STA 247 — Solutions to Assignment #1, Part I

1a) Prove that if $P(A \cap C) > 0$ and $P(B \cap C) > 0$, then

$$P(A|B \cap C) \;=\; \frac{P(B|A \cap C)P(A|C)}{P(B|C)}$$

Substituting for the definition of conditional probability, the left side is

$$\frac{P(A \cap B \cap C)}{P(B \cap C)}$$

Similarly, the right side is

$$\frac{P(B \cap A \cap C)}{P(A \cap C)}\,\frac{P(A \cap C)}{P(C)} \Big/ \frac{P(B \cap C)}{P(C)} \;=\; \frac{P(B \cap A \cap C)}{P(B \cap C)}$$

This is equal to the left side, so the result is proved.

1b) Prove that if $P(C) > 0$, then $P(A|C) \;=\; P(A \cap B|C) \;+\; P(A \cap B^c|C)$.

Again, substituting for the definition of conditional probability, we find that the right side is

$$\frac{P(A \cap B \cap C)}{P(C)} + \frac{P(A \cap B^c \cap C)}{P(C)} \;=\; \frac{P(A \cap B \cap C) \;+\; P(A \cap B^c \cap C)}{P(C)}$$

Since $A \cap B \cap C$ and $A \cap B^c \cap C$ are mututally exlusive, and their union is $A \cap C$, we can rewrite this as

$$\frac{P(A \cap C)}{P(C)}$$

which is the definition of $P(A|C)$, as desired.

2) Suppose that A, B, and C are mutually independent events in some sample space, S. (a) Prove that $A \cap B$ and C are independent. (b) Prove that $A \cup B$ and C are independent.

From the definition of mutually independence, $P(A \cap B) = P(A)P(B)$ and $P(A \cap B \cap C) = P(A)P(B)P(C)$. Combining these, we get that $P(A \cap B \cap C) = P(A \cap B)P(C)$, which is the definition of independence of $A \cap B$ and C.

To show that $A \cup B$ and C are independent, we need to show that

$$P((A \cup B) \cap C) = P(A \cup B)P(C)$$

The left side can be rewritten as follows, using the formula for the probability of the union of events:

$$\begin{aligned}
P((A \cup B) \cap C) &= P((A \cap C) \cup (B \cap C)) \\
&= P(A \cap C) \;+\; P(B \cap C) \;-\; P(A \cap C \cap B \cap C) \\
&= P(A \cap C) \;+\; P(B \cap C) \;-\; P(A \cap B \cap C)
\end{aligned}$$

From this, using the mututal independence of A, B, and C, we get

$$
\begin{aligned}
P((A \cup B) \cap C) &= P(A)P(C) \; + \; P(B)P(C) \; - \; P(A)P(B)P(C) \\
&= (P(A) + P(B) - P(A)P(B))\, P(C)
\end{aligned}
$$

Finally, applying the formula for the probability of the union of events in reverse, we get

$$
P((A \cup B) \cap C) \;=\; P(A \cup B)P(C)
$$

which completes the proof.

3) You have three urns containing red and black balls. Urn 1 initially contains 3 red and 4 black balls. Urn 2 initially contains 3 red and 2 black balls. Urn 3 initially contains 2 red and 2 black balls. You randomly select a ball from Urn 1 and place it in Urn 2. You then select a ball at random from Urn 2 and place it in Urn 3. Finally, you select a ball at random from Urn 3. You see that this ball is black, but you didn't look at the balls you selected earlier. Given this, what is the probability that the ball you took from Urn 1 was black?

Let B_1, B_2, and B_3 be the events of drawing a black ball from Urn 1, Urn 2, and Urn 3, respectively. We need to find $P(B_1|B_3)$, which we can rewrite as follows:

$$
\begin{aligned}
P(B_1|B_3) &= \frac{P(B_1 \cap B_3)}{P(B_3)} \\
&= \frac{P(B_1 \cap B_2 \cap B_3) \; + \; P(B_1 \cap B_2^c \cap B_3)}{P(B_1 \cap B_2 \cap B_3) \; + \; P(B_1 \cap B_2^c \cap B_3) + P(B_1^c \cap B_2 \cap B_3) \; + \; P(B_1^c \cap B_2^c \cap B_3)}
\end{aligned}
$$

From the description of the problem, we can find these probabilities as follows:

$$
\begin{aligned}
P(B_1 \cap B_2 \cap B_3) &= P(B_1)P(B_2|B_1)P(B_3|B1 \cap B2) &= (4/7)(3/6)(3/5) \\
P(B_1 \cap B_2^c \cap B_3) &= P(B_1)P(B_2^c|B_1)P(B_3|B1 \cap B2^c) &= (4/7)(3/6)(2/5) \\
P(B_1^c \cap B_2 \cap B_3) &= P(B_1^c)P(B_2|B_1)P(B_3|B1^c \cap B2) &= (3/7)(2/6)(3/5) \\
P(B_1^c \cap B_2^c \cap B_3) &= P(B_1^c)P(B_2^c|B_1^c)P(B_3|B1^c \cap B2) &= (3/7)(4/6)(2/5)
\end{aligned}
$$

Substituting these values int the formula above, we get the answer $P(B_1|B_3) \;=\; 0.588235$.

There are probably many other ways of solving this problem as well.

4) A computer has been set up with two redundant disk drives, called drive 1 and drive 2, with all data being "mirrored" on both drives, so that if one fails the computer can continue to operate without interruption and without loss of data. Computers can also fail for other reasons, of course. Suppose that drive 1 working correctly, drive 2 working correctly, and the rest of the computer working correctly are mutually independent events. Suppose also that, on this day, the probability of drive 1 failing is 0.01, the probability of drive 2 failing is 0.02, and the probability of some other failure is 0.03. Find the probability that the computer will fail to operate this day (because either both drives fail or something else goes wrong).

Let W_1 be the event that drive 1 works, W_2 be the event that drive 2 works, and W_0 be the event that the rest of the computer works. Let F be the event that the computer fails to work. From *the description of the system, $F = ((W_1 \cup W_2) \cap W_0)^c$ — ie, the computer fails unless at least one drive works and the rest of the computer also works. Hence the probability of failure is*

$$P(F) \;=\; 1 - P((W_1 \cup W_2) \cap W_0)$$

Using the result from question 2b above, we see that $(W_1 \cup W_2)$ and W_0 are independent. Using this, and the formula for the probability of a union of events, and the independence of W_1 and W_2, we get

$$\begin{aligned} P(F) &= 1 - P(W_1 \cup W_2)\, P(W_0) \\ &= 1 - (P(W_1) + P(W_2) - P(W_1 \cap W_2))\, P(W_0) \\ &= 1 - (P(W_1) + P(W_2) - P(W_1)P(W_2))\, P(W_0) \\ &= 1 - (0.99 + 0.98 - 0.99 \times 0.98) \times 0.97 \quad = \quad 0.030194 \end{aligned}$$

So by mirroring the two drives, the probability of failure has been reduced to only slightly more than the probability of failure from some other cause.