What If We Don't Know σ?

We've seen how to compute a confidence interval or P-value from the sample mean if we know σ, the standard deviation of one observation. But what if we don't?

We can use the sample standard deviation, s, as an estimate of σ. We can then estimate the standard deviation of $\bar{x}$ by $s/\sqrt{n}$. This is the (estimated) *standard error* for $\bar{x}$ as an estimate of the population mean, μ.

When can use this estimated standard error to standardize $\bar{x}$, producing the t *statistic* for a test of $H_0 : \mu = \mu_0$:

$$t \;=\; \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

If the observations are independent and normally distributed, this has what's called called a t *distribution*, with $n-1$ "degrees of freedom" — regardless of what σ really is.

A t Test for $\mu = 0$

If $x_1, \ldots, x_n$ are independent and normally distributed, we can use the t statistic in a hypothesis test of $H_0 : \mu = 0$ versus $H_a : \mu \neq 0$.

We know that under H_0 this statistic has the $t(n-1)$ distribution. For a two-tailed test, the P-value is the probability than $|t|$ is as big as we observed, according to this distribution.

Table B.3 in the book lets us check if this P-value is less than several values for α, so we can tell whether to reject H_0 at that significance level.

Example:

$H_0 : \mu = 0$, $H_a : \mu \neq 0$
$n = 5$, $\bar{x} = 0.3$, $s = 0.2$
$t = 0.3/(0.2/\sqrt{5}) = 3.354$, with 4 df

The table tells us we should reject H_0 at the $\alpha = 0.05$ level, but not at $\alpha = 0.01$ (recall, it's a two-sided test). The actual P-value is 0.0285.

Finding Confidence Intervals When σ is Unknown Using the t Distribution

We can also use the t distribution to find a confidence interval for the mean when we don't know σ.

To find a level C confidence interval for the mean, μ, from n observations, we find the t^* for which $P(|t| \leq t^*) = C$, or equivalently,

$$P(t > t^*) \;=\; P(t < -t^*) \;=\; (1-C)/2$$

assuming that t has the $t(n-1)$ distribution. Again, we use Table B.3 in the book.

The level C confidence interval for μ is then

$$(\bar{x} - t^* s/\sqrt{n},\ \bar{x} + t^* s/\sqrt{n})$$

Why? We can see that

$$P(\mu < \bar{x} - t^* s/\sqrt{n}) = P\left(\frac{\bar{x} - \mu}{s/\sqrt{n}} > t^*\right) = \frac{1-C}{2}$$

since $(\bar{x} - \mu)/(s/\sqrt{n})$ has the $t(n-1)$ distribution, and similarly, the other way.

Confidence Interval for the Light Experiment, With σ Unknown

Let's suppose we *don't* know σ for the light speed experiment. We'll calculate a 90% confidence interval for the mean difference in upward and downward times, from the same data as before:

$$0.21,\ -0.15,\ 0.19,\ 0.02$$

for which $n = 4$, $\bar{x} = 0.0675$, $s = 0.1682$.

For a 90% confidence interval when df $= 3$, we find $t^* = 2.353$. The confidence interval is therefore

$$(0.0675 - 2.353 * 0.1682/\sqrt{4},\ 0.0675 + 2.353 * 0.1682/\sqrt{4})$$

which works out to $(-0.130,\ 0.265)$.