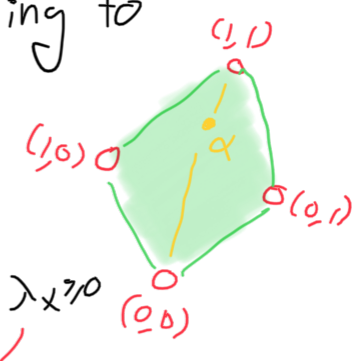
Last Time:

Sherali-Adams (SA): A method for obtaining a sequence of successively better LP-approximations to the integer hull of a given polytope P, $\text{hull}(P) = \text{Conv}(P \cap \{0,1\}^n)$

$$P = SA_0(P) \supseteq SA_1(P) \supseteq \cdots \supseteq SA_n(P) = \text{hull}(P)$$

→ View SA_d as a set of tests for trying to between $\alpha \in \text{hull}(P)$ from $\alpha \notin \text{hull}(P)$



$\alpha = \frac{3}{4}(1,1) + \frac{1}{4}(0,0)$

- For simplicity assume $\text{hull}(P) = [0,1]^n$

→ $\alpha \in \text{hull}(P) \Rightarrow \alpha = \sum_{x \in \{0,1\}^n \cap \text{hull}(P)} x \lambda_x, \ \sum \lambda_x = 1, \lambda_x \geq 0$

→ SA_d is a set of tests that tries to ensure a given $\alpha \in \mathbb{R}^n$ defines a distribution over $P \cap \{0,1\}^n$ by looking at the marginal distributions of $\mu^{(\alpha)}$

α defines a probability distribution $\mu^{(\alpha)}$ over $\{0,1\}^n$ where $\mu^{(\alpha)}(x) = \lambda_x$

Marginal Distribution: For $S \subseteq [n]$, distribution μ, $\mu_S : \{0,1\}^n \to \mathbb{R}$ is defined as

$$\mu_S(y) = \sum_{\substack{x \in \{0,1\}^n \\ x \restriction S = y}} \mu(x) \qquad \forall y \in \{0,1\}^S$$

ie $\mu_S(y) = \Pr_{x \sim \mu}[x_i = y_i \ \forall i \in S]$

Clm: $\mu : \{0,1\}^n \to \mathbb{R}$ is a probability distribution iff $\forall S \subseteq T \subseteq [n]$, $\forall y \in \{0,1\}^S$

1. $\mu_S(y) = \sum_{\substack{x \in \{0,1\}^T \\ x \restriction S = y}} \mu_T(x)$ consistent marginals

2. $\mu_S(y) \geq 0$
3. $\mu_\emptyset = 1$

non-negativity
normalized

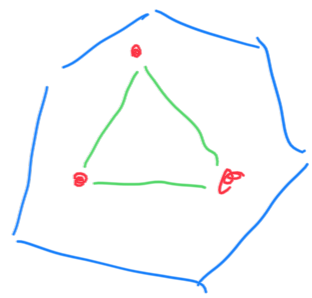
→ SA tests whether μ is a distribution by checking whether (1)-(3) hold.

→ Exponentially many such checks to do... Inefficient!

→ Level-d SA only tests the marginals of at most d variables

Degree-d Pseudo-Distribution: Iff $\forall S \subseteq T \subseteq [n]$ with $|T| \leq d$, $\forall y \in \{0,1\}^S$

1. $\mu_S(y) = \sum_{\substack{x \in \{0,1\}^T \\ x \restriction S = y}} \mu_T(x)$
2. $\mu_S(y) \geq 0$
3. $\mu_\emptyset = 1$



SA Tests that these conditions hold!

I claimed that SA was a family of LPs \ polytopes

→ These tests can be encoded as LP constraints!

Conjunct: $J_{S,T} = \prod_{i \in S} x_i \prod_{j \in T} (1 - x_j)$

Let $W \subseteq [n]$ and $z \in \{0,1\}^W$, $S = \{i \mid z_i = 1\}$, $T = \{j \mid z_j = 0\}$

$$\mu_W(z) = \Pr_{x \sim \mu}[x \restriction W = z] = \Pr\left[\prod_{i \in S} x_i \prod_{j \in T} (1 - x_j) = 1\right] = \mathbb{E}[J_{S,T}]$$

Conjuncts are tests for the marginal distributions of μ!

Degree-d Pseudo-Distribution: $\tilde{\mathbb{E}}$ iff $\forall S, T \subseteq [n]$, $S \cap T = \emptyset$, $|S \cup T| \leq d$

(Pseudo-Expectation)

a) $\hat{\mathbb{E}}[J_{S,T}(x)] \geq 0$

b) $\tilde{\mathbb{E}}[1] = 1$

Claim: These two definitions are equivalent

Pf: Def 1 $\rightarrow$ def 2 is straight forward

Def 2 $\Rightarrow$ def 1:

Let $W \subseteq [n]$ $x \in \{0,1\}^W$, $S = \{i \mid x_i = 1\}$, $T = \{i \mid x_i = 0\}$

define

$$\mu_W(x) = \tilde{\mathbb{E}}_{\mu}[J_{S,T}] \geq 0. \text{ (non-negative)}$$

Then $\mu_\emptyset = 1$ (normalized)

Fix $Z \supseteq W$, then (consistent marginals)

$$\sum_{\substack{y \in \{0,1\}^Z \\ y \restriction W = x}} \mu_Z(y) = \sum_{Z' \subseteq Z \setminus W} \tilde{\mathbb{E}}[J_{S,T} \cdot J_{Z', Z \setminus (W \cup Z')}]$$

$$= \tilde{\mathbb{E}}[J_{S,T} \underbrace{\sum_{Z' \subseteq Z \setminus W} J_{Z', Z \setminus (W \cup Z')}}_{= 1 \text{ e.g } x + (1-x) = 1}]$$

$$= \tilde{\mathbb{E}}[J_{S,T}] = \mu_W(x) \quad \square$$

We can encode the constraints of a degree-d PE as an LP!

- Think of $\tilde{\mathbb{E}}$ as a n^d vector with one entry for each monomial $J_{S,\emptyset}$ of degree $\leq d$
- The variables of our LP will be y_S for $S \subseteq [n]$, $|S| \leq d$, which encode $\mathbb{E}[J_{S,\emptyset}]$ $\mathbb{E}[J_{S,\emptyset}]$

Degree-d SA LP: Defined in stages.

- For all $S, T \subseteq [n]$, $S \cap T = \emptyset$, $|S \cup T| \leq d$, add a constraint

$$\prod x_i \prod (1 - x_i) \geq 0$$

$\prod_{i \in S} \prod_{j \in T}$

- Linearize: Replace every monomial $J_{S,\emptyset}$ with y_S.

→ $n^{O(d)}$ constraints & variables.

→ A solution in $\{0,1\}^n$ can be obtained by projecting $y \in \{0,1\}^{n^d}$ to $y' \in \{0,1\}^n$ ie, $y'_i = y_{\{i\}}$

→ So far we have focused on testing whether a point $\alpha \in \mathrm{conv}(\{0,1\}^n)$. We actually want $\alpha \in \mathrm{conv}(P \cap \{0,1\}^n) = \mathrm{hull}(P)$.

→ Modify our pseudo expectation to test whether α defines a distribution on $\{0,1\}^n \cap P$, rather than $\{0,1\}^n$

- Idea: Marginals also satisfy constraints of P

* Throughout, we work modulo $\langle x_i^2 - x_i = 0 \rangle$, ie $x_i^2 = x_i$

Pseudo-Expectation For P: Let $P = \{p_1 \geq 0, \dots, p_m \geq 0\}$, $\tilde{\mathbb{E}}$ is a PE for P if $\forall S,T \subseteq [n]$, $S \cap T = \emptyset$, $|S \cup T| \leq d$

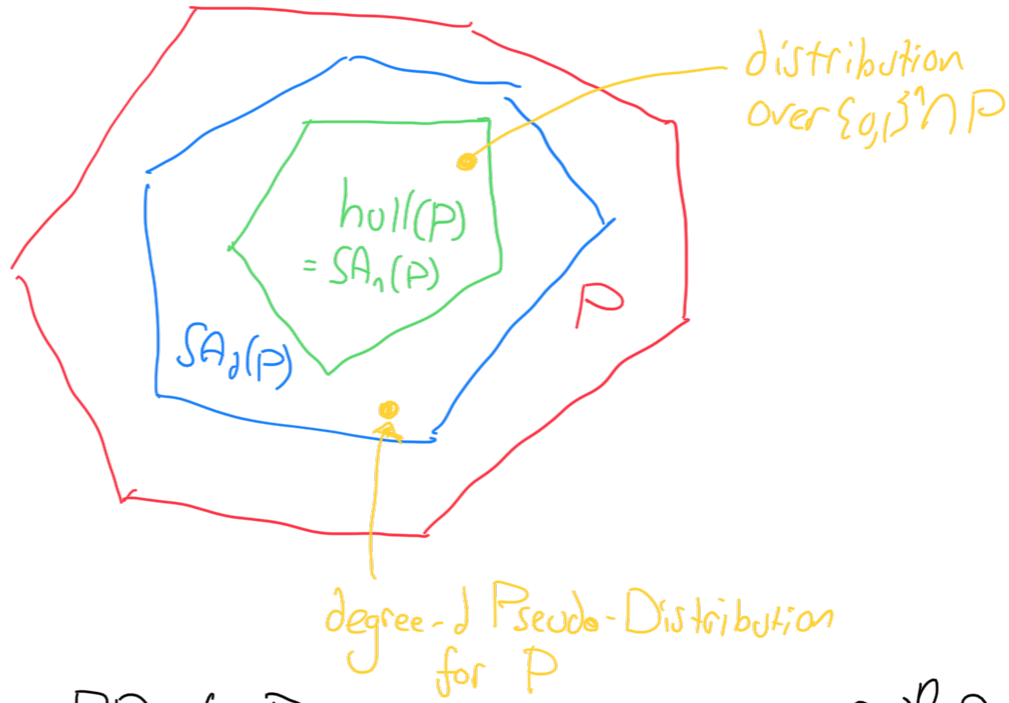
1. $\tilde{\mathbb{E}}[p_i J_{S,T}] \geq 0 \quad \forall p_i \in P$
2. $\tilde{\mathbb{E}}[J_{S,T}] \geq 0$
3. $\tilde{\mathbb{E}}[1] = 1$

Degree-d SA LP ($SA_d(P)$): Defined in stages.

- For all $S,T \subseteq [n]$, $S \cap T = \emptyset$, $|S \cup T| \leq d$, add a constraint, $p \in P \cup \{1 \geq 0\}$

$$p \prod_{i \in S} x_i \prod_{j \in T} (1 - x_j) \geq 0$$

- Multilinearize: replace x_i^2 by x_i everywhere. Replace each monomial $J_{S,\emptyset}$ by y_S.



→ Degree-n PDs for P are true distributions over $\{0,1\}^n \cap P$

⇒ $\text{proj}(SA_n(P)) = \text{hull}(P)$

projection to $x_1, \ldots, x_n$

$SA_d(P)$ is an LP with $n^{O(d)}$ constraints & variables

→ Can be solved in time $n^{O(d)}$!

Sherali-Adams Proofs

Suppose $P \cap \{0,1\}^n = \emptyset$. What degree of the Sherali-Adams hierarchy is needed to witness this?

– ie for what d is $SA_d(P) = \emptyset$?

A Sherali-Adams proof of $P = \{p_1 \geq 0, \ldots, p_m \geq 0\}$ certifies that $P \cap \{0,1\}^n = \emptyset$!

Conical Junta: $J = \sum_i \lambda_i J_i$ where J_i is a conjunct, $\lambda_i \geq 0$

* Through out we work modulo $\langle x_i^2 - x_i = 0 \rangle$ ie $x_i^2 = x_i$

Sherali-Adams Proof: A SA proof of a polynomial q from P is an expression of the form

$$\sum p_i J_i + J = q$$

where J_i, J are conical juntas

SA Refutation: $\sum p_i J_i + J = -1$

non-negative negative

only possible if no $x \in \{0,1\}^n$ simultaneously satisfies $p_i(x) \geq 0$ $\forall i \in [m]$

Degree: Maximum degree of J_i, J

Thm: There is a degree-d SA refutation of P iff $SA_d(P) = \emptyset$

Proof: Suppose $SA_d(P) \neq \emptyset$ then $\exists \alpha \in SA_d(P)$.
α defines a degree-d PE $\tilde{\mathbb{E}}$

Suppose $\sum p_i J_i + J = -1$ is a degree-d SA refutation of P. Then

$$-1 = \tilde{\mathbb{E}}[-1] = \tilde{\mathbb{E}}[\sum p_i J_i + J] = \sum \underbrace{\tilde{\mathbb{E}}[p_i J_i]}_{\geq 0} + \underbrace{\tilde{\mathbb{E}}[J]}_{\geq 0} \geq 0$$

Contradiction! $\square$