbol{\theta}^{(k)}),\end{aligned}$$

  where $\alpha = \lambda^{-1}$

- The vector $\tilde{\nabla}\mathcal{J}(\boldsymbol{\theta}) = \mathbf{F}_{\boldsymbol{\theta}}^{-1}\nabla\mathcal{J}(\boldsymbol{\theta})$ is the natural gradient, and the update rule is natural gradient descent
- The term natural indicates that the update direction is equivalent in any coordinate system. This works because KL divergence is an intrinsic dissimilarity function.

Function Space Distance

Function Space Distance

- Motivations
  - "Output space gradient descent" can be more efficient
  - Natural gradient descent is invariant to parameterization
- Can we do the same for neural nets?
- **Idea:** define a metric on the outputs of a network and pull it back to weight space
- Pullback of a function:

$$f^* g(\mathbf{x}_1, \ldots, \mathbf{x}_K) = g(f(\mathbf{x}_1), \ldots, f(\mathbf{x}_K))$$

Function Space Distance

- We compute the metric matrix by taking a second-order Taylor approximation to $f^*\rho$ around the current point
- The decomposition is similar to the Gauss-Newton Hessian (Lecture 2), except this time it's exact

$$\begin{aligned}
\mathbf{G}_{\mathbf{x}} &= \nabla^2_{\mathbf{x}} f^*\rho(\mathbf{x}, \mathbf{x}_0)\big|_{\mathbf{x}=\mathbf{x}_0} \\
&= \mathbf{J}_{\mathbf{zx}}^\top \left[\nabla^2_{\mathbf{z}} \rho(\mathbf{z}, \mathbf{z}_0)\big|_{\mathbf{z}=\mathbf{z}_0}\right] \mathbf{J}_{\mathbf{zx}} + \underbrace{\sum_a \frac{\partial \rho}{\partial z_a} \nabla^2_{\mathbf{x}} [f(\mathbf{x})]_a}_{=\mathbf{0}} \\
&= \mathbf{J}_{\mathbf{zx}}^\top \mathbf{G}_{\mathbf{z}} \mathbf{J}_{\mathbf{zx}}
\end{aligned}$$

Function Space Distance

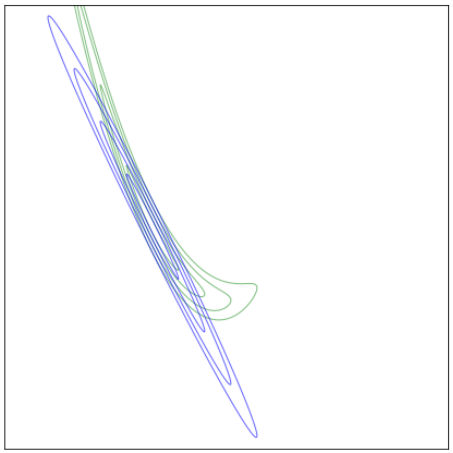
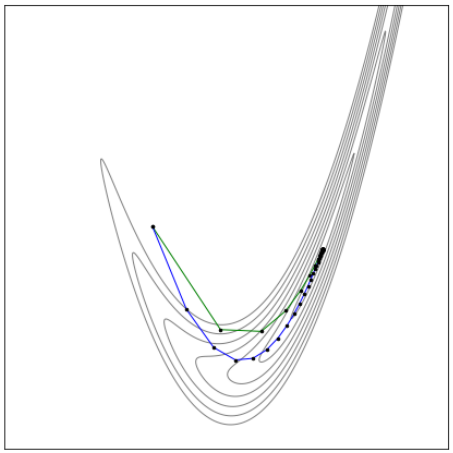
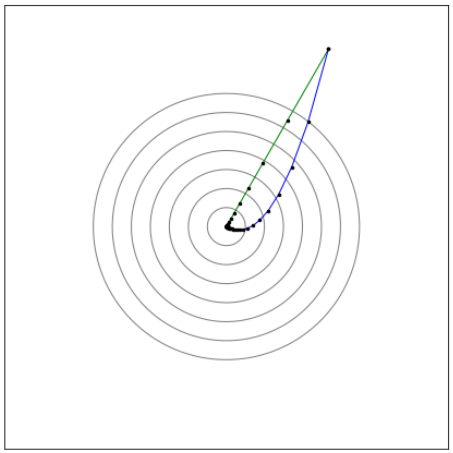
- Let's revisit the Rosenbrock function. Instead of doing output space gradient descent, put a Euclidean metric on the outputs and pull it back to parameter space.

$$\rho_{\text{euc}}(\mathbf{z}, \mathbf{z}') = \tfrac{1}{2}\|\mathbf{z} - \mathbf{z}'\|^2$$
$$f^*\rho_{\text{euc}}(\mathbf{x}, \mathbf{x}') = \tfrac{1}{2}\|f(\mathbf{x}) - f(\mathbf{x}')\|^2$$

- $\mathbf{G}_\mathbf{z} = \mathbf{I}$, so

$$\mathbf{G}_\mathbf{x} = \mathbf{J}_{\mathbf{zx}}^\top \mathbf{G}_\mathbf{z} \mathbf{J}_{\mathbf{zx}} = \mathbf{J}_{\mathbf{zx}}^\top \mathbf{J}_{\mathbf{zx}}$$

- Approximating the proximal point update with this metric:

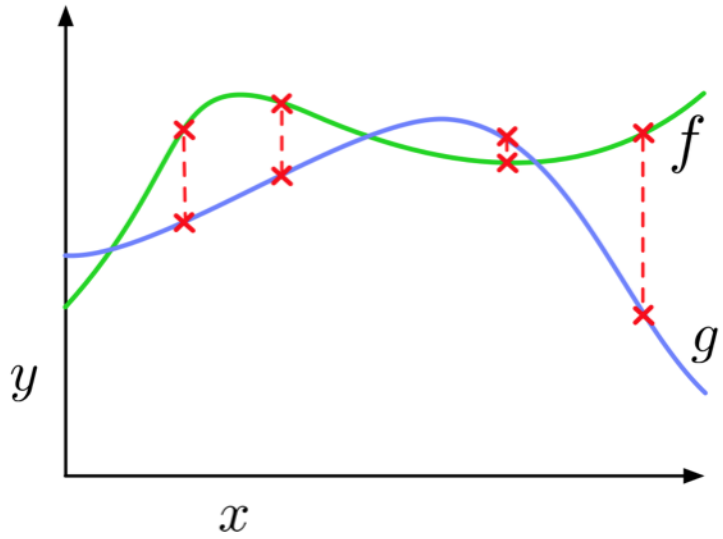
Function Space Distance

- For neural nets, function space distance measures how dissimilar the outputs are *in expectation*:

$$\rho_{\text{pull}}(\mathbf{w}, \mathbf{w}') = \mathbb{E}_{\mathbf{x}}[f_{\mathbf{x}}^* \rho(\mathbf{w}, \mathbf{w}')] = \mathbb{E}_{\mathbf{x}}[\rho(f(\mathbf{w}, \mathbf{x}), f(\mathbf{w}', \mathbf{x}))]$$

or the finite sample version:

$$\rho_{\text{pull}}(\mathbf{w}, \mathbf{w}') = \frac{1}{N} \sum_{i=1}^{N} [\rho(f(\mathbf{w}, \mathbf{x}^{(i)}), f(\mathbf{w}', \mathbf{x}^{(i)}))]$$



Function Space Distance

- Second-order Taylor approximation:

$$\begin{aligned}\mathbf{G}_{\mathbf{w}} &= \nabla^2_{\mathbf{w}} \rho_{\text{pull}}(\mathbf{w}, \mathbf{w}')\big|_{\mathbf{w}=\mathbf{w}'} \\ &= \mathbb{E}_{\mathbf{x}}\left[\nabla^2_{\mathbf{w}} \rho(f(\mathbf{w}, \mathbf{x}), f(\mathbf{w}', \mathbf{x}))\right] \\ &= \mathbb{E}_{\mathbf{x}}[\mathbf{J}_{\mathbf{zw}}^\top \mathbf{G}_{\mathbf{z}} \mathbf{J}_{\mathbf{zw}}].\end{aligned}$$

- I'll call this the pullback metric (there isn't a standard term in our field)
- We'll overload notation by using $\mathbf{G}$ for both the pullback metric and the Gauss-Newton Hessian. This is OK since the two matrices often coincide.

Matrix-Vector Products

- Computing matrix-vector products:

```
def pullback_mvp(f, rho, w, v):
    z0, R_z = jvp(f, (w,), (v,))
    rho_z0 = lambda z: rho(z, z0)
    R_gz = hvp(rho_z0, z0, R_z)
    _, f_vjp = vjp(f, w)
    return f_vjp(R_gz)[0]
```

- Compare with Gauss-Newton Hessian-vector products:

```
def gnhvp(f, L, w, v):
    z, R_z = jvp(f, (w,), (v,))
    R_gz = hvp(L, z, R_z)
    _, f_vjp = vjp(f, w)
    return f_vjp(R_gz)[0]
```

Connection to Gauss-Newton Hessian

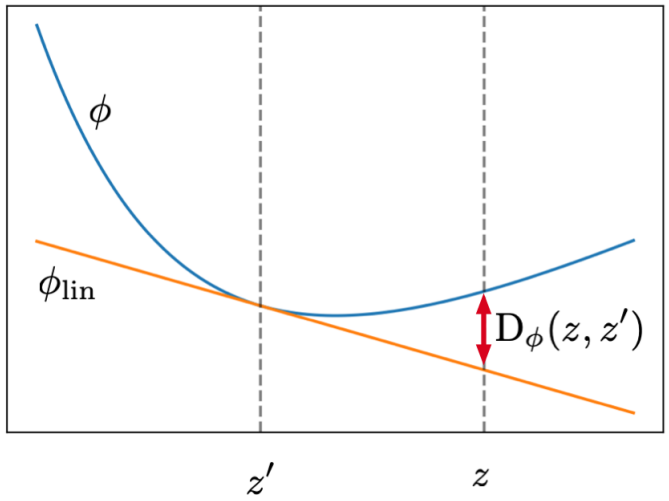
Connection to Gauss-Newton Hessian

- Recall:
  - Gauss-Newton Hessian: $\mathbf{G} = \mathbb{E}_{\mathbf{x}}[\mathbf{J}_{\mathbf{zw}}^\top \mathbf{H}_{\mathbf{z}} \mathbf{J}_{\mathbf{zw}}]$
  - Pullback metric: $\mathbf{G} = \mathbb{E}_{\mathbf{x}}[\mathbf{J}_{\mathbf{zw}}^\top \mathbf{G}_{\mathbf{z}} \mathbf{J}_{\mathbf{zw}}]$
- These are equal if $\mathbf{G}_{\mathbf{z}} = \mathbf{H}_{\mathbf{z}}$. When does this happen?
- **Special case:** squared error loss, with a Euclidean metric on the outputs. Then $\mathbf{H}_{\mathbf{z}} = \mathbf{G}_{\mathbf{z}} = \mathbf{I}$, and $\mathbf{G} = \mathbb{E}_{\mathbf{x}}[\mathbf{J}_{\mathbf{zw}}^\top \mathbf{J}_{\mathbf{zw}}]$ is the Gauss-Newton matrix

Bregman Divergence

- Let ϕ be a strictly convex function of $\mathbf{z}$
- Bregman divergence:

$$\mathrm{D}_\phi(\mathbf{z}, \mathbf{z}') = \phi(\mathbf{z}) - \phi(\mathbf{z}') - \nabla\phi(\mathbf{z}')^\top(\mathbf{z} - \mathbf{z}')$$



- **Examples:**
  - (squared) Euclidean distance:

$$\phi(\mathbf{z}) = \tfrac{1}{2}\|\mathbf{z}\|^2 \qquad \Longrightarrow \qquad \mathrm{D}_\phi(\mathbf{z}, \mathbf{z}') = \tfrac{1}{2}\|\mathbf{z} - \mathbf{z}'\|^2$$

  - $\mathbf{z}$ are natural parameters of an exponential family, ϕ is the log partition function $\Rightarrow \mathrm{D}_\phi$ is KL divergence

Bregman Divergence

- The Hessian of a Bregman divergence D_ϕ is just the Hessian of ϕ:

$$\begin{aligned}\nabla^2_{\mathbf{z}}\mathrm{D}_\phi(\mathbf{z},\mathbf{z}')\big|_{\mathbf{z}=\mathbf{z}'} &= \nabla^2_{\mathbf{z}}\left[\phi(\mathbf{z}) - \phi(\mathbf{z}') - \nabla\phi(\mathbf{z}')^\top(\mathbf{z}-\mathbf{z}')\right]\Big|_{\mathbf{z}=\mathbf{z}'} \\ &= \nabla^2_{\mathbf{z}}[\phi(\mathbf{z})]\big|_{\mathbf{z}=\mathbf{z}'} \\ &= \nabla^2\phi(\mathbf{z}')\end{aligned}$$

- If the output loss $\mathcal{L}$ is convex, we can choose $\phi = \mathcal{L}$
- Then $\mathbf{G}_{\mathbf{z}} = \mathbf{H}_{\mathbf{z}}$, and therefore the GN Hessian equals the metric matrix.

Fisher Information Matrix

Fisher Information Matrix

- An important special case of pullback metrics is when the outputs parameterize a probability distribution and ρ is KL divergence
  - Then $\mathbf{G_z} = \nabla^2 \rho$ is the Fisher information matrix $\mathbf{F_z}$
  - Pullback metric is $\mathbb{E}_{\mathbf{x}}[\mathbf{J}_{\mathbf{zw}}^\top \mathbf{F_z} \mathbf{J_{zw}}]$ as usual
- While the Fisher metric has many convenient properties, there's nothing *that* special about it when it comes to neural nets. For most algorithms, you're free to choose another output space metric.

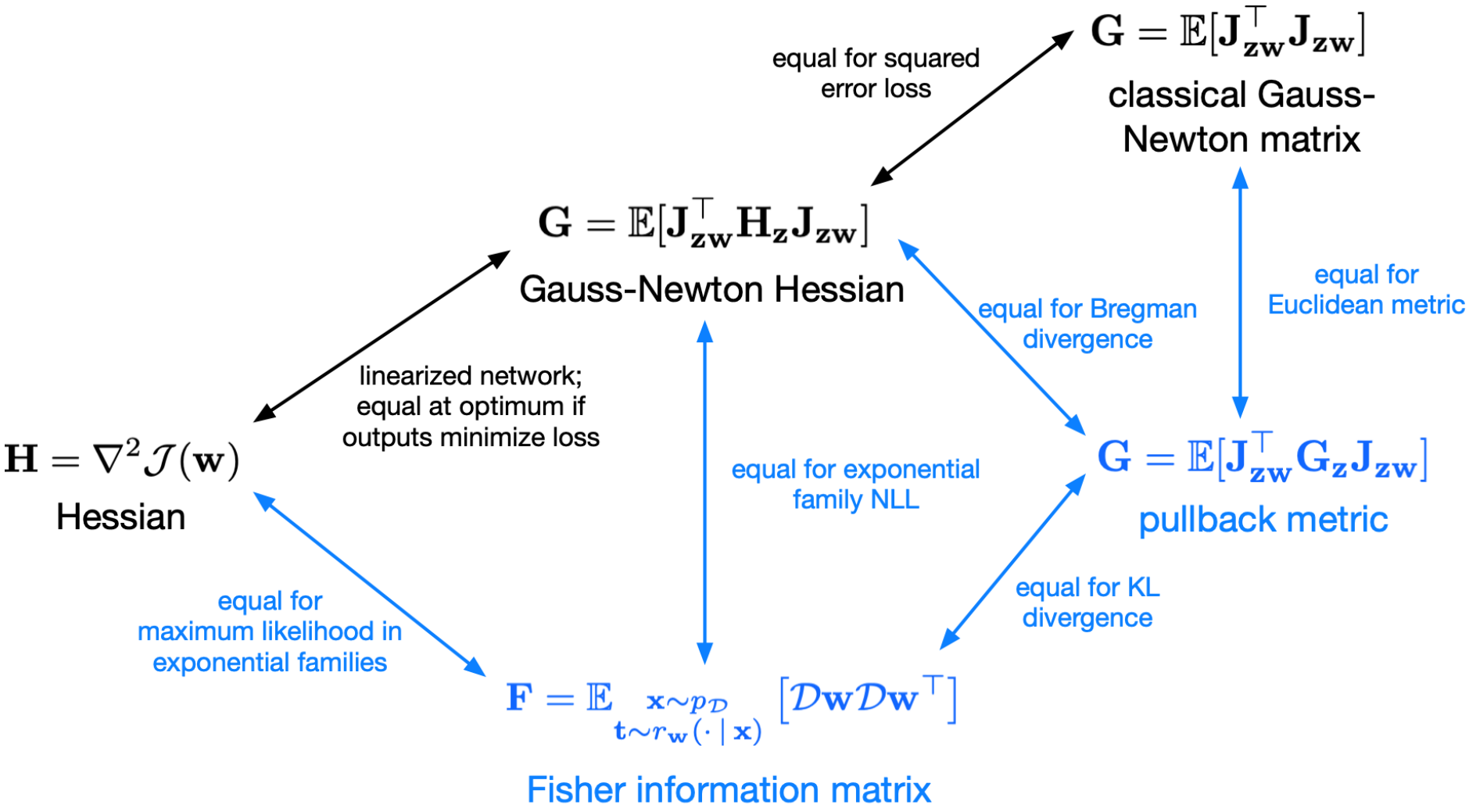
Fisher Information Matrix

- Let $\mathcal{D}\mathbf{z} = \nabla_{\mathbf{z}} \log p(\mathbf{t} \mid \mathbf{z})$
- We can simplify the pullback metric:

$$\begin{aligned}
\mathbf{F}_{\mathbf{w}} &= \mathbb{E}_{\mathbf{x} \sim p_{\text{data}}} \left[\mathbf{J}_{\mathbf{zw}}^{\top} \mathbf{F}_{\mathbf{z}} \mathbf{J}_{\mathbf{zw}} \right] && \text{(def'n)} \\
&= \mathbb{E}_{\mathbf{x} \sim p_{\text{data}}} \left[\mathbf{J}_{\mathbf{zw}}^{\top} \mathbb{E}_{\mathbf{t} \sim r(\cdot \mid \mathbf{x})} [\mathcal{D}\mathbf{z} \mathcal{D}\mathbf{z}^{\top}] \mathbf{J}_{\mathbf{zw}} \right] && \text{(def'n)} \\
&= \mathbb{E}_{\substack{\mathbf{x} \sim p_{\text{data}} \\ \mathbf{t} \sim r(\cdot \mid \mathbf{x})}} \left[\mathbf{J}_{\mathbf{zw}}^{\top} \mathcal{D}\mathbf{z} \mathcal{D}\mathbf{z}^{\top} \mathbf{J}_{\mathbf{zw}} \right] \\
&= \mathbb{E}_{\substack{\mathbf{x} \sim p_{\text{data}} \\ \mathbf{t} \sim r(\cdot \mid \mathbf{x})}} [\mathcal{D}\mathbf{w} \mathcal{D}\mathbf{w}^{\top}] && \text{(Chain Rule)}
\end{aligned}$$

- Since the final formula closely resembles that of the Fisher information matrix, we call this the Fisher information matrix for the network.

Fisher Information Matrix



Fisher Information Matrix

$$\mathbf{F} = \mathbb{E}_{\substack{\mathbf{x}\sim p_{\text{data}} \\ \mathbf{t}\sim r(\cdot\,|\,\mathbf{x})}}[\mathcal{D}\mathbf{w}\mathcal{D}\mathbf{w}^\top]$$

- **Caution:** don't confuse the Fisher matrix with the empirical Fisher matrix:

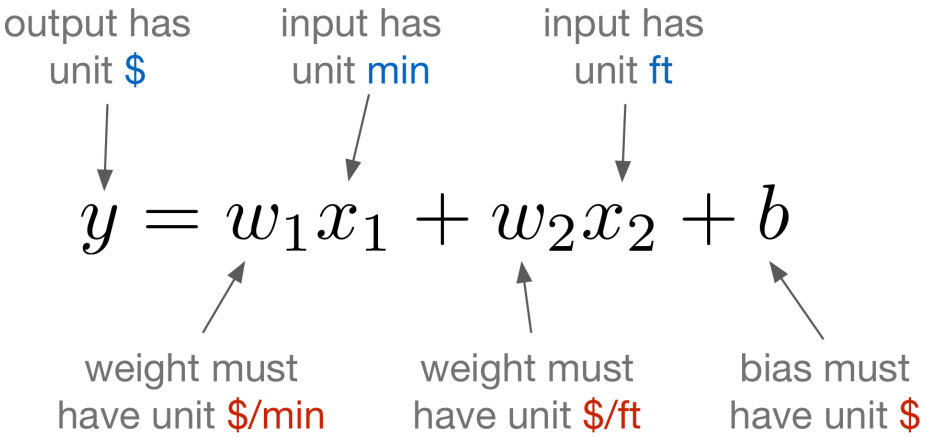
$$\mathbf{F}_{\text{emp}} = \mathbb{E}_{\mathbf{x},\mathbf{t}\sim p_{\text{data}}}[\mathcal{D}\mathbf{w}\mathcal{D}\mathbf{w}^\top]$$

- The difference is that the true Fisher matrix samples the targets from the model's predictions, while the empirical Fisher uses the training targets.
- Only the true Fisher is related to the Hessian. Using the empirical Fisher to approximate the Hessian is a common mistake (even in published papers!)

Invariance

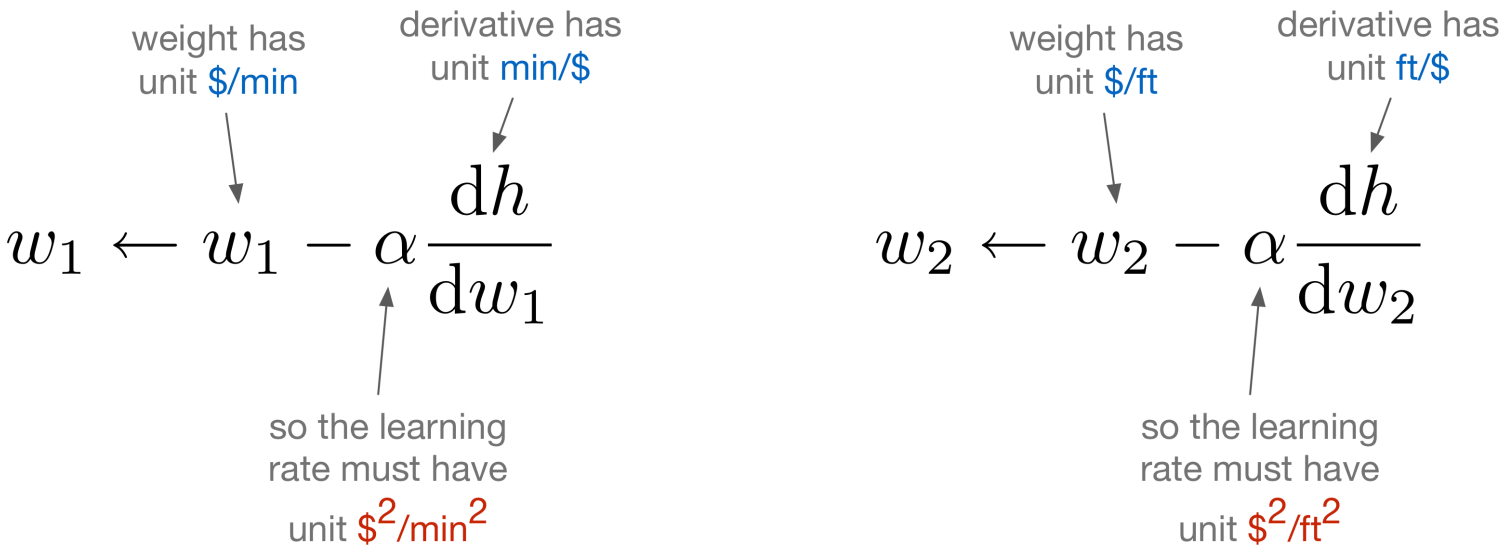
Invariance

- Various fields have bookkeeping devices to prevent us from performing nonsensical operations, e.g.
  - dimensional analysis
  - types in programming
- Recall from Lecture 1: gradient descent applied to linear regression is not invariant to affine transformations of the inputs
- What if we try to assign units to a linear regression problem:



Invariance

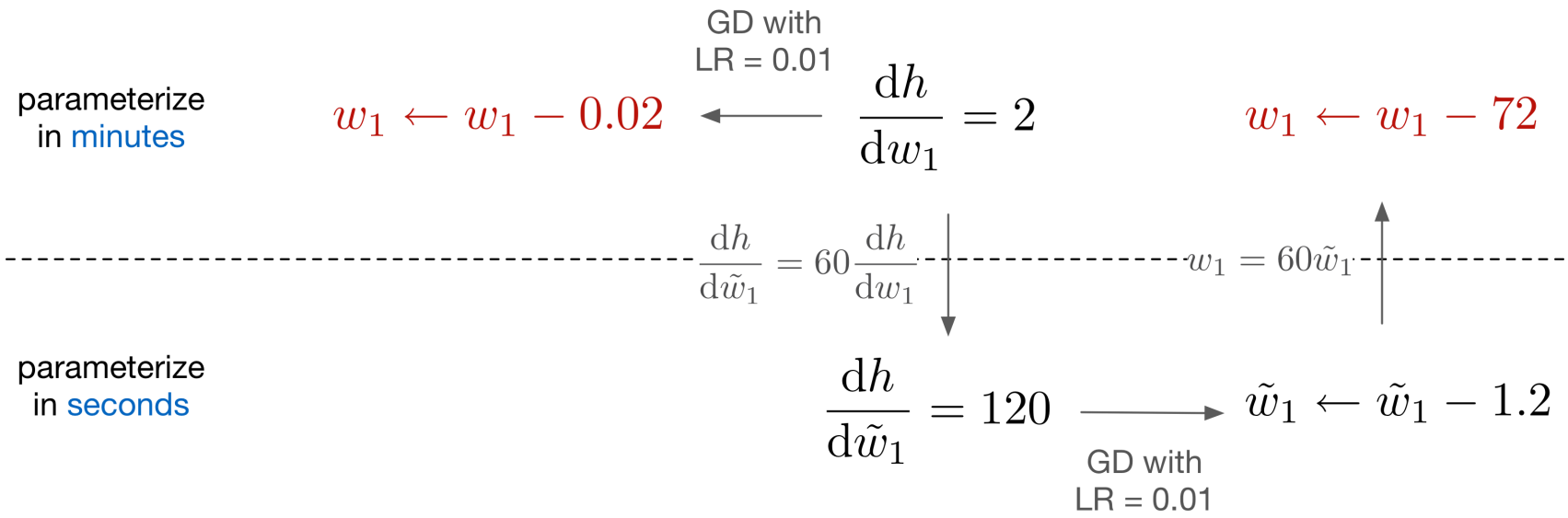
- Trying to attach units to the gradient descent update:



- No consistent assignment of dimensions!

Invariance

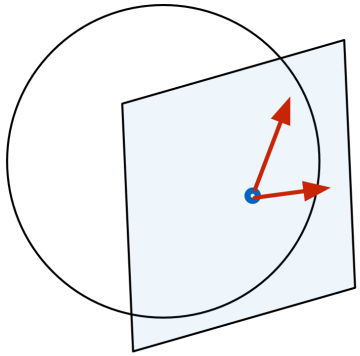
- Look what happens to the gradient descent update when you change from minutes to seconds:



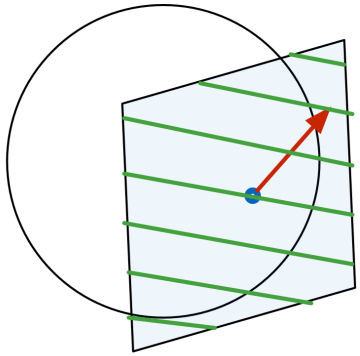
- The gradient gets 3600 times larger!

Invariance

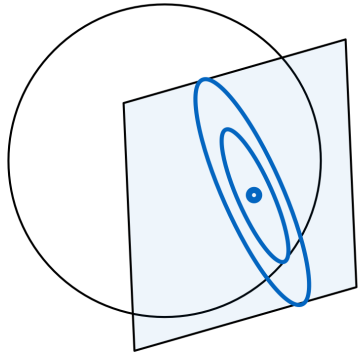
- Mathematicians have developed lots of abstractions to ensure that things are invariant to the choice of coordinate system.
- Vectors vs. covectors



Two vectors in the tangent space to M.



A vector in the tangent space, and a covector, visualized as a linear functional. When we apply this covector to this vector, it evaluates to 2 (count the contours).

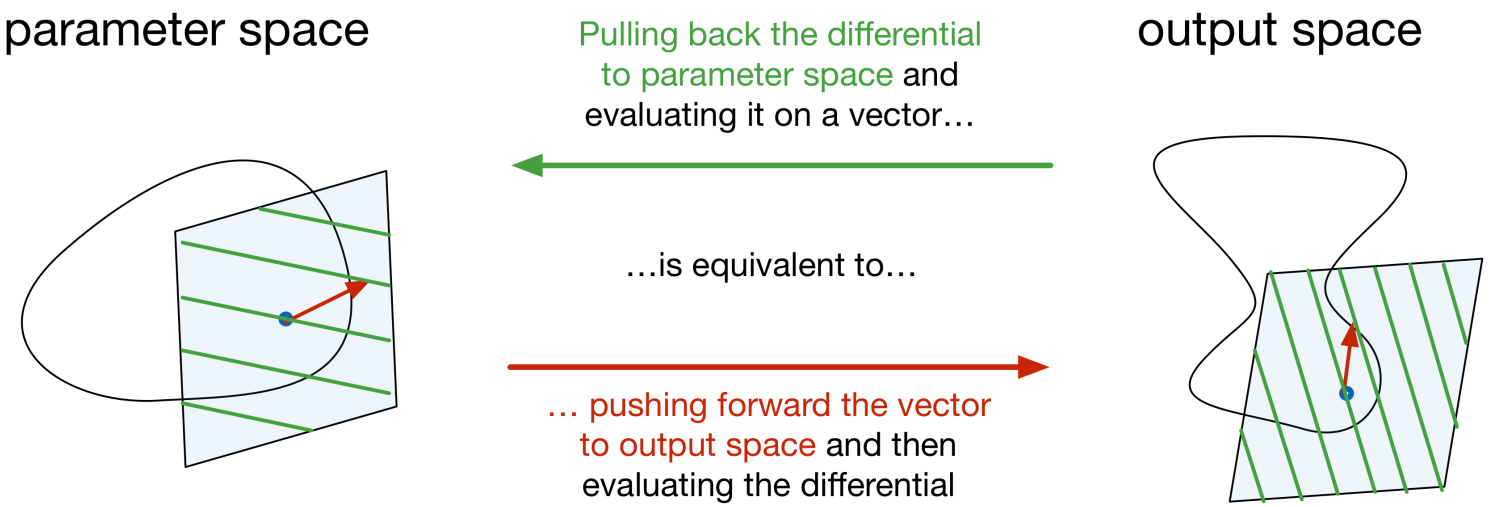


A Riemmanian metric gives an inner product on the tangent space at every point. This induces a norm.

- Important distinction: vectors can be pushed forward, and covectors can be pulled back.

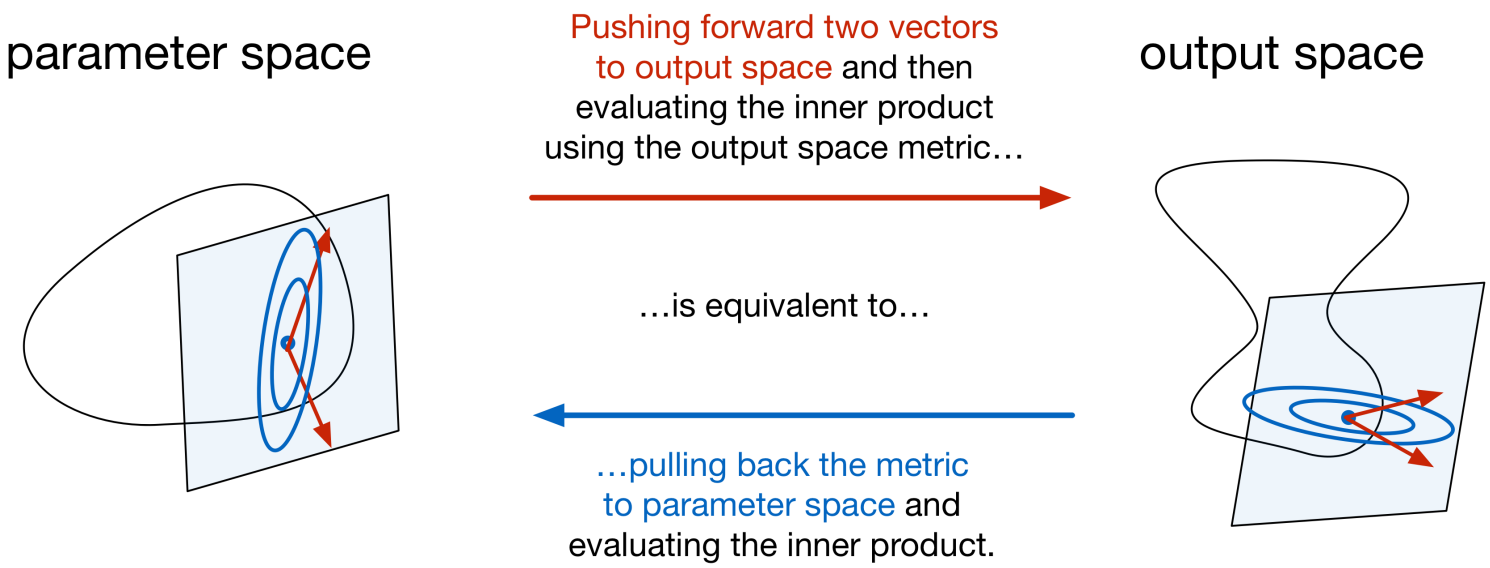
Invariance

- Pulling back a covector:



Invariance

- Pulling back a metric:



Invariance

- The thing we call the "gradient" is really a covector called the differential. We know it's a covector because you can pull it back (i.e. backprop).
- In order to update the parameters, you need a vector.
- Using a Riemannian metric, you can convert between covectors and vectors (this is the musical isomorphism)
- If the metric is defined in a coordinate-independent way (e.g. pullback from output space), then it is natural.
- When computed in a coordinate system, this coincides with our formula for the natural gradient:

$$\tilde{\nabla}\mathcal{J}(\mathbf{w}) = \mathbf{G}^{-1}\nabla\mathcal{J}(\mathbf{w})$$

Invariance

- To the first order, the natural gradient update is invariant to differentiable reparameterizations of the model.
  - I.e., the natural gradient updates in two coordinate systems will be equivalent (to first order) in terms of the functions represented by the network
  - If the coordinate systems are related by an affine transformation, then the invariance is exact
- In our field, "natural gradient" is often assumed to mean the Fisher metric, but there are lots of other natural metrics (and Amari considered other examples in his paper)
  - E.g., the pullback of a Euclidean metric on outputs is independent of how the network is parameterized (except for the output layer)