Last Time:

**Sherali-Adams (SA):** A method for obtaining a sequence of successively better LP-approximations to the integer hull of a given polytope $P$, $\text{hull}(P) = \text{Conv}(P \cap \{0, 1\}^n)$

\[ P = \text{SA}_0(P) \supseteq \text{SA}_1(P) \supseteq \ldots \supseteq \text{SA}_n(P) = \text{hull}(P) \]

- View $\text{SA}_j$ as a set of tests for trying to between $\alpha \in \text{hull}(P)$ from $\alpha \not\in \text{hull}(P)$

  - For simplicity, assume $\text{hull}(P) = [0, 1]^n$
  
  \[ \alpha \in \text{hull}(P) \Rightarrow \alpha = \sum_{x \in \{0, 1\}^n \cap \text{hull}(P)} x \lambda_x, \quad \sum \lambda_x = 1, \lambda_x \geq 0 \]

  \[ \alpha = \frac{3}{4} (1) + \frac{1}{4} (0) \]

- $\text{SA}_j$ is a set of tests that tries to ensure a given $\alpha \in \mathbb{R}^n$ defines a distribution over $P \cap \{0, 1\}^n$ by looking at the marginal distributions of $\nu(\omega)$

**Marginal Distribution:** For $S \subseteq [n]$, distribution $\mu$, $\nu_S : \{0, 1\}^S \to \mathbb{R}$ is defined as

\[ \nu_S(y) = \sum_{x \in \{0, 1\}^n : x_S = y} \mu(x) \quad \forall y \in \{0, 1\}^S \]

\[ \mu_S(y) = \frac{1}{\text{Pr}_x [x_S = y, \forall i \notin S]} \]

- $\nu_S : \{0, 1\}^S \to \mathbb{R}$ is a probability distribution iff $\forall S \subseteq T \subseteq [n]$

\[ \nu_T(y) = \sum_{x \in \{0, 1\}^S : x_T = y} \nu_S(x) \]

\[ \mu_T(y) \geq \sum_{x \in \{0, 1\}^S : x_T = y} \frac{\nu_S(x)}{\text{Pr}_x [x_S = x, \forall i \notin S]} \]

- $\mu_S(y) \geq \mu_T(y)$

**Consistent Marginals**
2. \( p(y) \geq 0 \)
3. \( \sum p(y) = 1 \)  
   **Non-negativity**
   **Normalized**

SA tests whether \( p \) is a distribution by checking whether (1)-(3) hold.

- Exponentially many such checks to do... *Inefficient!*
- Level-1 SA only tests the marginals of at most \( d \) variables

**Degree-1 Pseudo-Distribution**

1. \( p(y) = \sum_{x \in S} p_T(x) \) s.t. \( x \in S \Rightarrow y \)
2. \( p(y) \geq 0 \)
3. \( \sum p(y) = 1 \)

SA Tests that these conditions hold!

I claimed that SA was a family of LPs \( \setminus \) polytopes

- These tests can be encoded as LP constraints!

**Conjunct:** \( \mathbf{J}_{S,T} = \prod_{i \in S} x_i \prod_{j \in T} (1-x_j) \)

Let \( W \subseteq \mathbb{R}^J \) and \( z \in \mathbb{N}^W \), \( s= \sum_i z_i = 1 \), \( T=\{ j \mid z_j = 0 \} \)

\( \mu_W(z) = \Pr_{x \in \mathbb{N}} \left[ x|W = z \right] = \prod_{i \in S} x_i \prod_{j \in T} (1-x_j) = \mathbb{E}[\mathbf{J}_{S,T}] \)

Conjuncts are tests for the marginal distributions of \( \mu \).

**Degree-1 Pseudo-Distribution**

\( \mathcal{E} \text{ iff } \forall S,T \subseteq \mathbb{N}, \mathbf{J}_{S,T} = \mathcal{E} \)  
   **(Pseudo-Expectation)**
a) $\hat{E}[J_{ST}(x)] \geq 0$

b) $\tilde{E}[1] = 1$

**Claim:** These two definitions are equivalent.

**Proof:**

Def 1 $\Rightarrow$ Def 2 is straightforward.

Def 2 $\Rightarrow$ Def 1:

Let $W \subseteq [n] \times \{0,1\}^w$, $S = \{ i \mid x_i = 1 \}, T = \{ i \mid x_i = 0 \}$

Define

$\nu_W(x) = \frac{\tilde{E}[J_{ST}, T]}{\nu} \geq 0$. (non-negative)

Then $\nu_\emptyset = 1$ (normalized)

Fix $Z \supseteq W$, then (consistent marginals)

$$\sum_{y \in \{0,1\}^Z} \nu_Z(y) = \sum_{y \in \{0,1\}^Z} \nu \tilde{E}[J_{ST}, J_z, z \times (\cup W U Z)] = \nu \tilde{E}[\sum_{z \subseteq Z \setminus W} J_{ST}, J_z, z \times (\cup W U Z)] = \nu \tilde{E}[J_{ST}] = \nu_W(x)$$

We can encode the constraints of a degree-$d$ PE as an LP,

- Think of $\tilde{E}$ as a $n^d$ vector with one entry for each monomial $J_S \emptyset$ of degree $\leq d$.

- The variables of our LP will be $y_S$ for $S \subseteq [n], |S| \leq d$, which encode $\tilde{E}[J_S \emptyset]$.

**Degree-$d$ SA LP:** Defined in stages.

- For all $S, T \subseteq [n], S \cap T = \emptyset, |S \cup T| \leq d$, add a constraint

$$\prod_{x \in S} \prod_{j \in T} (1-x_j) \geq 0$$
- Linearize: Replace every monomial $J_{s,\phi}$ with $y_s$.

- $n$ (size of) constraints & variables.

- A solution in $\{0,1\}^n$ can be obtained by projecting $y \in \{0,1\}^n$ to $y' \in \{0,1\}^n$, i.e., $y'_i = y_i$.

- So far we have focused on testing whether a point $\alpha \in \text{conv}(\{0,1\}^n)$. We actually want $\alpha \in \text{conv}(\text{PNECG}) = \text{hull} (\mathcal{P})$.

- Modify our pseudo expectation to test whether $\alpha$ defines a distribution on $\{0,1\}^n \setminus \mathcal{P}$, rather than $\{0,1\}^n$.

- Idea: Marginals also satisfy constraints of $\mathcal{P}$

* Throughout, we work modulo $\langle x_i^2 - x_i, 0 \rangle$, i.e. $x_i^2 = x_i$.

**Pseudo-Expectation For $\mathcal{P}$**: Let $\mathcal{P} = \{p_1 > 0, \ldots, p_m > 0\}$, $\mathcal{E}$ is a PE for $\mathcal{P}$ if $\forall S \subseteq [n], S \cap T = \emptyset, \sum_{i \in S} p_i \leq 1$.

1. $\mathcal{E}[p, J_{s,T}] \geq 0 \quad \forall p, \alpha \in \mathcal{P}$
2. $\mathcal{E}[J_{s,T}] \geq 0$
3. $\mathcal{E}[1] = 1$  

$(\text{SAR}^\alpha(\mathcal{P}))$

**Degree-3 $\text{SA LP}$**: Defined in stages.

- For all $S, T \subseteq [n], \text{SNT} = \emptyset, |\text{SUT}| = 1$, add a constraint $\prod_{i \in S} p_i \prod_{i \in T} (1-x_i) \geq 0$.

- Multilinearize: replace $x_i^2$ by $x_i$ everywhere. Replace each monomial $J_{s,\phi}$ by $y_s$. 


Degree-

PDs for \( P \) are true distributions over \( \mathbb{R}^n \setminus P \)

\[ \text{proj}(SA_n(P)) = \text{hull}(P) \]

projection to \( x_1, \ldots, x_n \)

\( SA_d(P) \) is an LP with \( n^\Omega(d) \) constraints & variables
- Can be solved in time \( n^{\Omega(d)} \)!

**Sherali-Adams Proofs**

Suppose \( P \cap \{0,1\}^n = \emptyset \). What degree of the Sherali-Adams hierarchy is needed to witness this?

- i.e. for what \( d \) is \( SA_d(P) = \emptyset \)?

A Sherali-Adams proof of \( P = \{p_1, \ldots, p_m \} \) certifies that \( P \cap \{0,1\}^n = \emptyset \)!
Conical Junta: $J = \sum x_i J_i$ where $J_i$ is a conjunct, $x_i \neq 0$

* Through out we work modulo $\langle x_i^2 - x_i = 0 \rangle$ ie $x_i^2 = x_i$

Shorli-Adams Proof: A SA proof of a polynomial $\mathcal{P}$ from $P$ is an expression of the form

$$\sum p_i J_i + J = \mathcal{P}$$

where $J_i, J$ are conical junctors

SA Refutation: $\sum p_i J_i + J = -1$

non-negative negative only possible if no $x \in [0, 1]^n$ simultaneously satisfies $p(x) > 0$

Degree: Maximum degree of $J_i, J$

Thm: There is a degree-$d$ SA refutation of $P$ iff $SA_d(P) = \emptyset$

Proof: Suppose $SA_d(P) \neq \emptyset$ then $\exists \alpha \in SA_d(P), \alpha$ defines a degree-$d$ PE $\hat{P}$

Suppose $\sum p_i J_i + J = -1$ is a degree-d SA refutation of $P$. Then

$$-1 = \hat{P}[-1] = \hat{P}[\sum p_i J_i + J] = \sum \hat{P}[p_i J_i] + \hat{P}[J] \geq 0$$

Contradiction!