Relating Response Variables to Explanatory Variables

We are often interesting in how a *response variable* relates to an *explanatory* variable (also called a *predictor* variable).

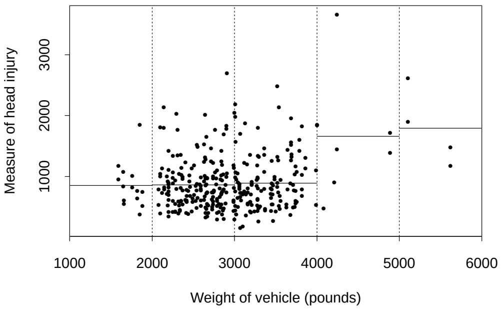
When our interest is in making predictions:

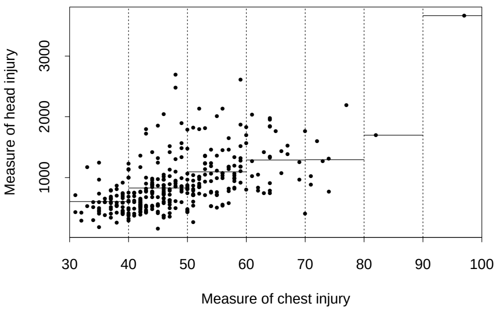
- The response variable is what we want to predict.
- The predictor variable is something we can measure to help us predict

When our interest is in cause-and-effect explanation:

- The response variable is what we think is the effect.
- The explanatory variable is what we think is the cause.

Prediction by Averaging Nearby Points

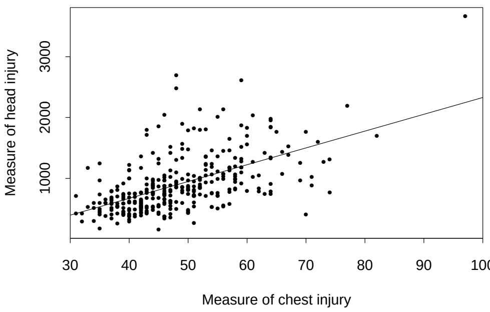




Linear Regression

Sometimes the response variable can be predicted from the explanatory variable by a straight line.

For the crashtest data:



The straight line gives the value to predict for the response variable for each value of the explanatory variable.

The "Least Squares" Regression Line

We can describe a line predicting y from x by the equation

$$\widehat{y} \;=\; a \,+\, b\,x$$

One criterion for the "best" line is the one that minimizes the sum of the squared prediction errors, that is

$$\sum_{i=1}^{n} (y_i - \widehat{y}_i)^2 \;=\; \sum_{i=1}^{n} (y_i - a - b\,x_i)^2$$

The values of a and b that minimize this are

$$a \;=\; \bar{y} - b\,\bar{x}, \qquad b \;=\; r\,\frac{s_y}{s_x} \;=\; \frac{s_{xy}}{s_x^2}$$

where $\bar{x}$ and $\bar{y}$ are the means, s_x and s_y are the standard deviations, s_{xy} is the covariance of x and y, and r is the correlation of x and y.

Derivation of the Least Squares Line

We need to find a and b that minimize

$$E \;=\; \sum_{i=1}^{n} \left(y_i - a - b\,x_i\right)^2$$

The minimum should be at a point where the derivative with respect to a is zero:

$$\frac{\partial E}{\partial a} \;=\; -2\sum_{i=1}^{n} \left(y_i - a - b\,x_i\right) \;=\; 0$$

This is equivalent to

$$\frac{1}{n}\sum_{i=1}^{n} \left(y_i - a - b\,x_i\right) \;=\; \bar{y} - a - b\,\bar{x} \;=\; 0$$

which implies $a = \bar{y} - b\,\bar{x}$.

One consequence: The regression line goes through the point $(\bar{x}, \bar{y})$.

Derivation of Least Squares (Continued)

At the minimum, the derivative with respect to b should also be zero:

$$\frac{\partial E}{\partial b} \;=\; -2\sum_{i=1}^{n} \left(y_i - a - b\,x_i\right)x_i \;=\; 0$$

Using $a = \bar{y} - b\bar{x}$, we get

$$-2\sum_{i=1}^{n} \left(y_i - (\bar{y} - b\bar{x}) - b\,x_i\right)x_i \;=\; 0$$

from which

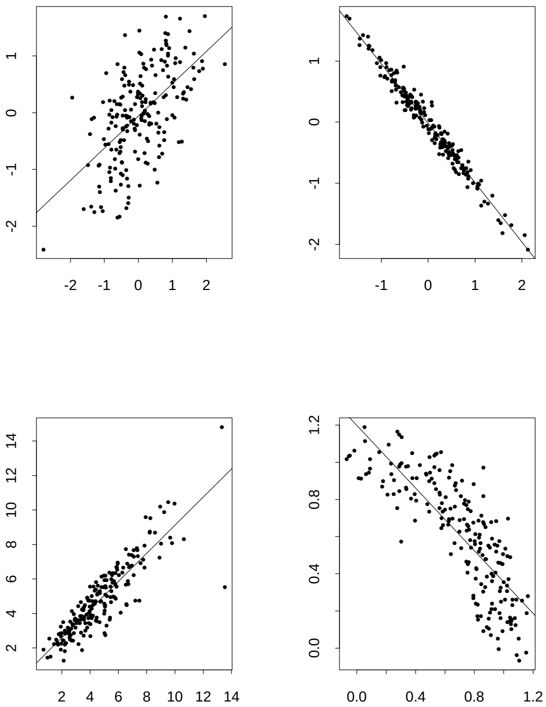
$$\frac{1}{n-1}\sum_{i=1}^{n} \Big((y_i - \bar{y}) - b(x_i - \bar{x})\Big)x_i \;=\; 0$$

Notice that $\sum(y_i - \bar{y}) = 0$, $\sum(x_i - \bar{x}) = 0$, and hence $\sum\Big((y_i - \bar{y}) - b(x_i - \bar{x})\Big)\bar{x} = 0$. So,

$$\frac{1}{n-1}\sum_{i=1}^{n} \Big((y_i - \bar{y}) - b(x_i - \bar{x})\Big)(x_i - \bar{x}) \;=\; 0$$

which is $s_{xy} - b s_x^2 \;=\; 0$, or $b = \dfrac{s_{xy}}{s_x^2}$.

Some Examples of Regression Lines

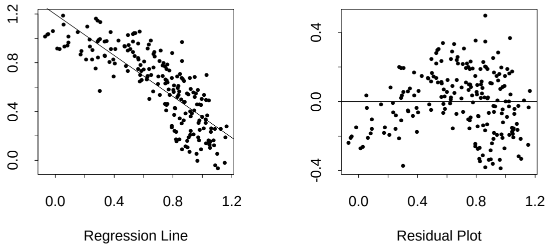


The Residuals for Observations

The *residual* is the difference between the observed response and the predicted response:

$$\text{residual for } y_i \;=\; y_i - \hat{y}_i \;=\; y_i - (a + bx_i)$$

The mean of the residuals from least squares regression is always zero, but there could be a pattern to them:



This pattern here (negative residual for large or small x, positive for x in the middle) shows that the relationship is not exactly linear.