Interpolation and Total Search Problems

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of size poly(s)

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$$F(x, y, z) = A(x, z) \wedge B(y, z)$$

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The Function Computed

Let $\alpha \in \{0,1\}^y$ be any assignment to $y \Longrightarrow A(x, \alpha)$ or $B(y, \alpha)$ is unsatisfiable

Define monotone "interpolant" function $I_F(z) = \begin{cases} 0 & \text{if } A(x, \alpha) \text{ is satisfiable} \\ 1 & \text{if } B(y, \alpha) \text{ is satisfiable} \end{cases}$

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Interpolation for CP [P97]: Let F be a split formula

CP proof of F
of size s

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Monotone Real Circuit computing I_F
of size poly(s)

Today: Interpolation for any Formula

Goal for today: remove the “split” assumption!

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→ Proofs and circuits as computations of search problems

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Given $x \in I$, output $y \in O$ such that $(x, y) \in S$

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Search Problem: A relation $S \subseteq I \times O$

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e.g.

Falsified Clause search problem: for an unsatisfiable CNF

$F = C_1 \wedge \ldots \wedge C_m$, given $x \in \{0,1\}^n$, output $i \in [m]$ such that $C_i(x) = 0$

Today: Interpolation for any Formula

First, we will characterize the complexity of circuits and proofs in terms of the complexity of search problems

Karchmer-Wigderson

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Intuition: any monotone circuit computing f must must differentiate between 0-inputs (y) and 1-inputs (x)

Karchmer-Wigderson and Circuits

The communication complexity of mKW will characterize monotone formulas

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Communication Complexity: Two players

Alice	Bob
Knows $x \in f^{-1}(1)$	Knows $y \in f^{-1}(0)$

Communicate bits of information to find a solution $i \in [m]$ such that $(x, y, i) \in \mathrm{mKW}_f$

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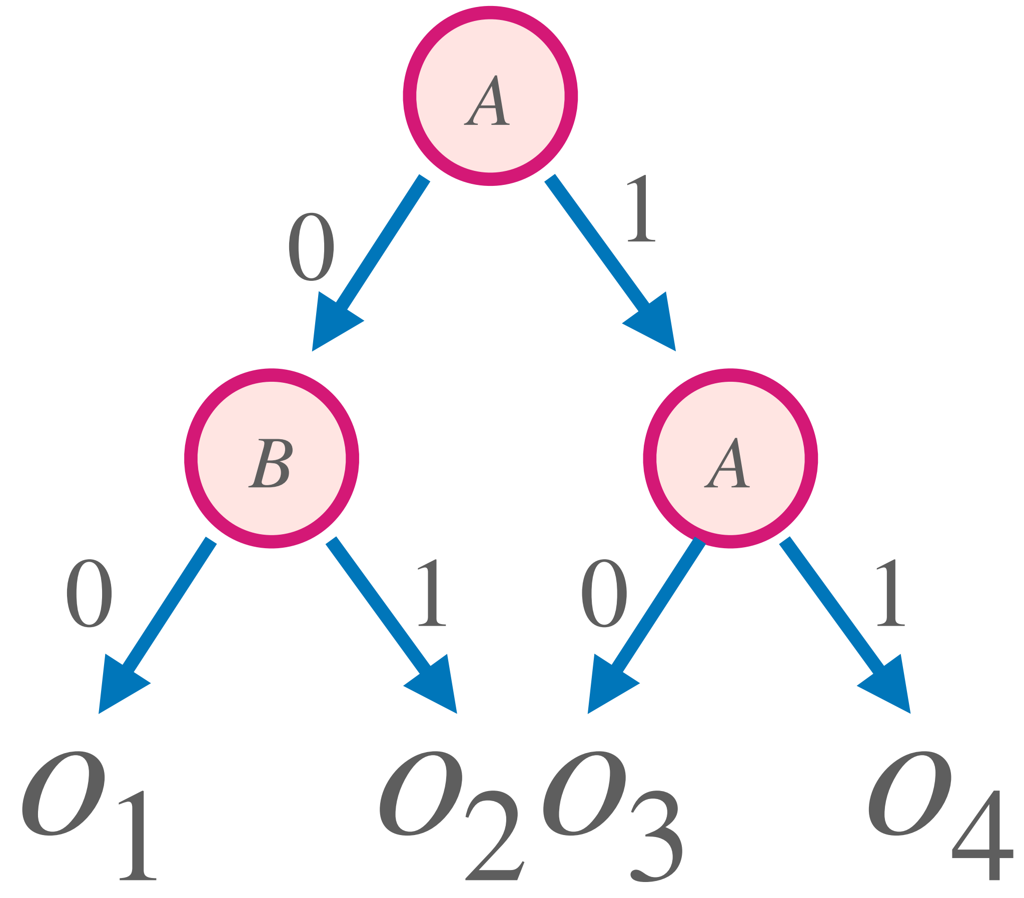

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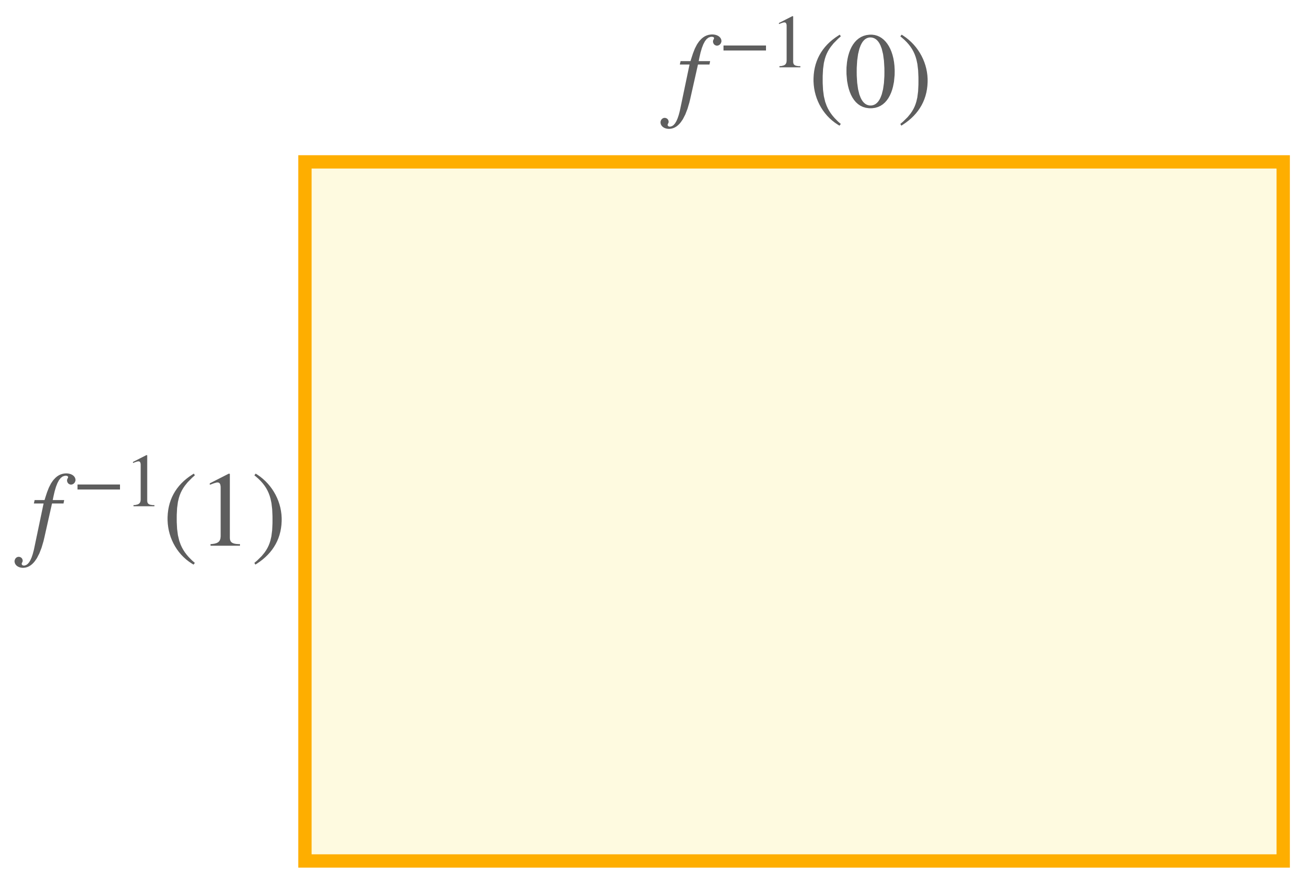

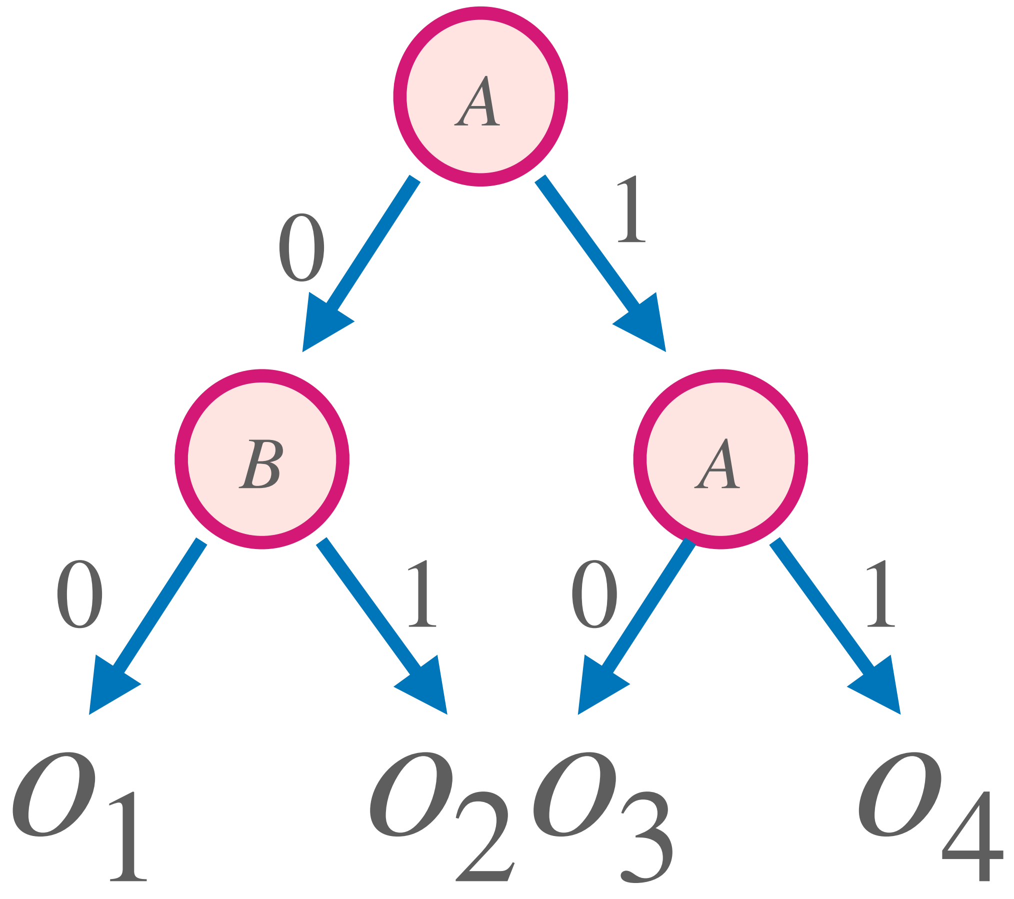

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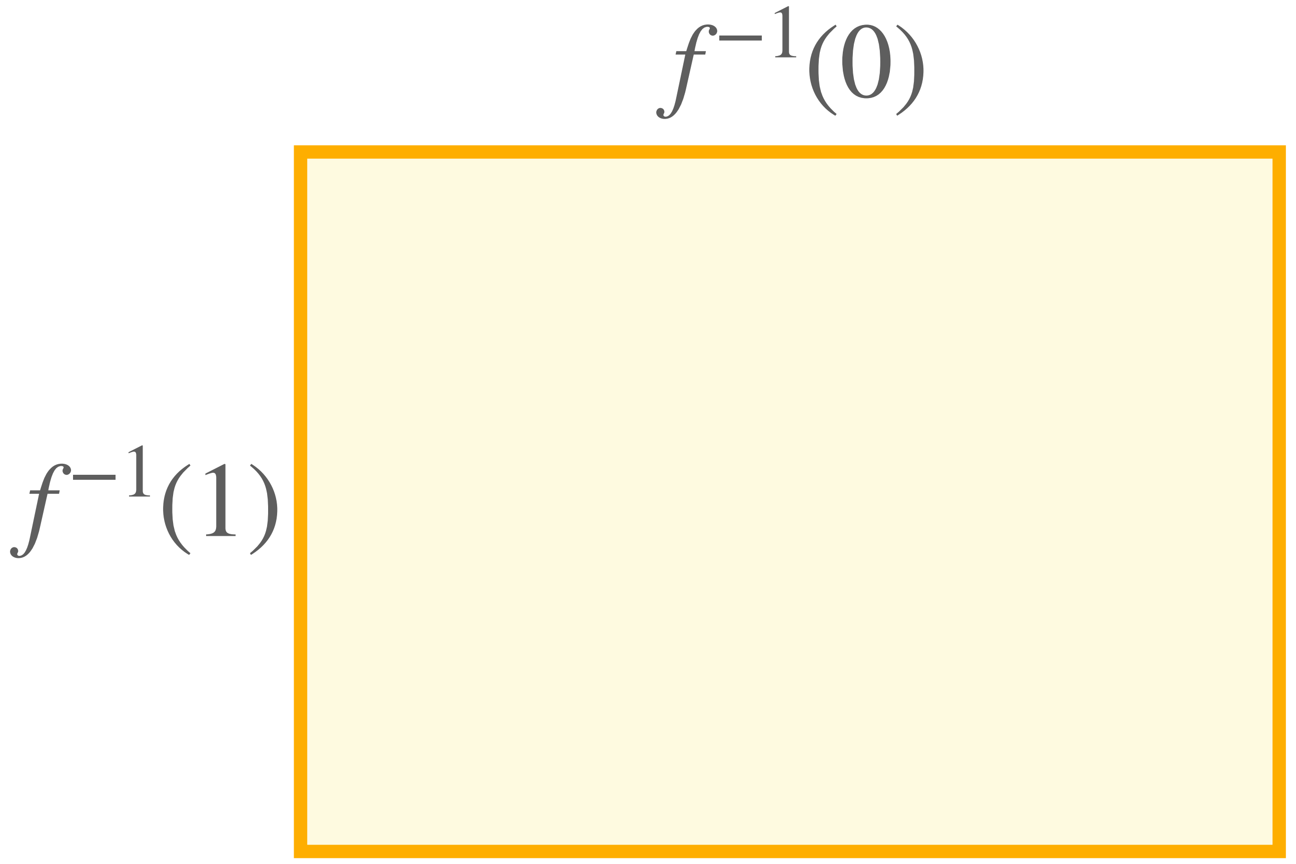

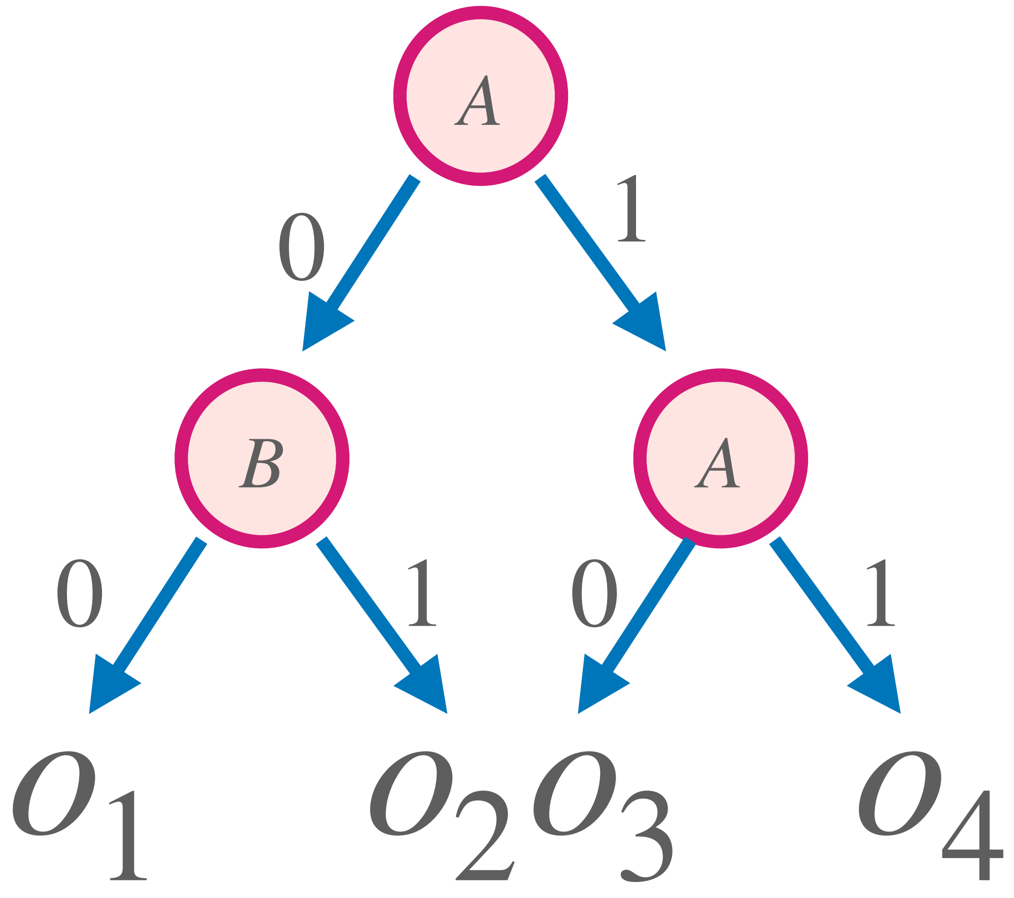

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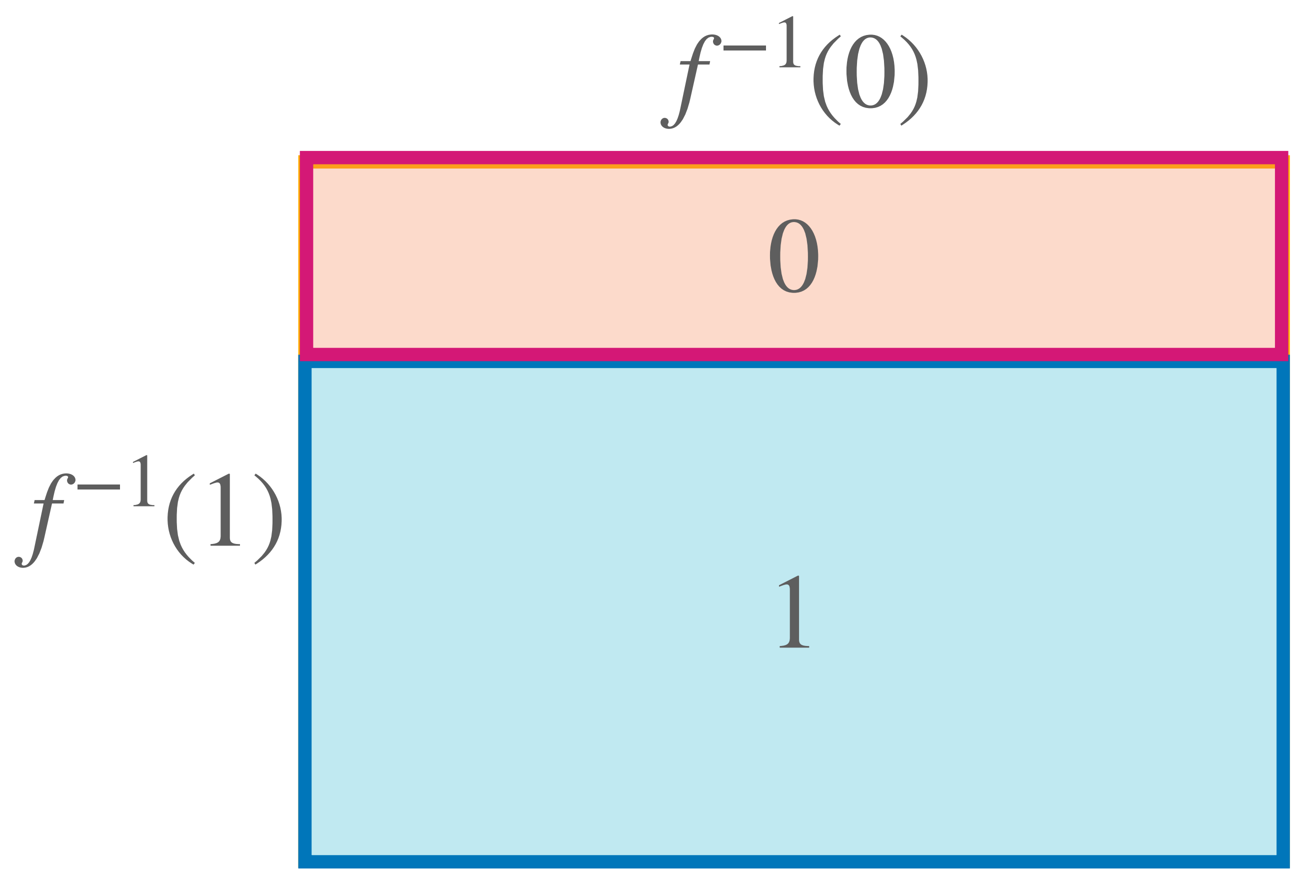

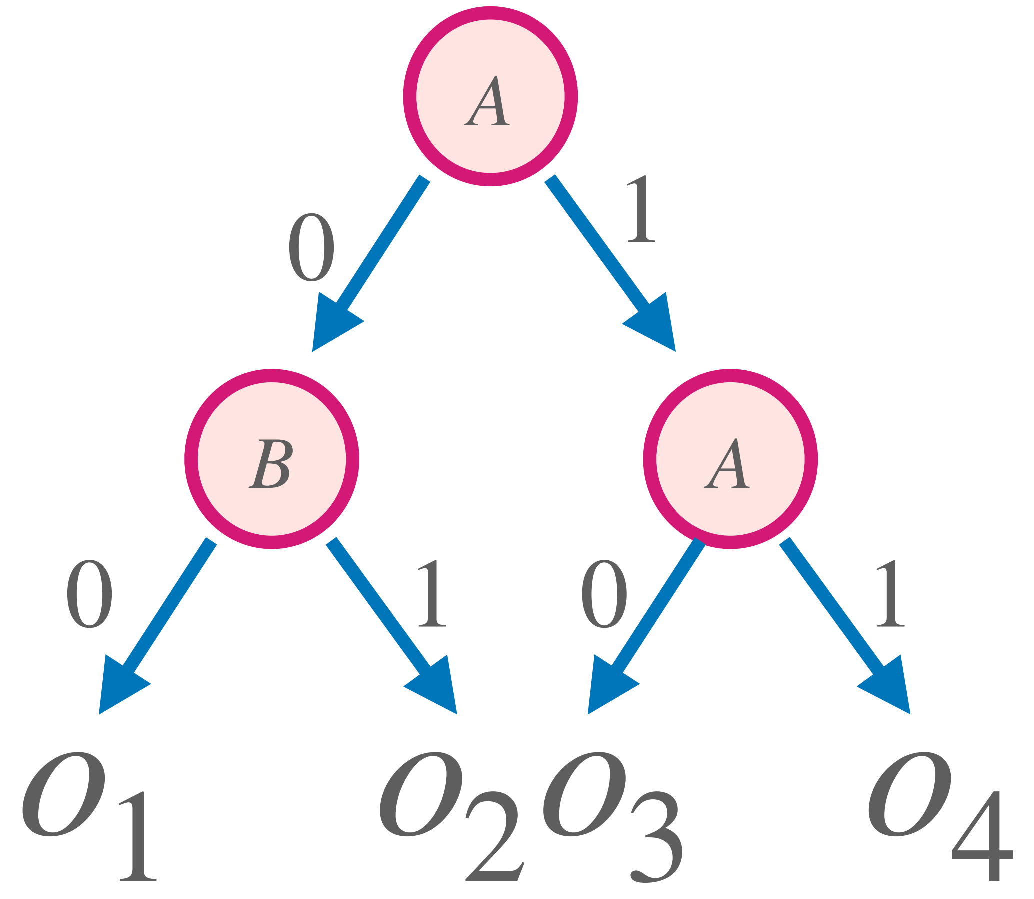

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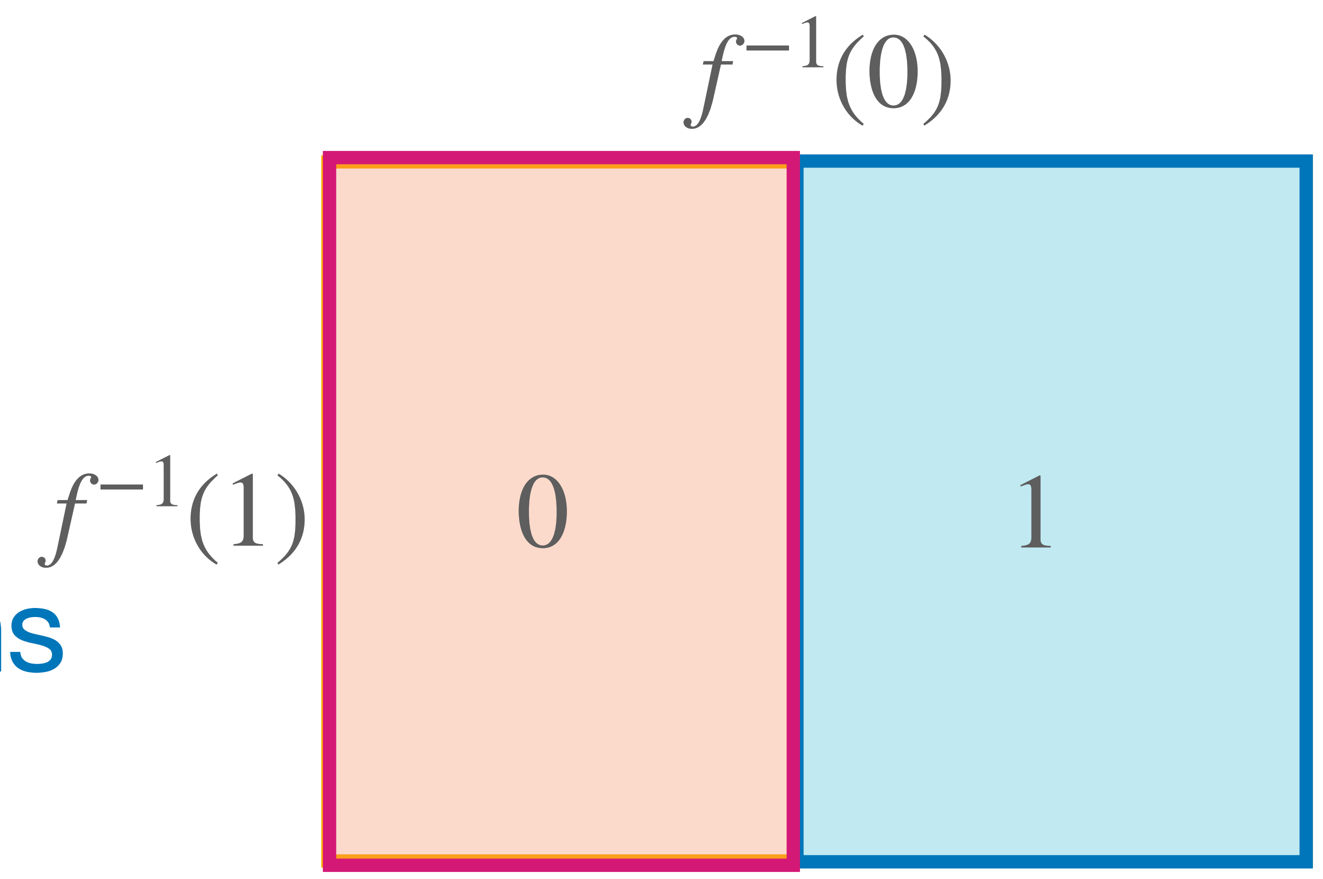

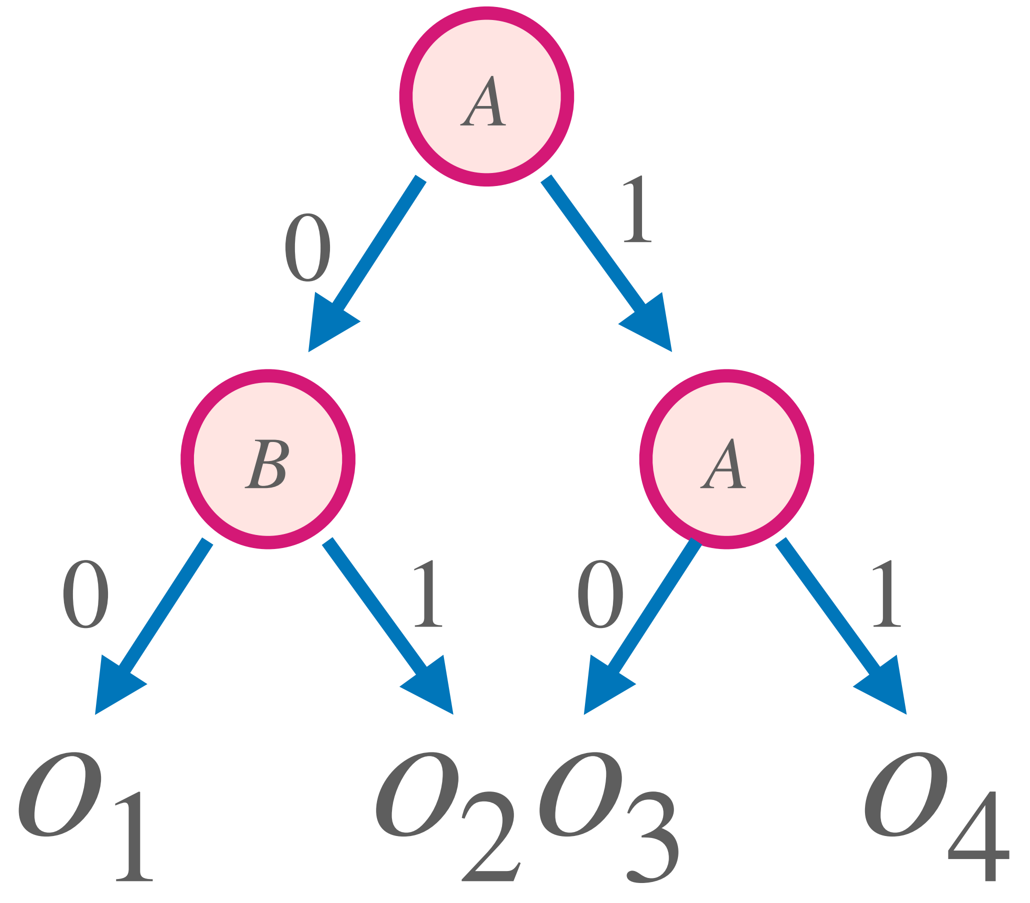

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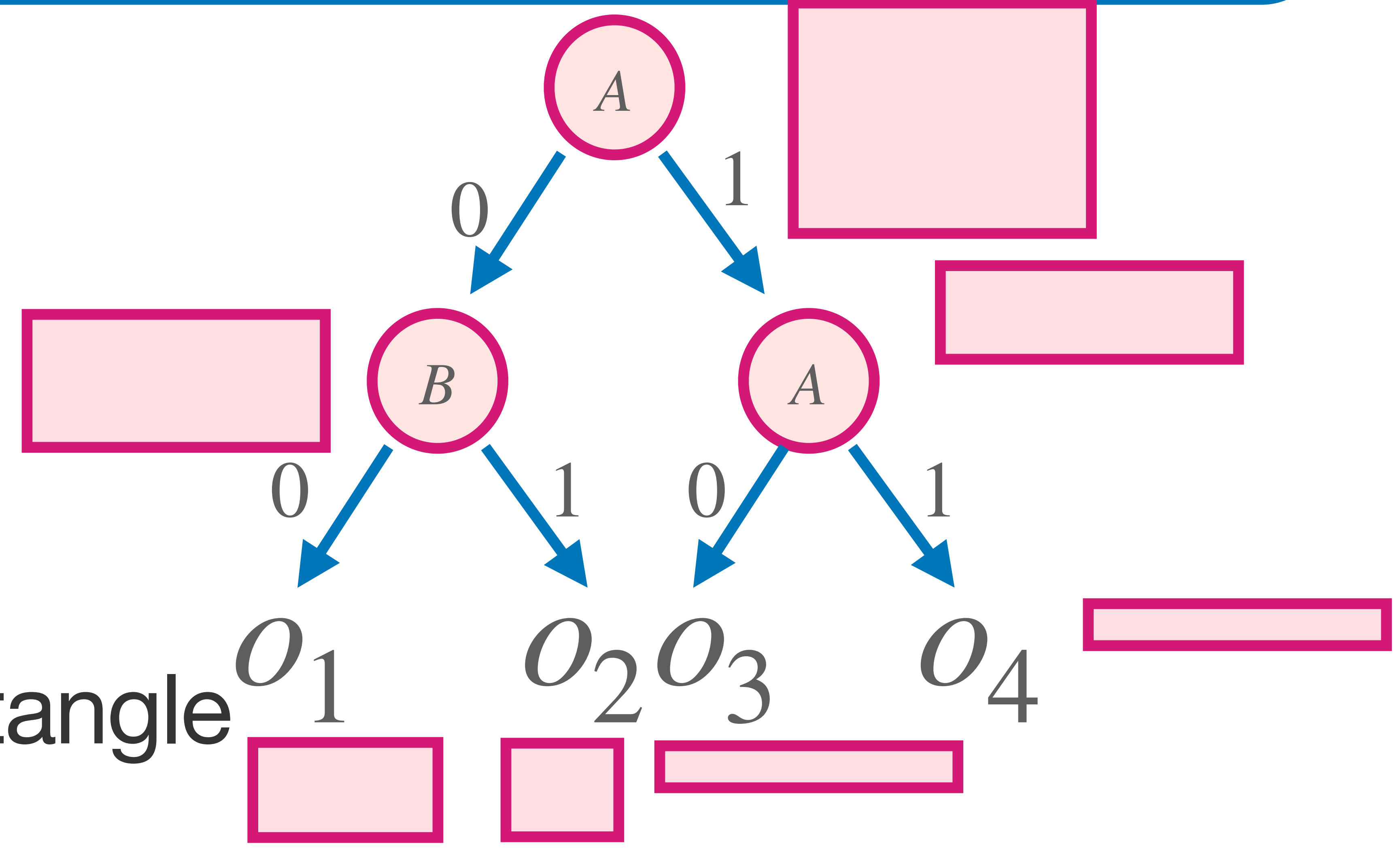

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$\implies$ Every node in the protocol tree is associated with a rectangle

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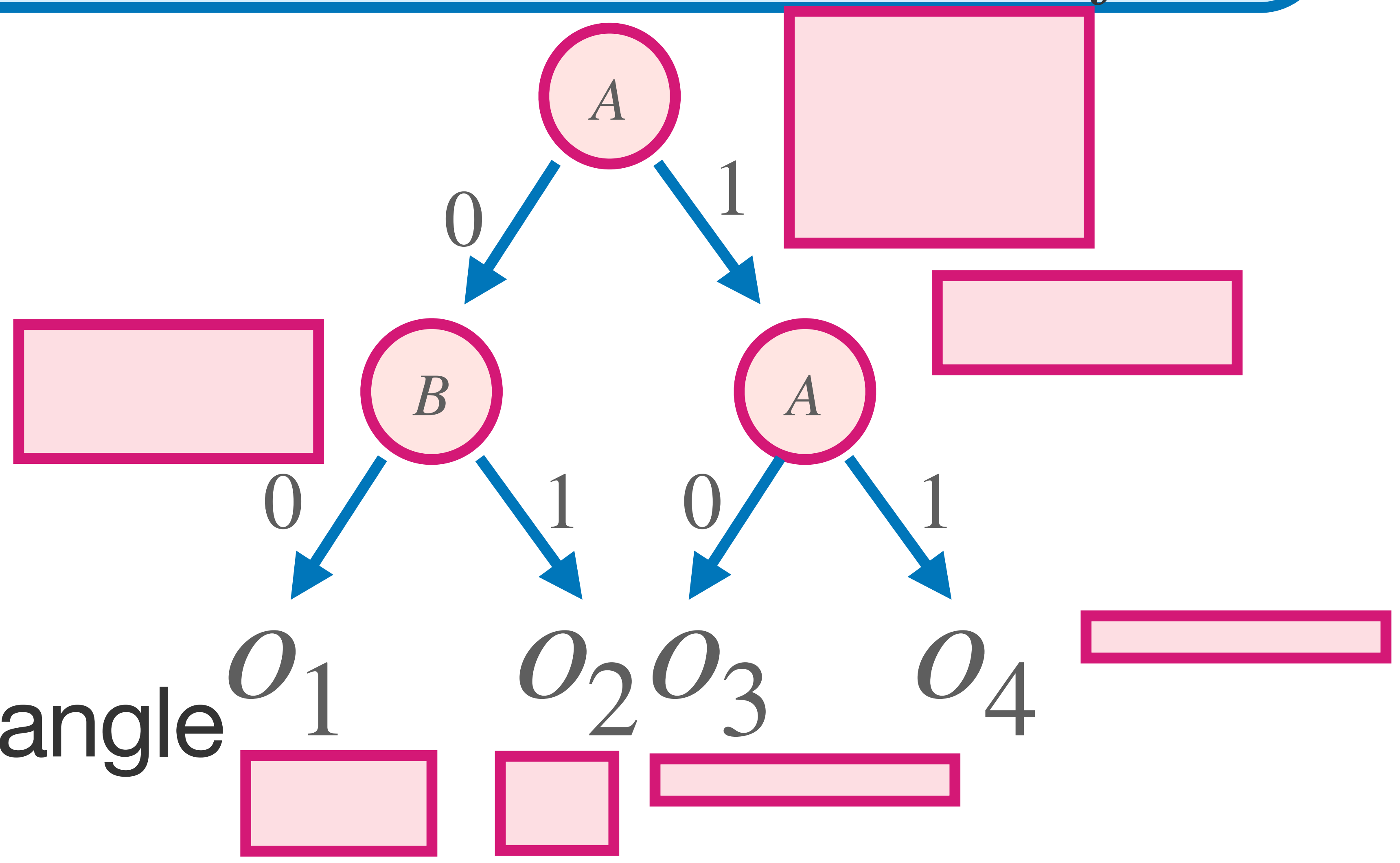
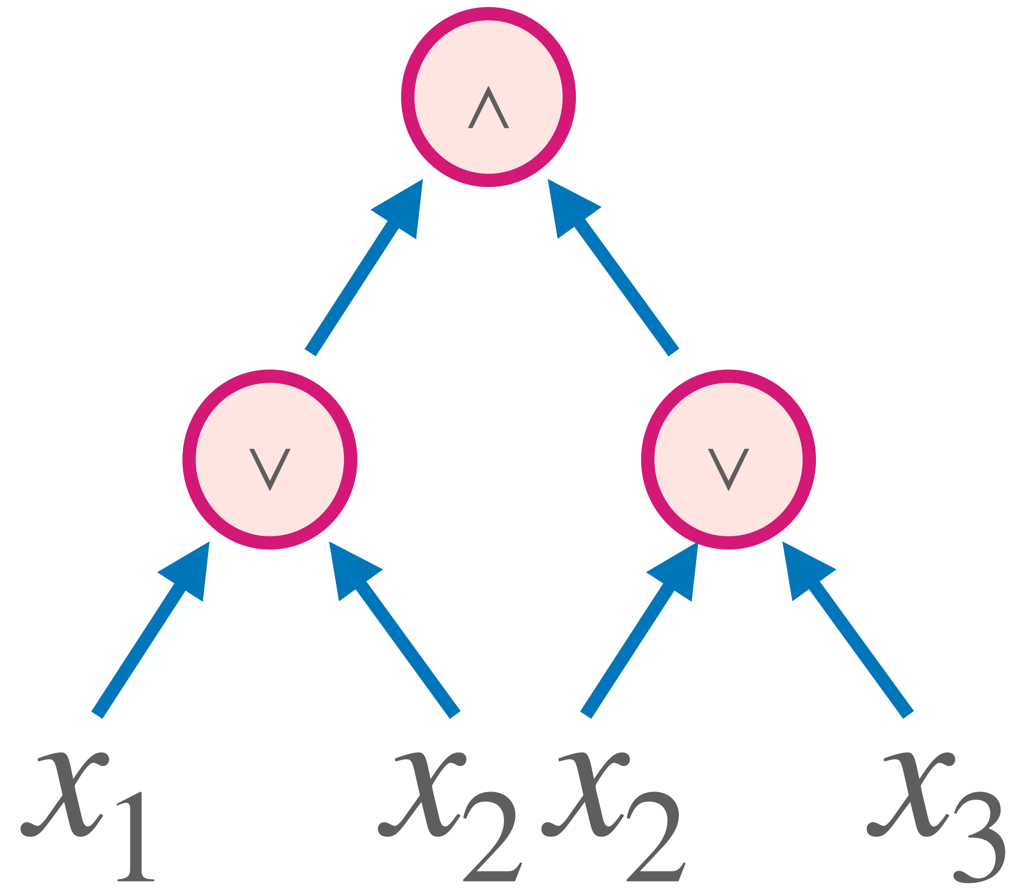
$\rightarrow$ Leaves are monochromatic rectangles



Karchmer-Wigderson and Circuits

Monotone Formula: A tree-like circuit using only $\wedge$ and $\vee$ gates

$\rightarrow$ Can only compute monotone functions



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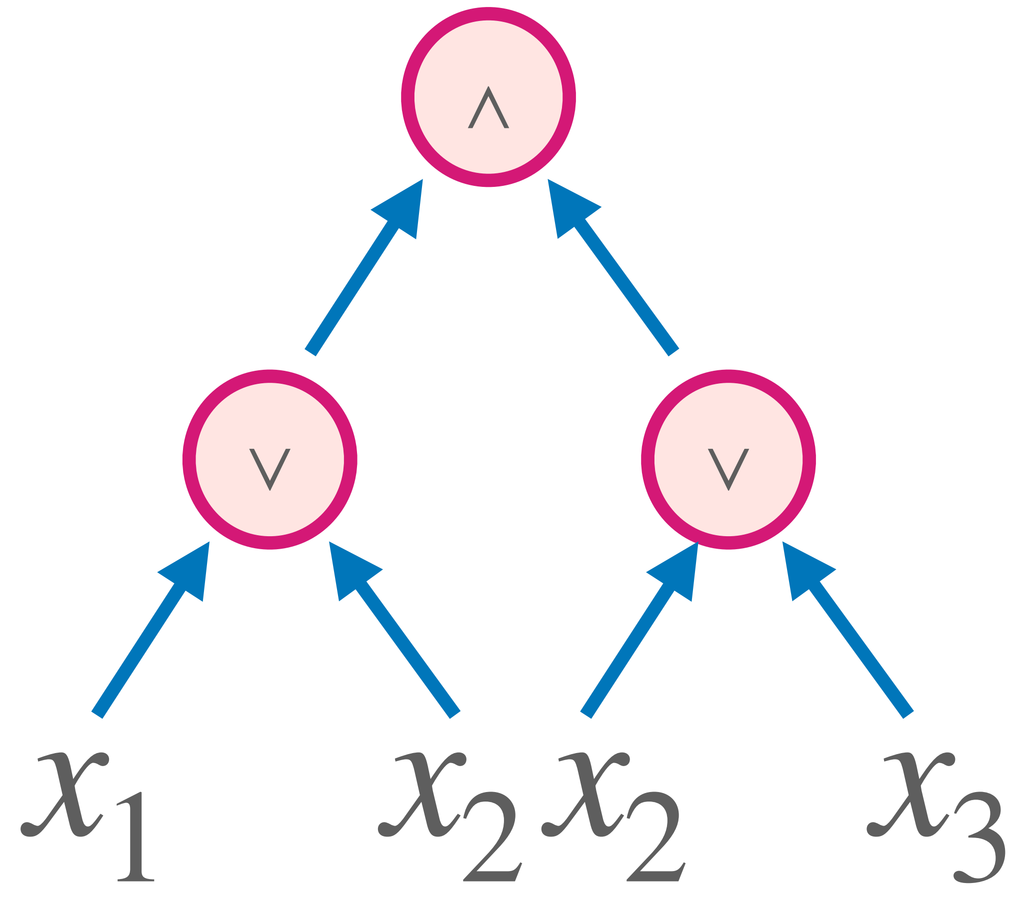

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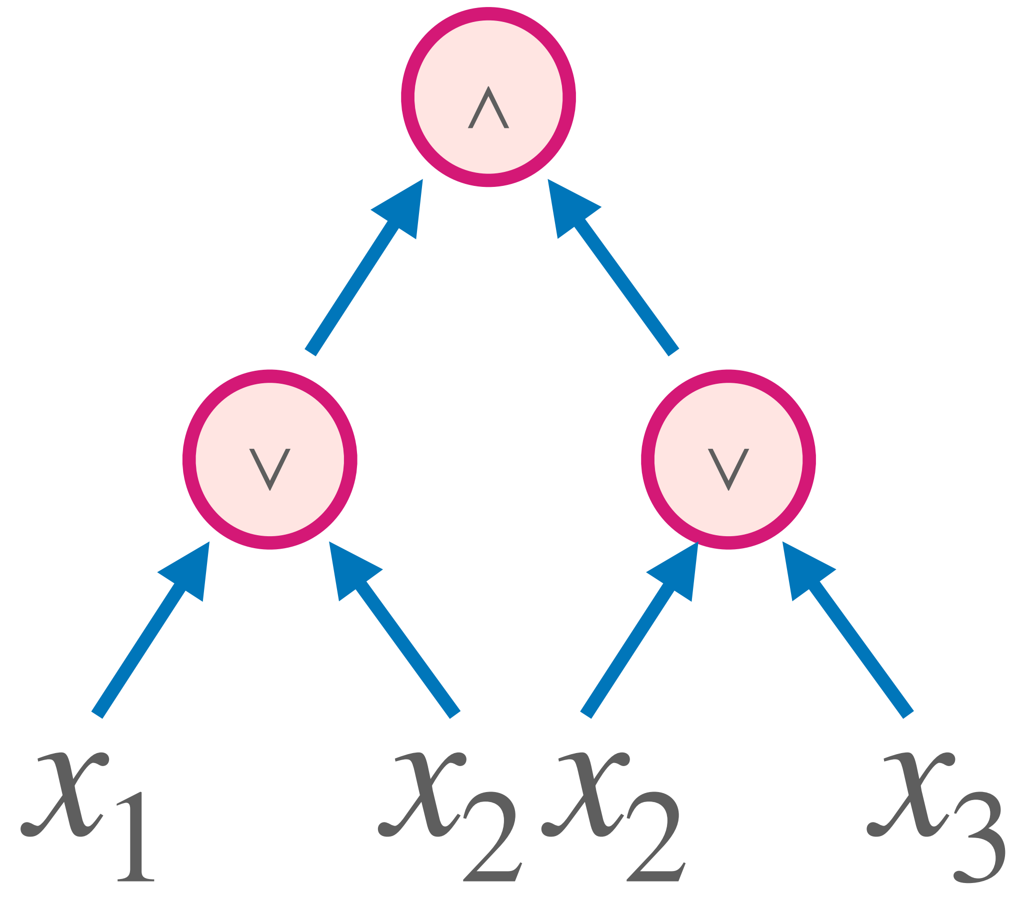

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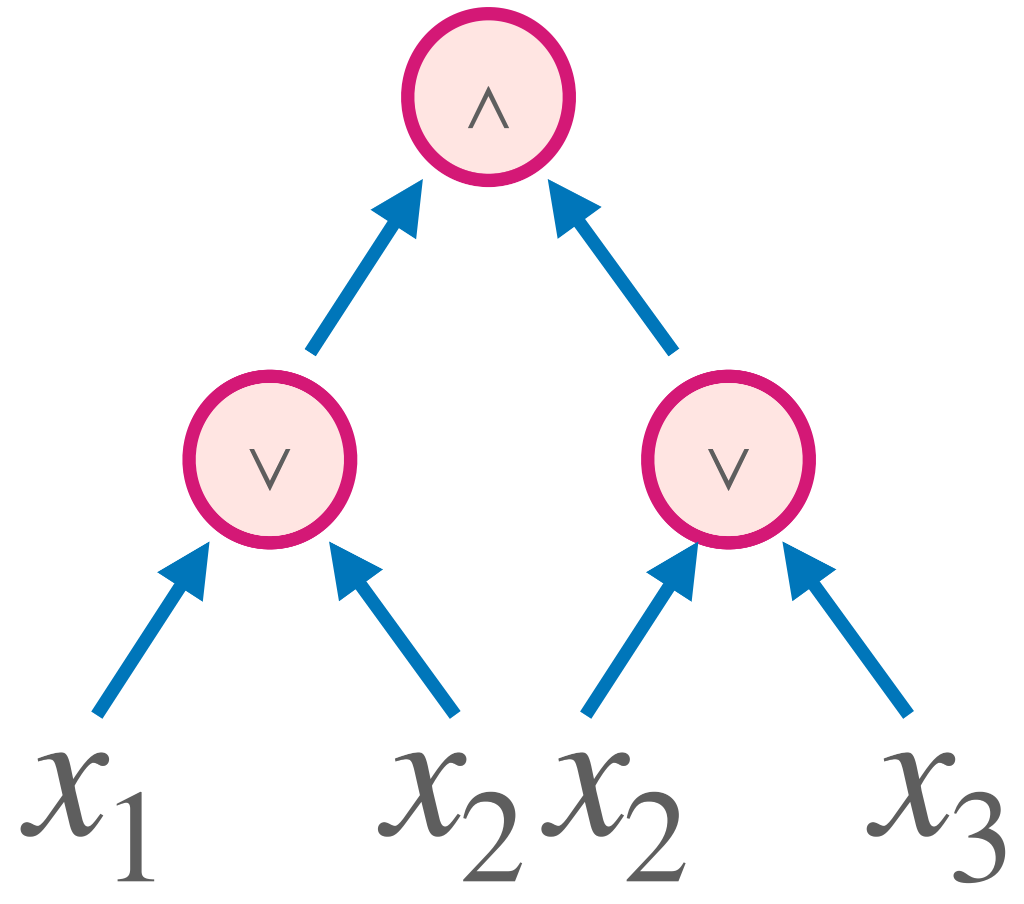

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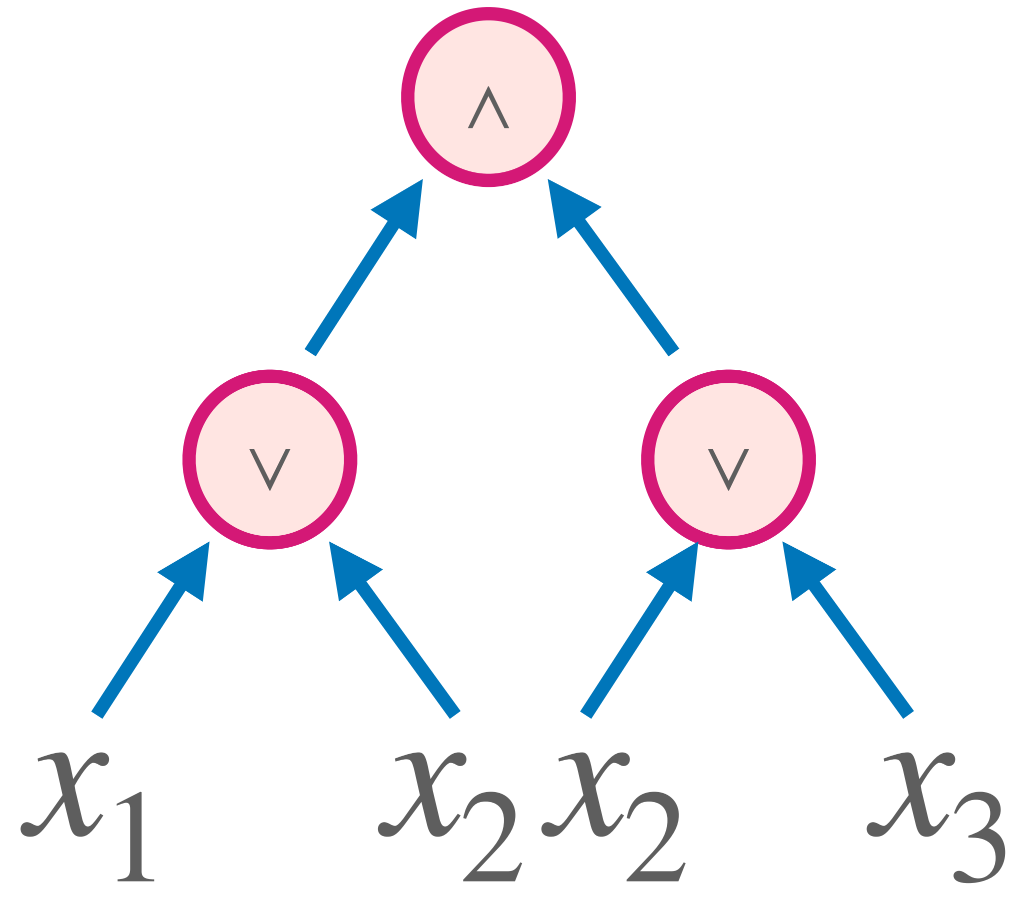

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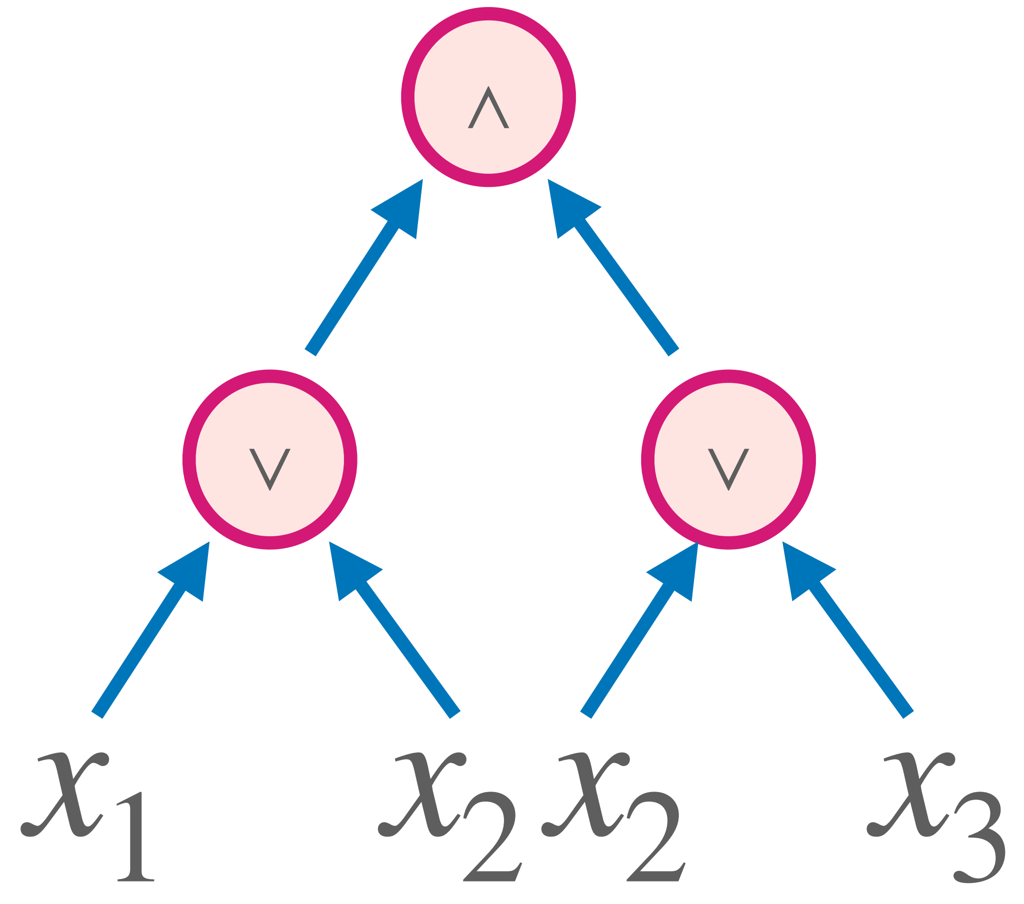

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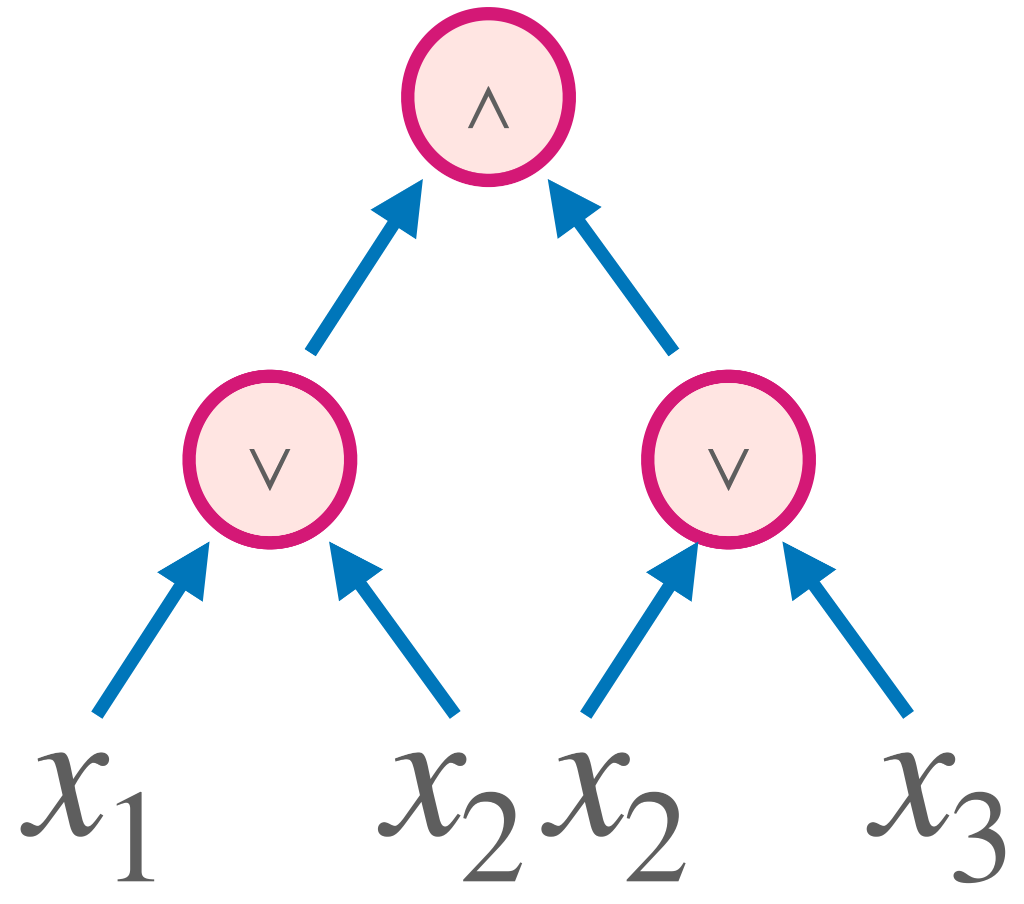

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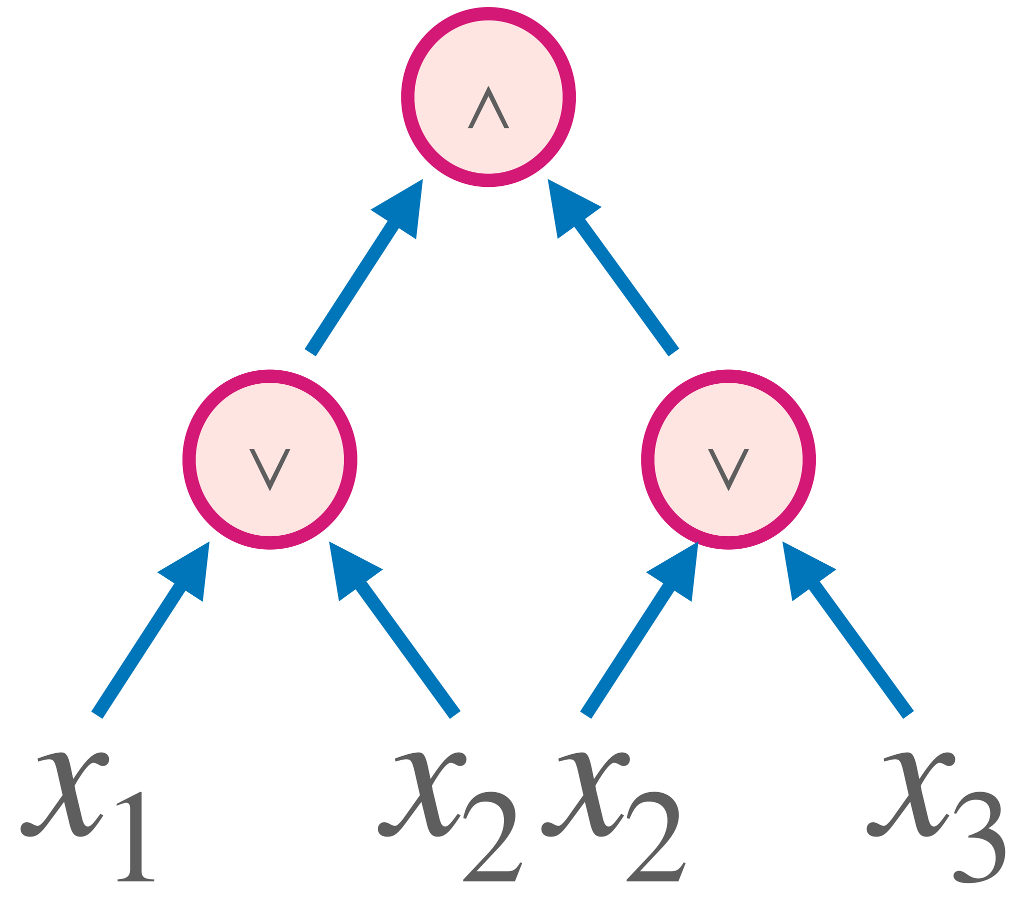

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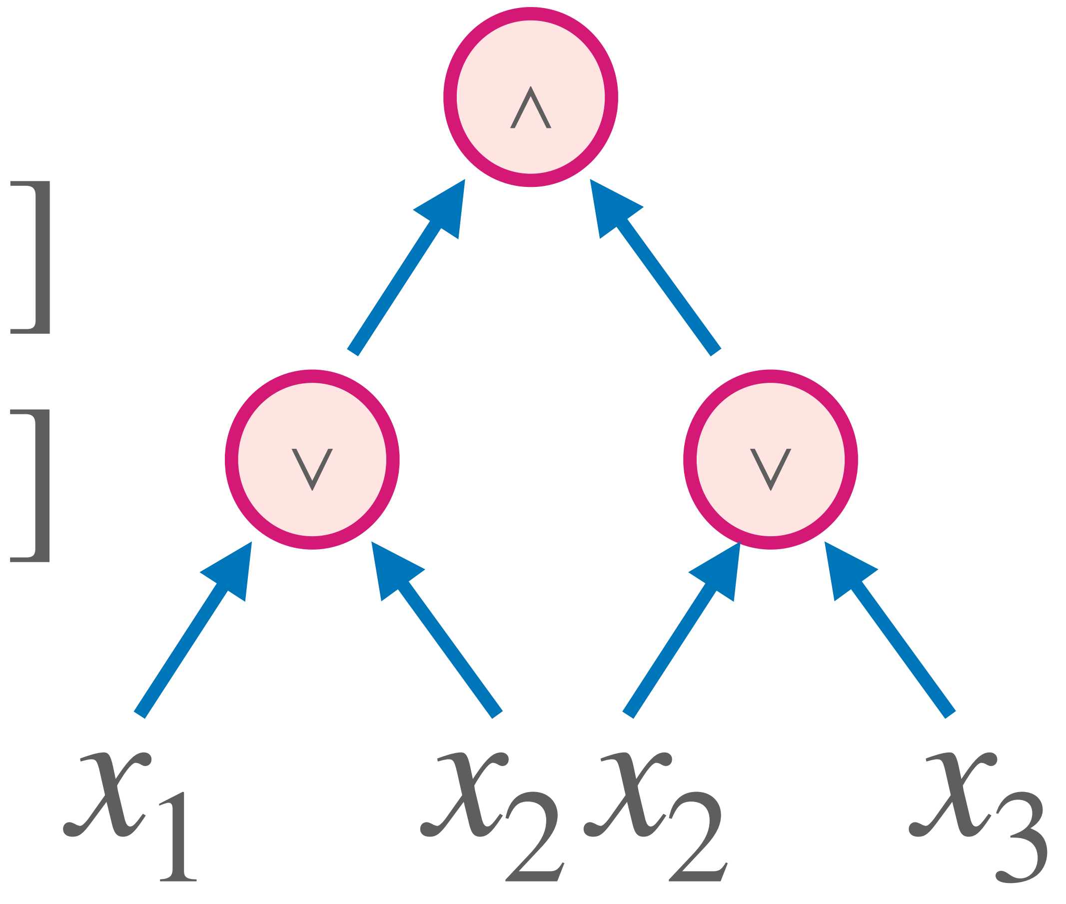

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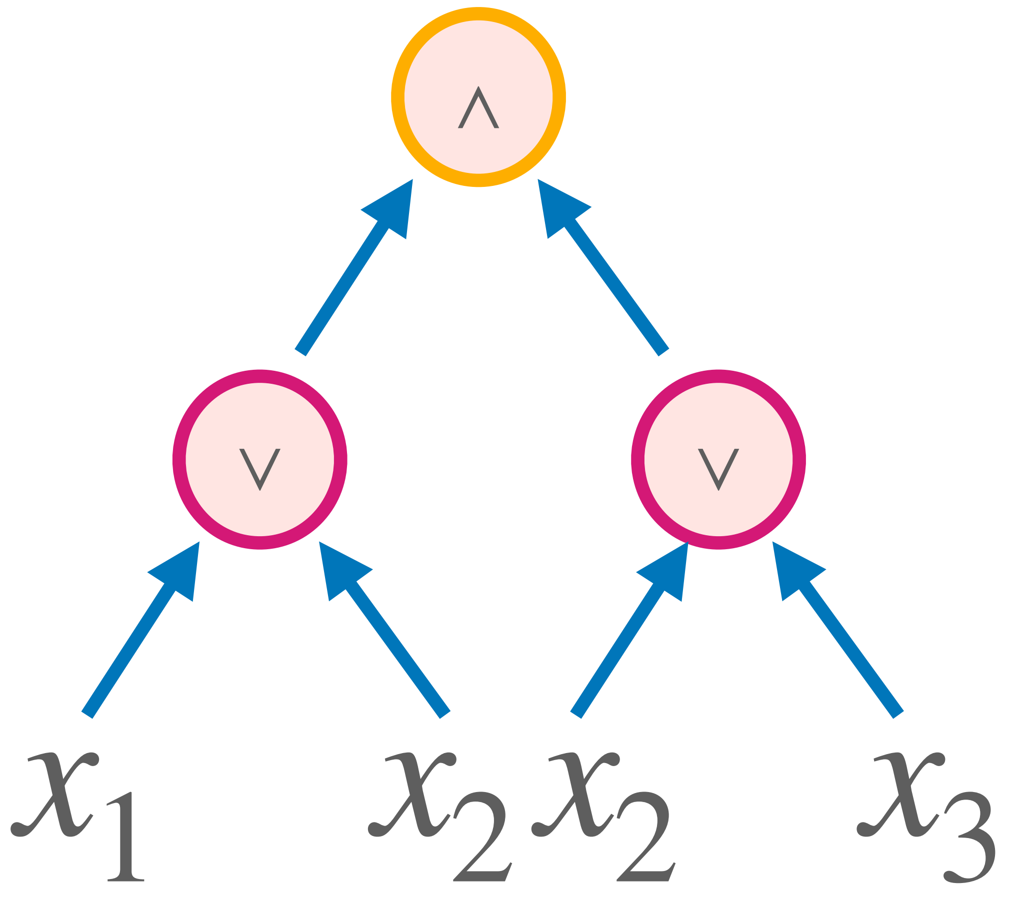

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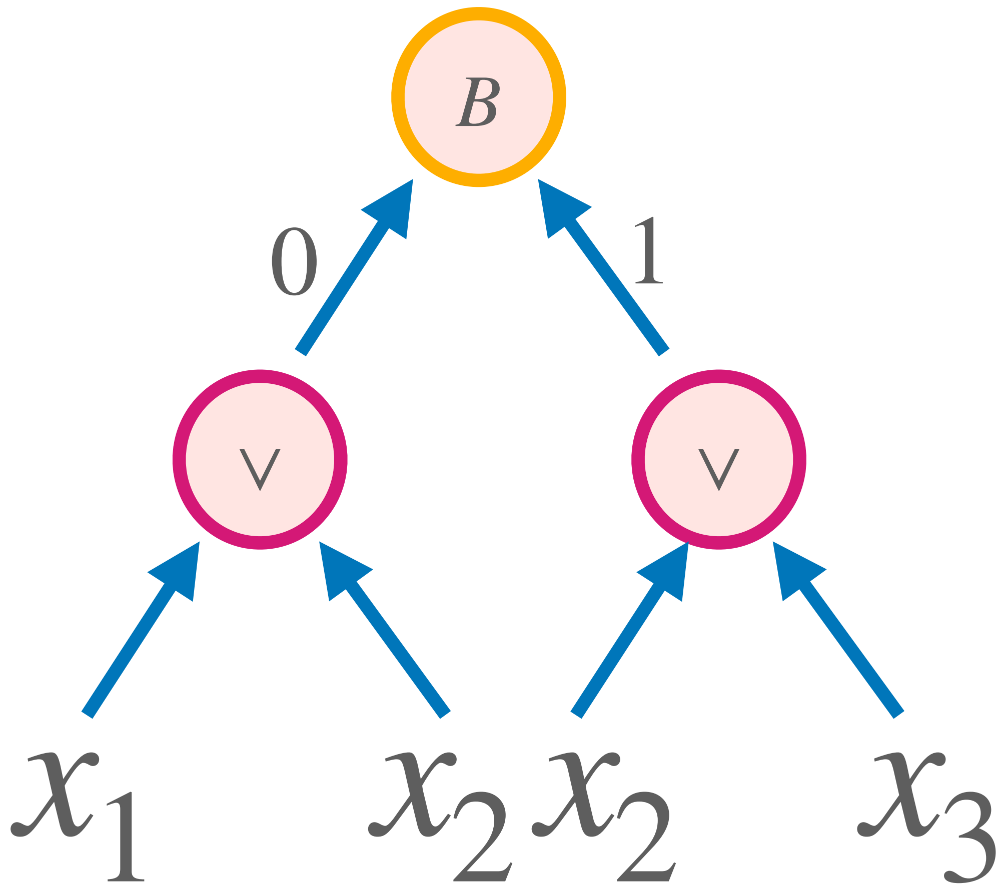

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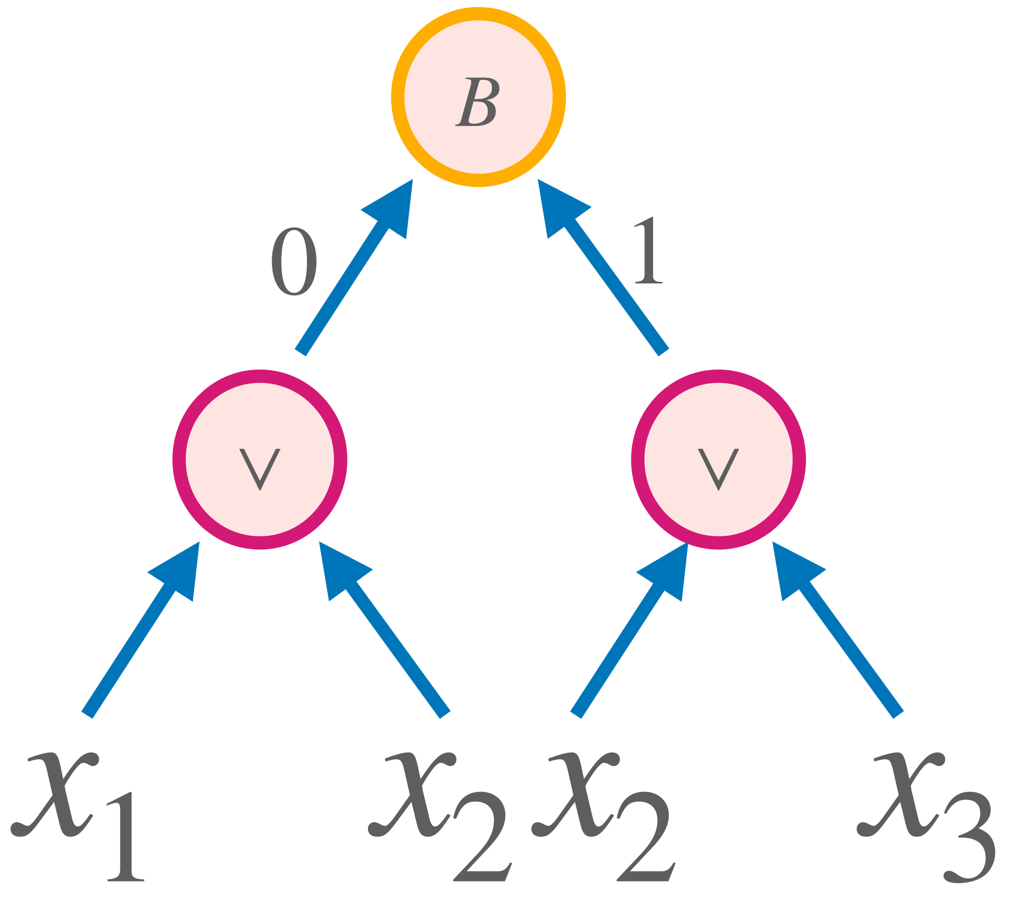

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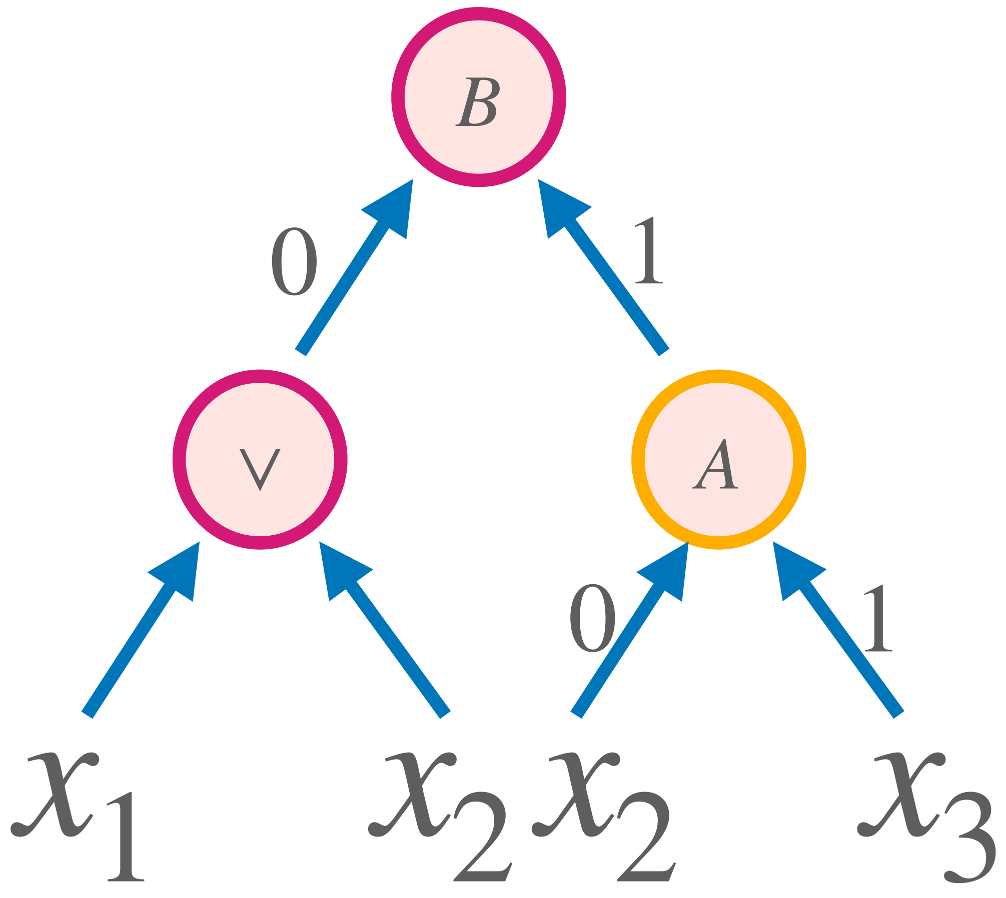

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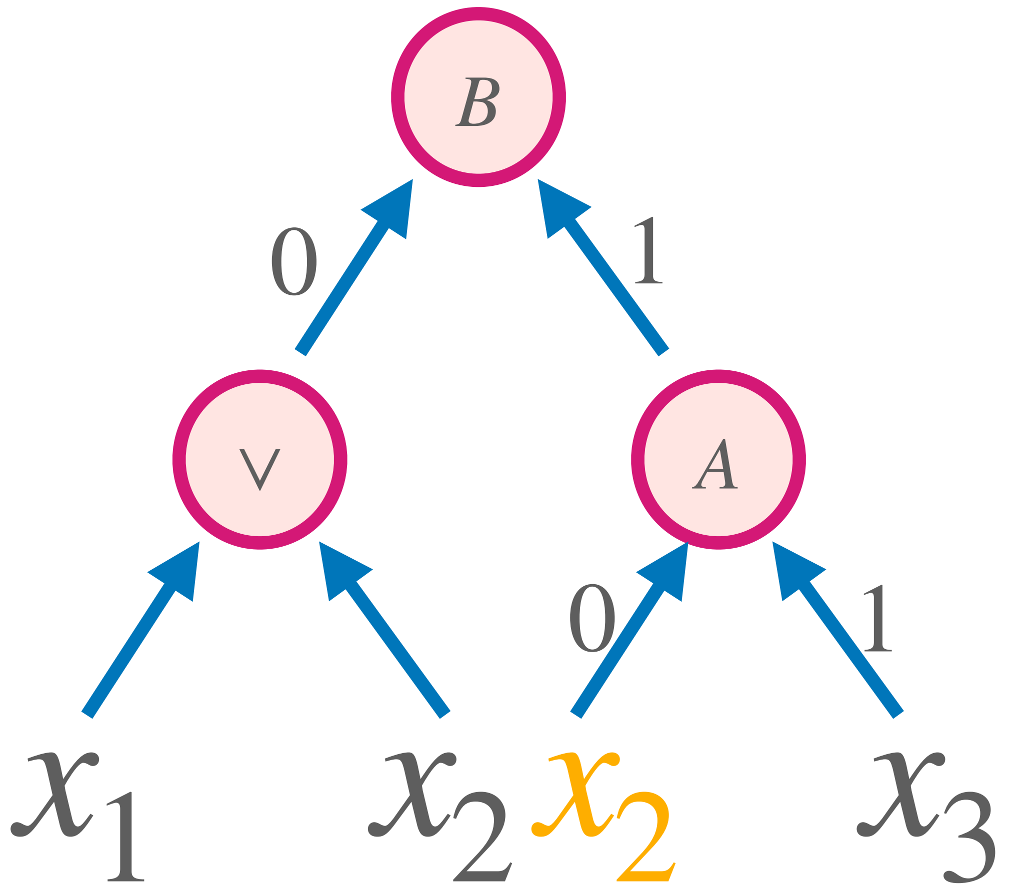

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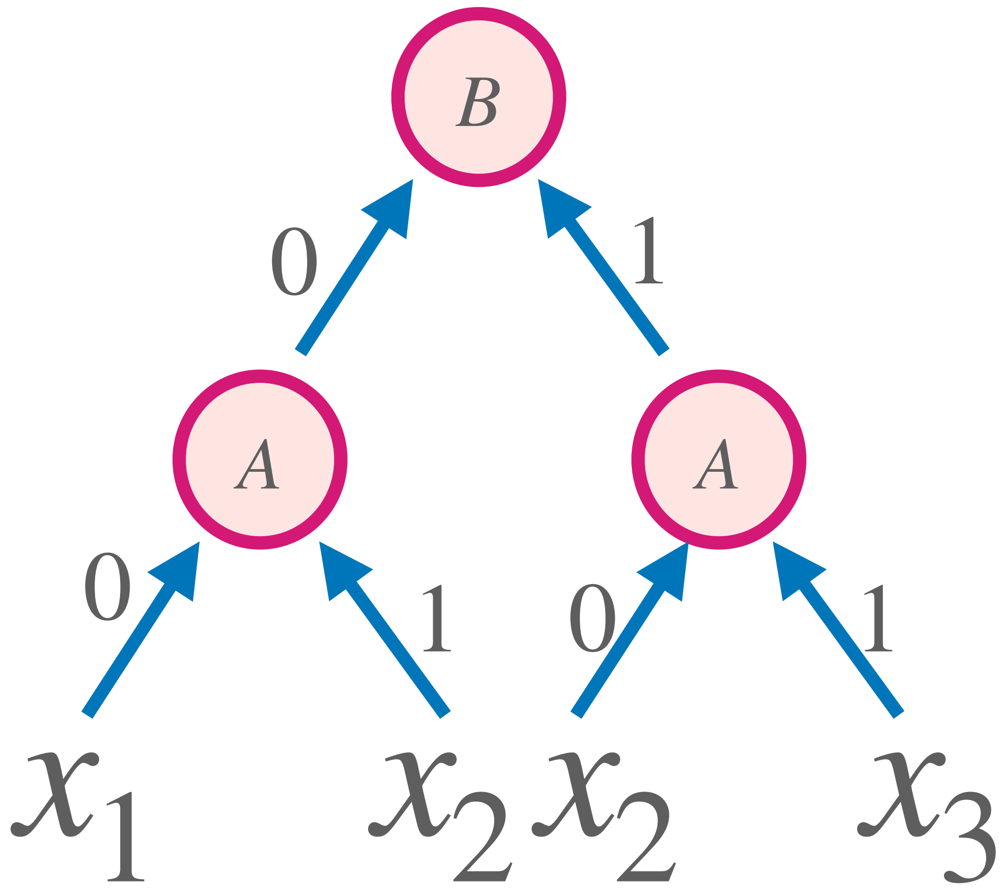

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- $\vee$: Alice sends 0 if $C_{u_1}(x) = 1$, 1 o.w. ($C_{u_2}(x) = 1$)

Karchmer-Wigderson and Circuits

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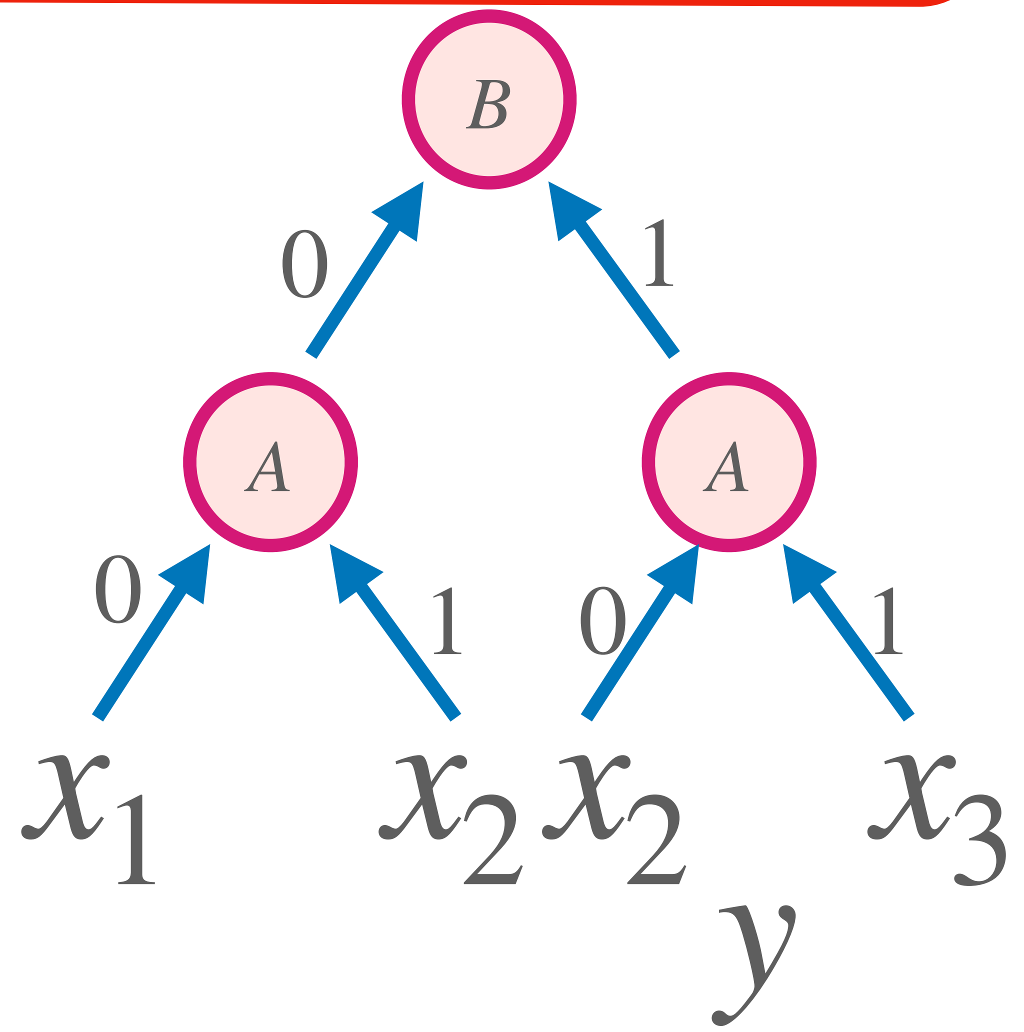

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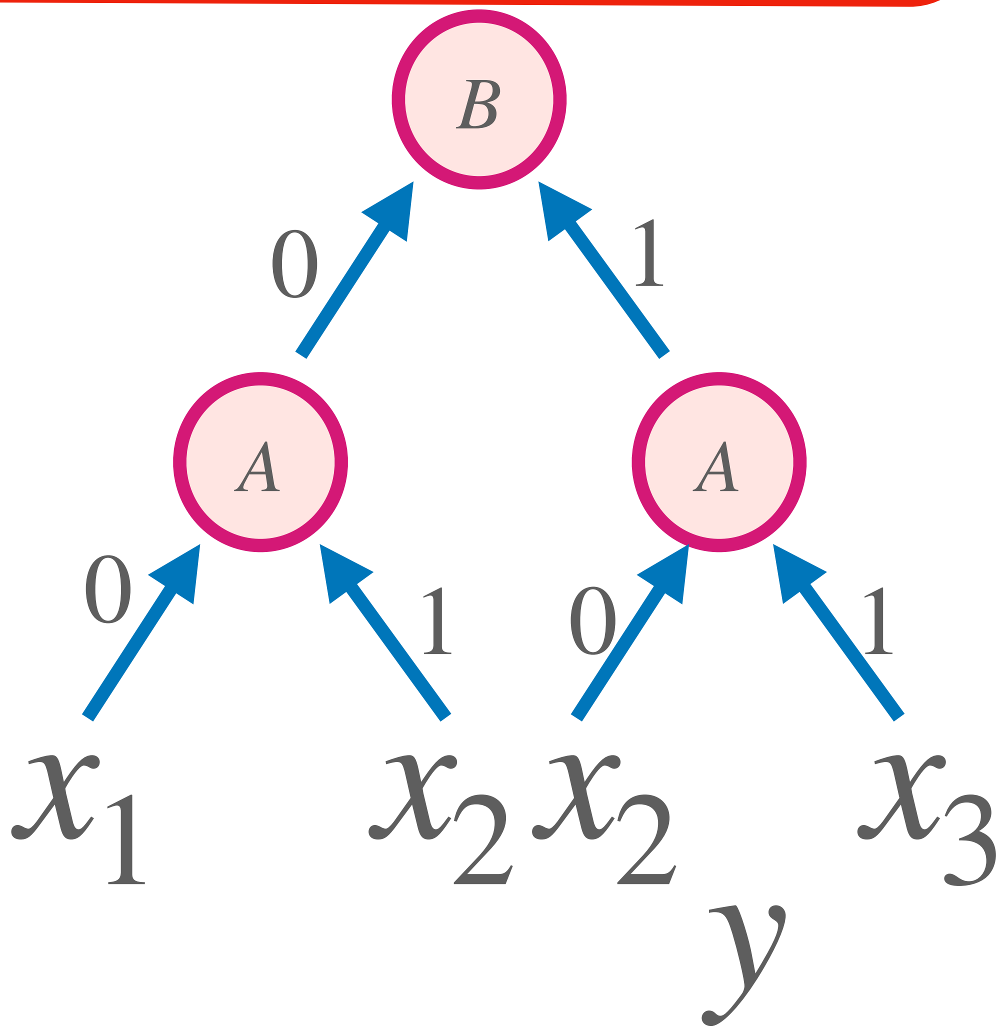

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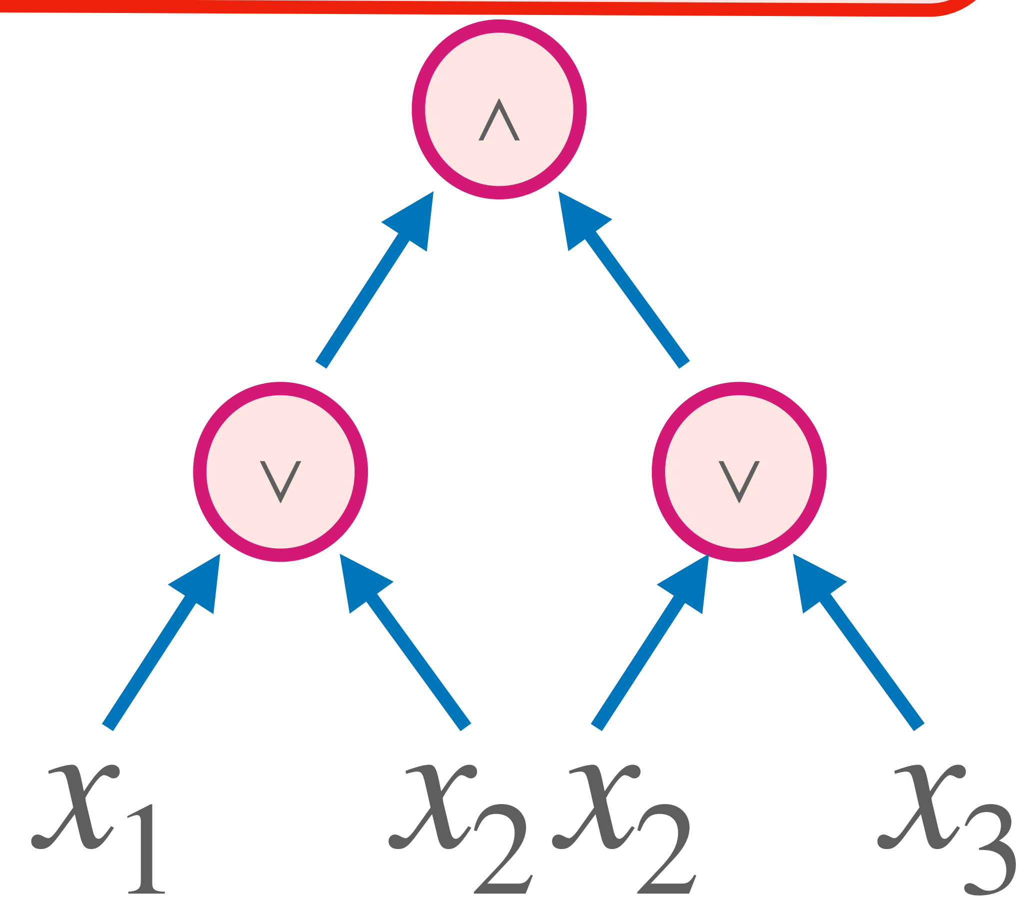

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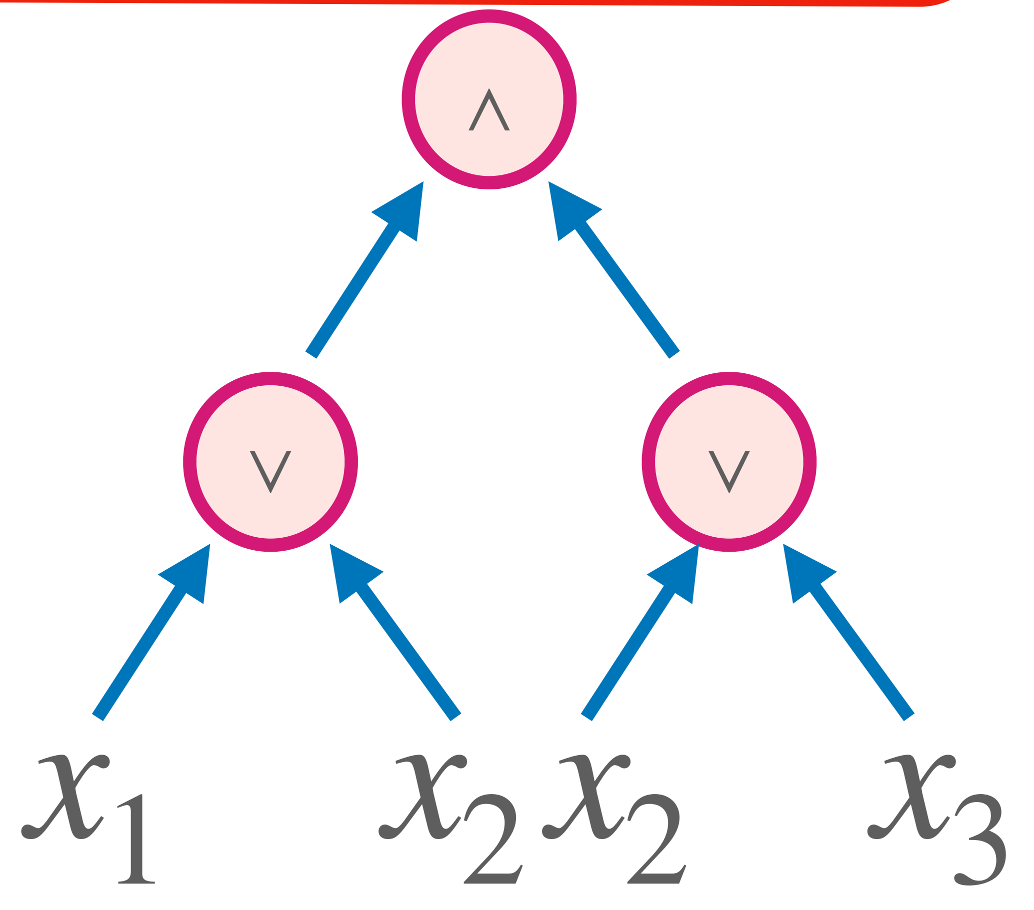

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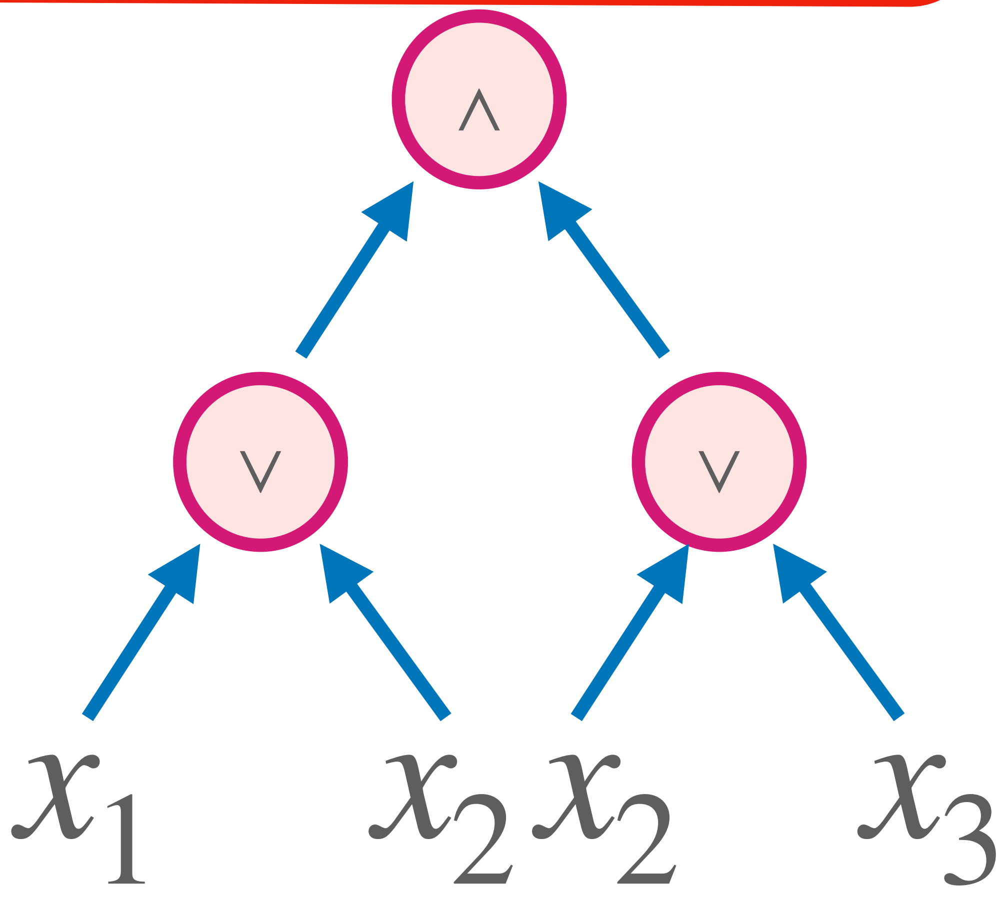

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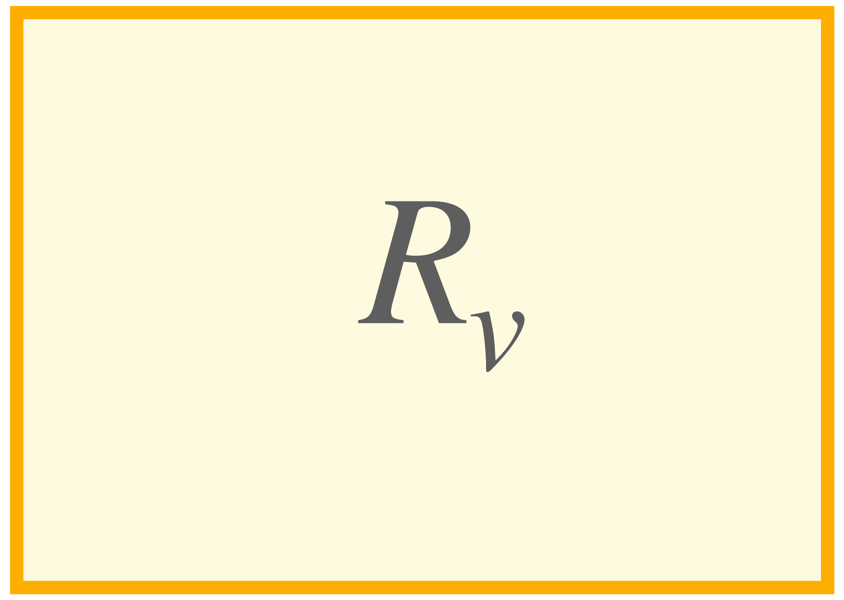

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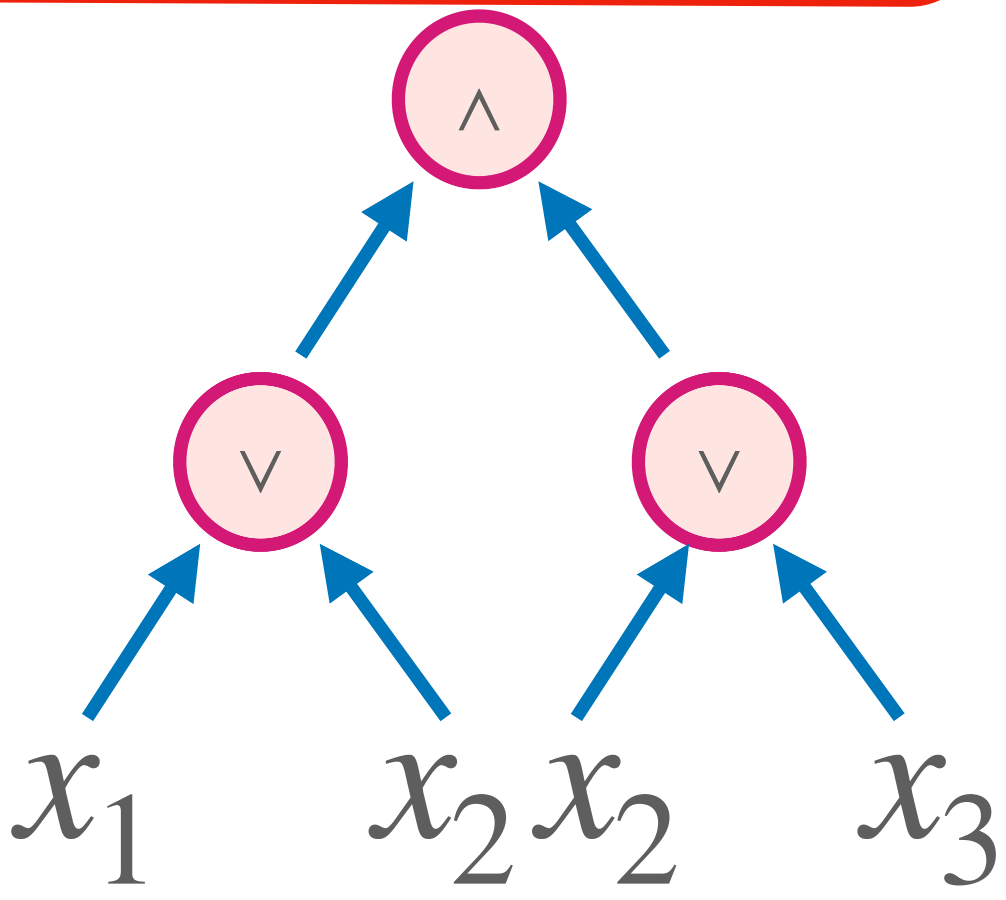

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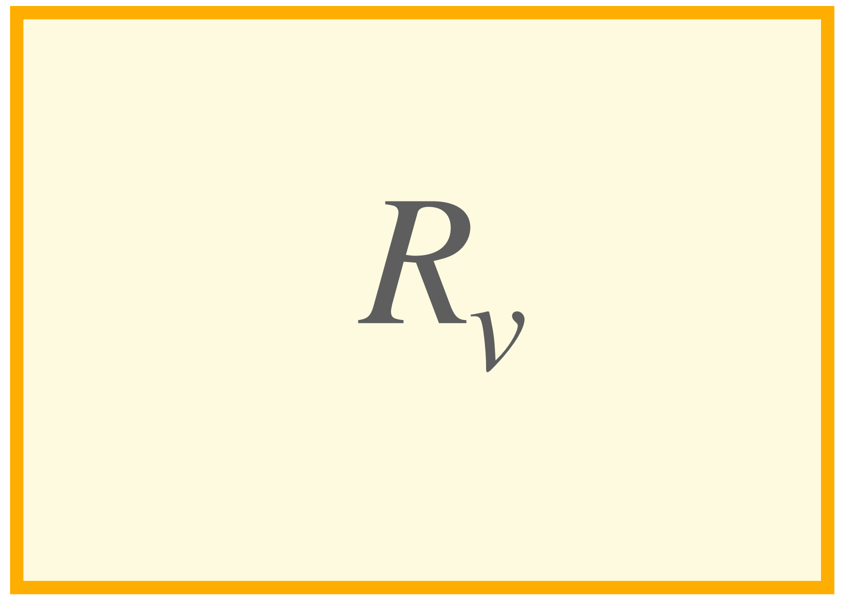

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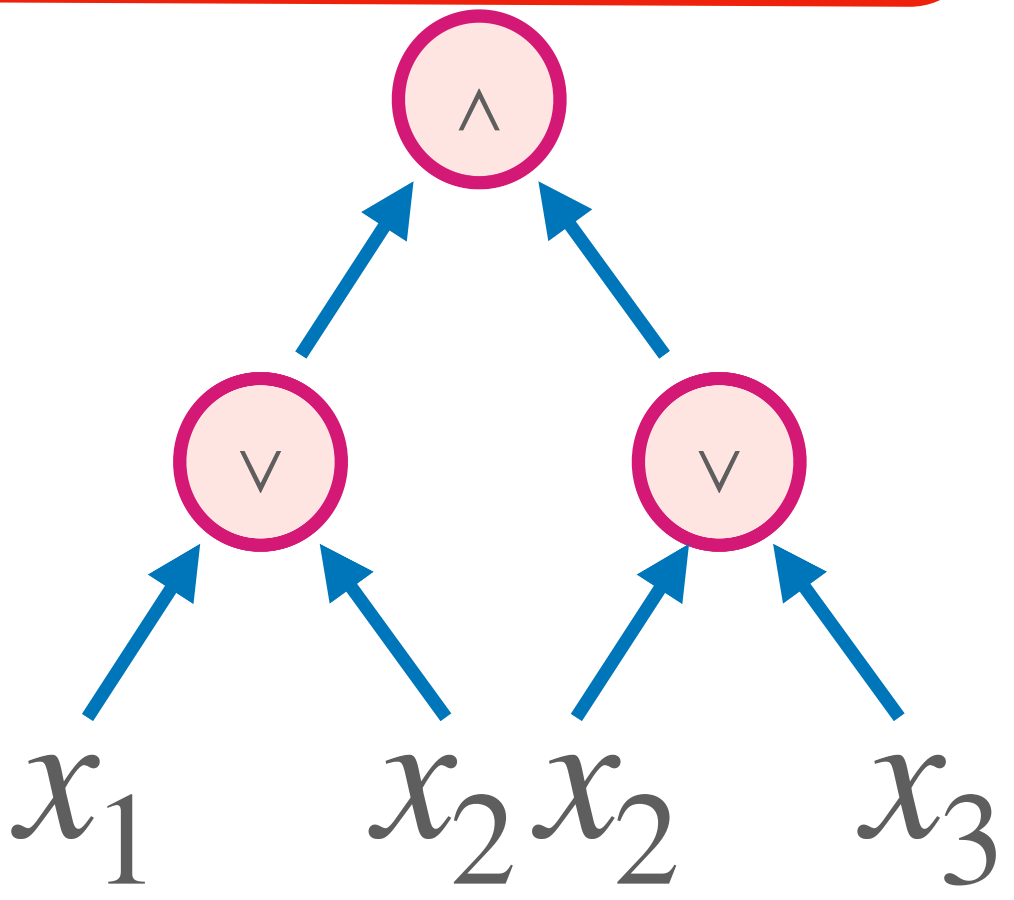

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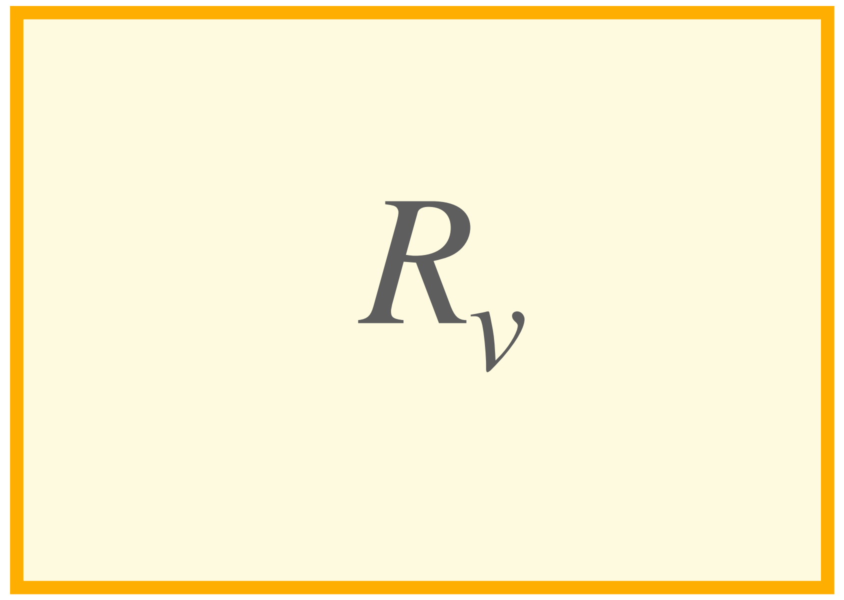

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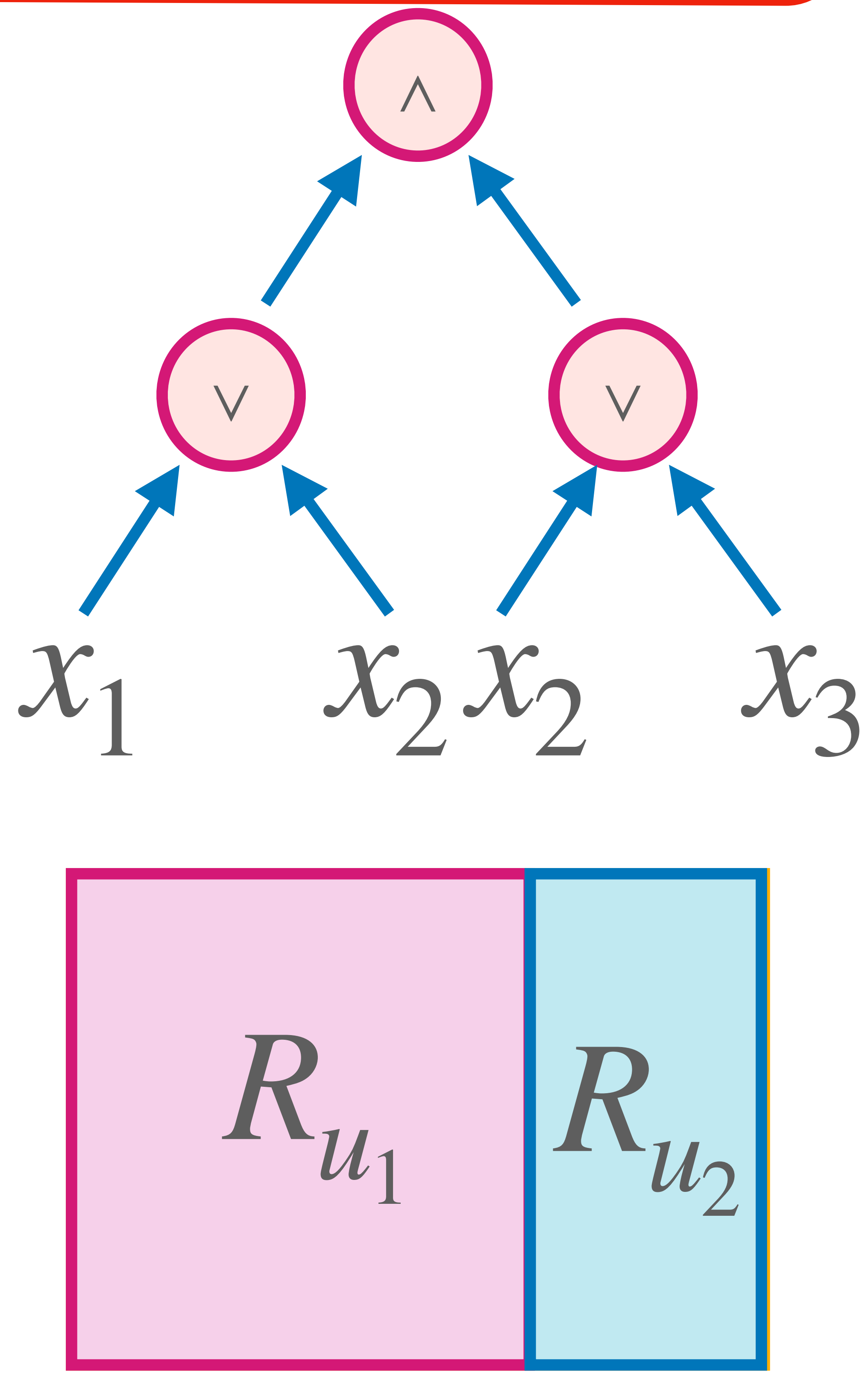

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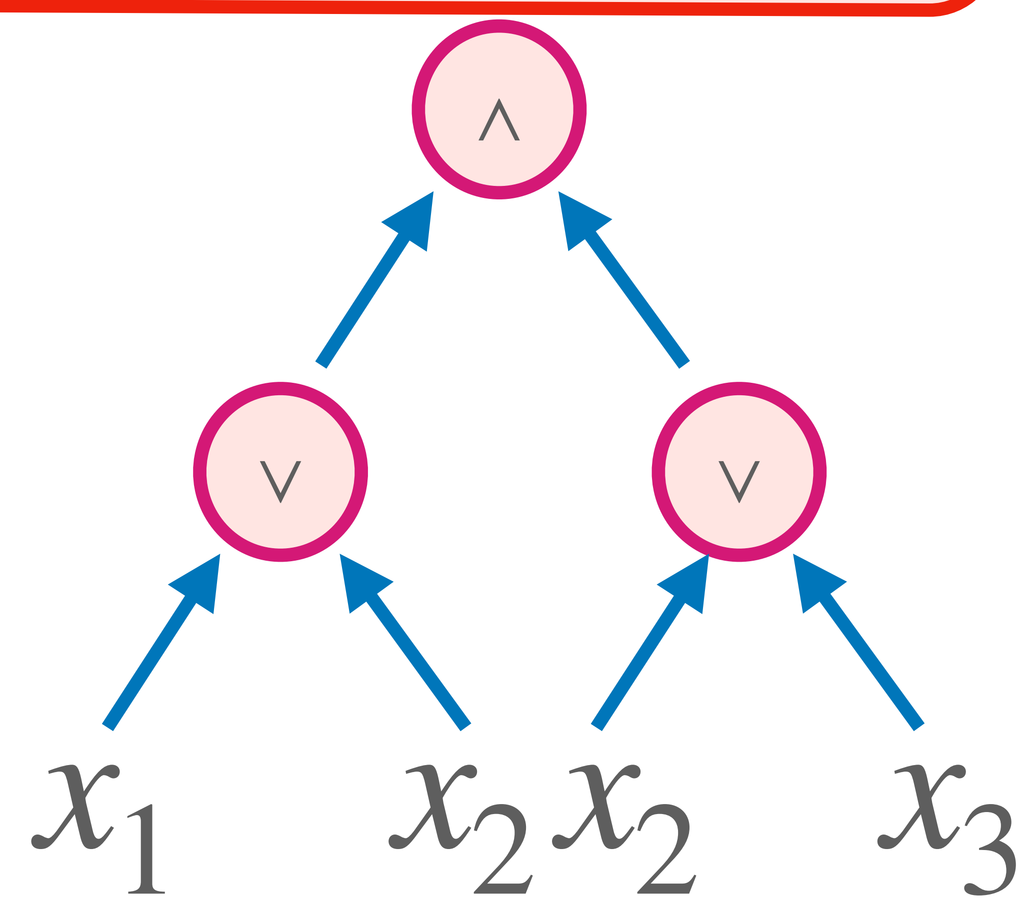

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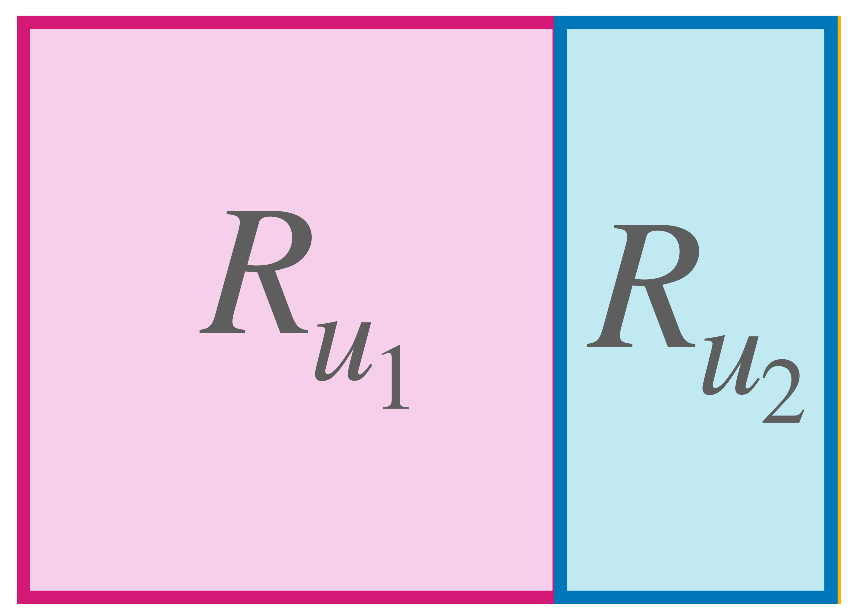

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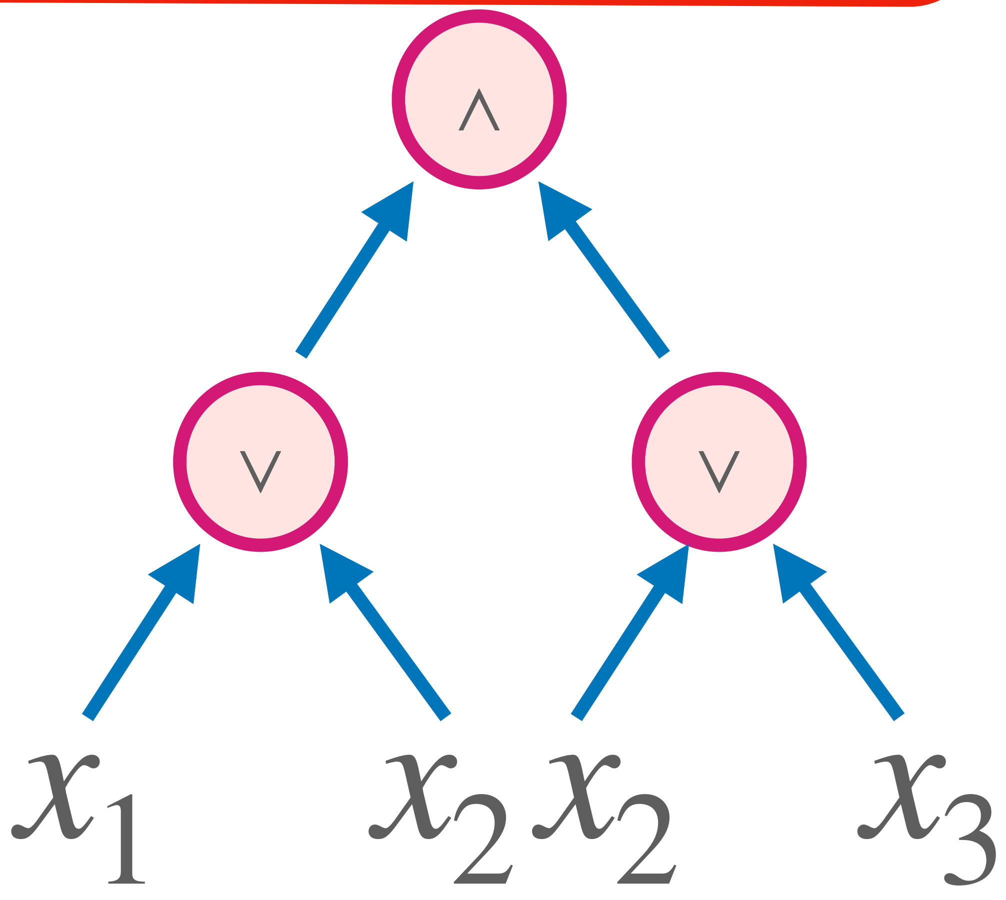

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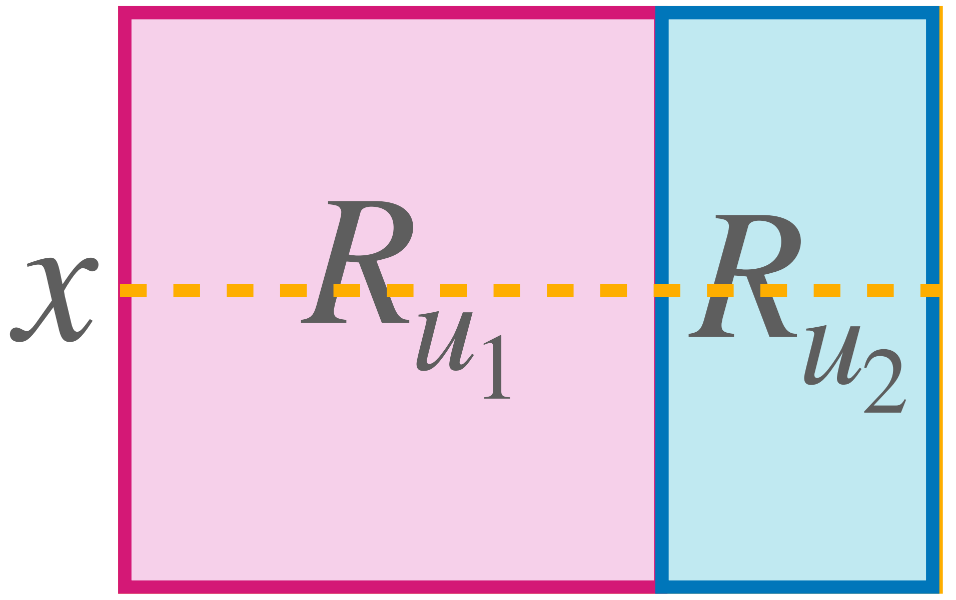

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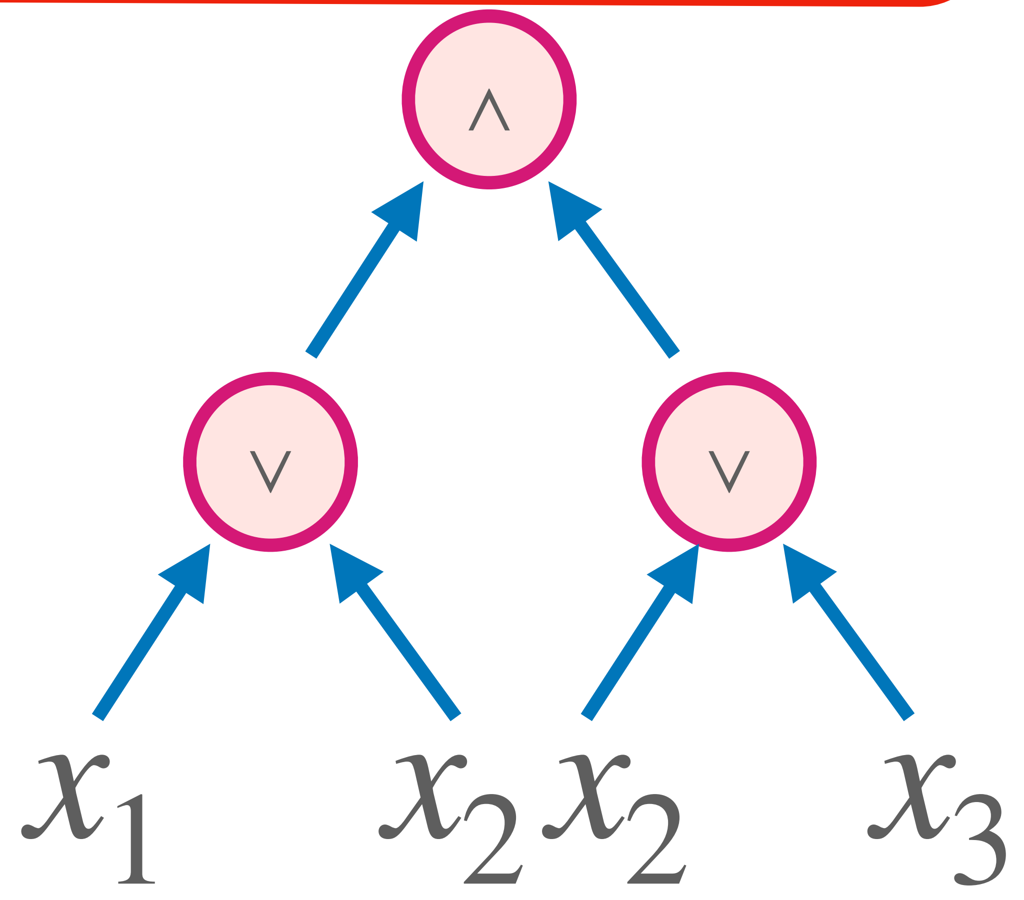

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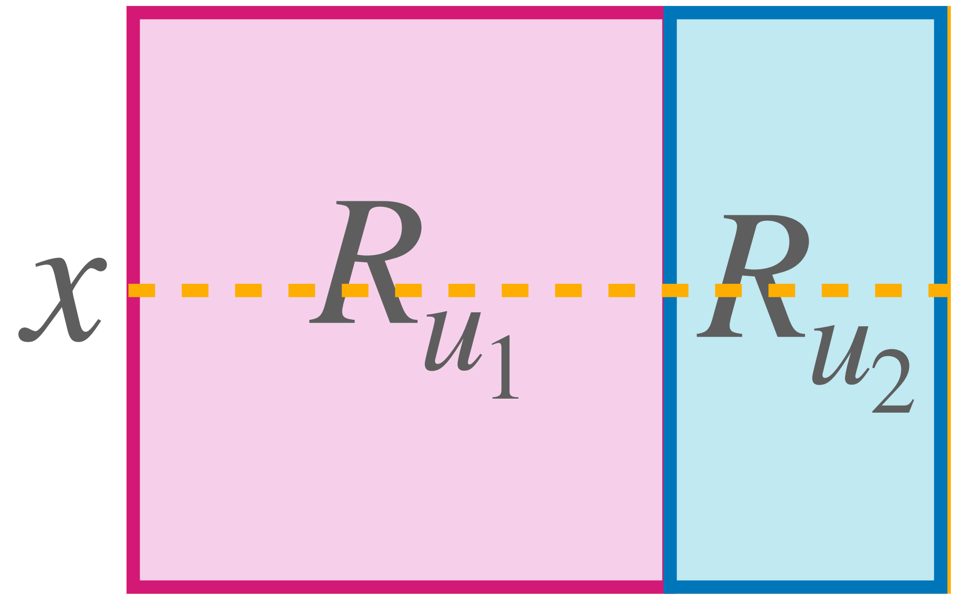

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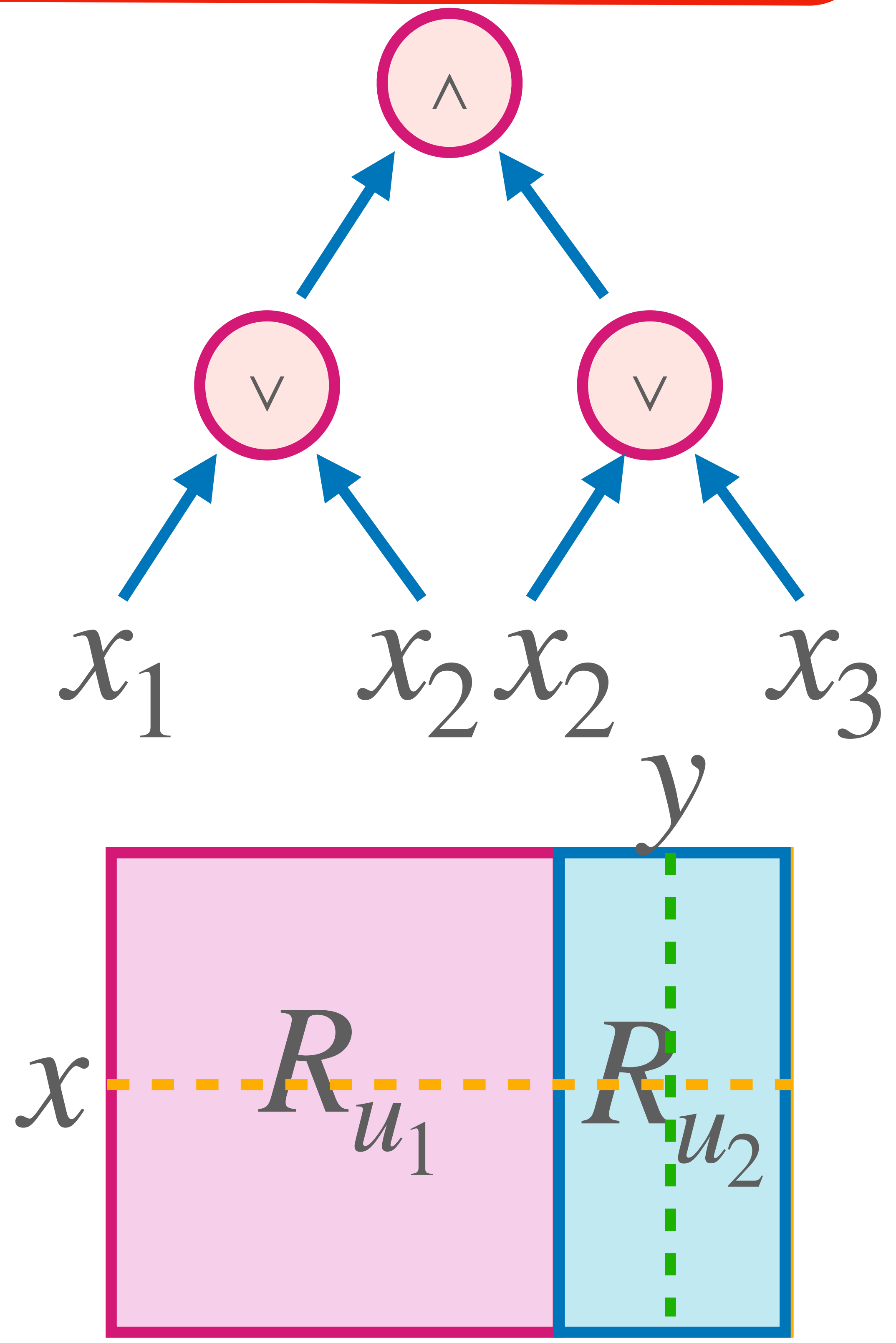

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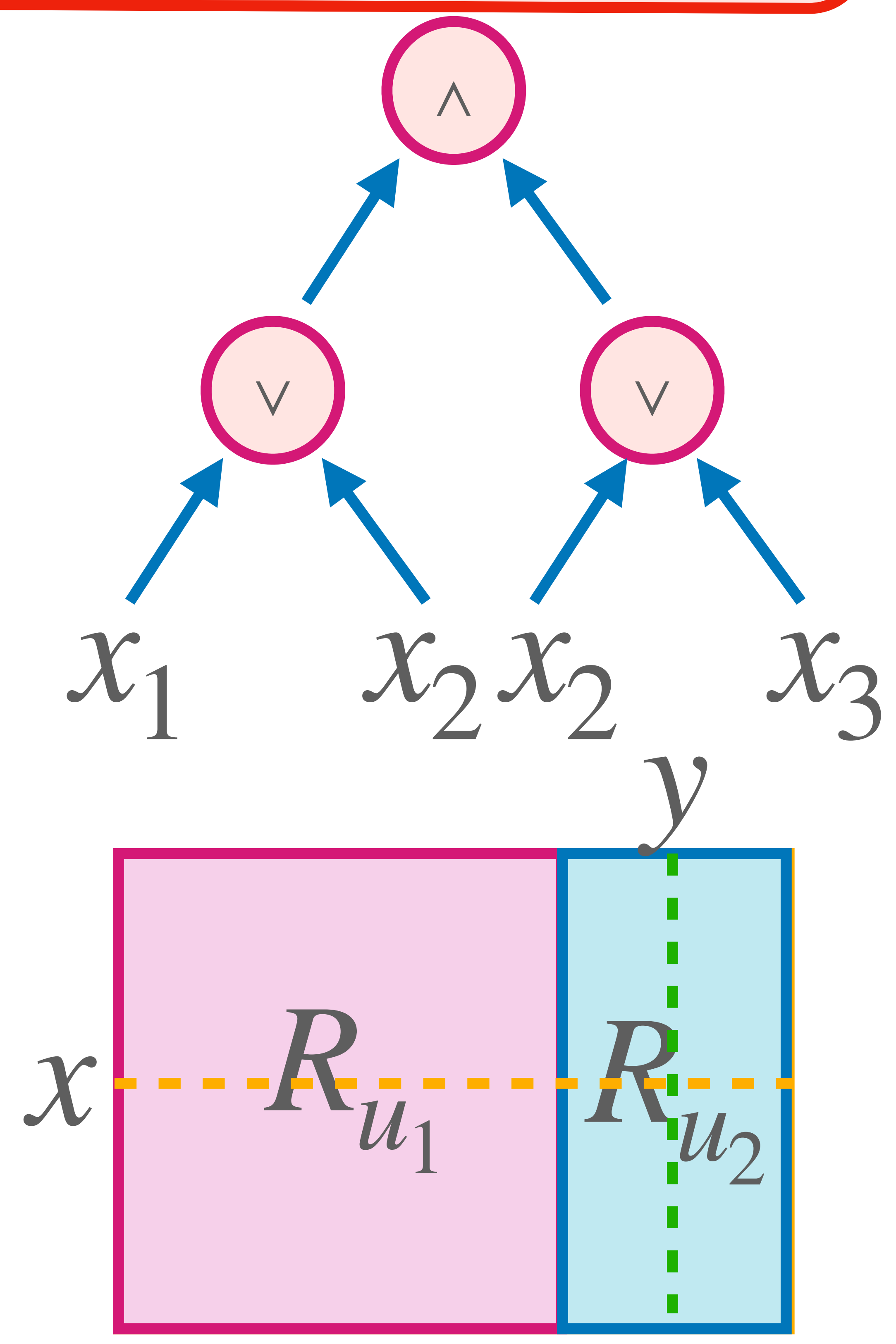

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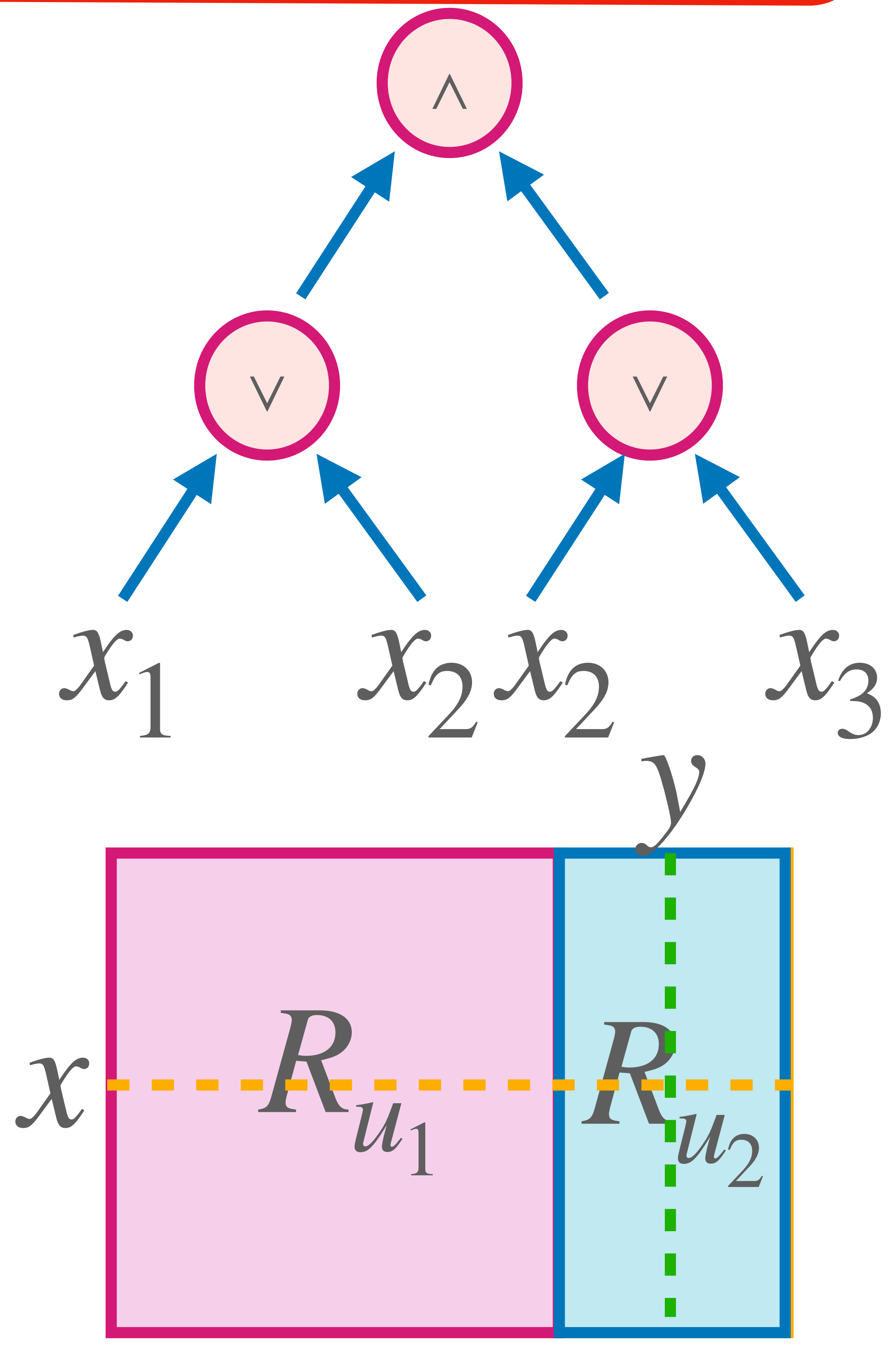
- If $v = \wedge$ Bob speaks, partitioning R_v on columns.
- If $v = \vee$ Alice speaks partitioning R_v on rows; symmetric argument.



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False Clause Search Problem: Let $F = C_1 \wedge \ldots \wedge C_m$ be an unsat CNF.
Let $X \times Y$ be a partition of $[n]$. $\text{Search}_F^{X,Y} \subseteq \{0,1\}^X \times \{0,1\}^Y \times [m]$:
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Intuition: Every refutation must prove that for every assignment x to F there is a falsified clause of F

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Derive new clauses from old ones using:

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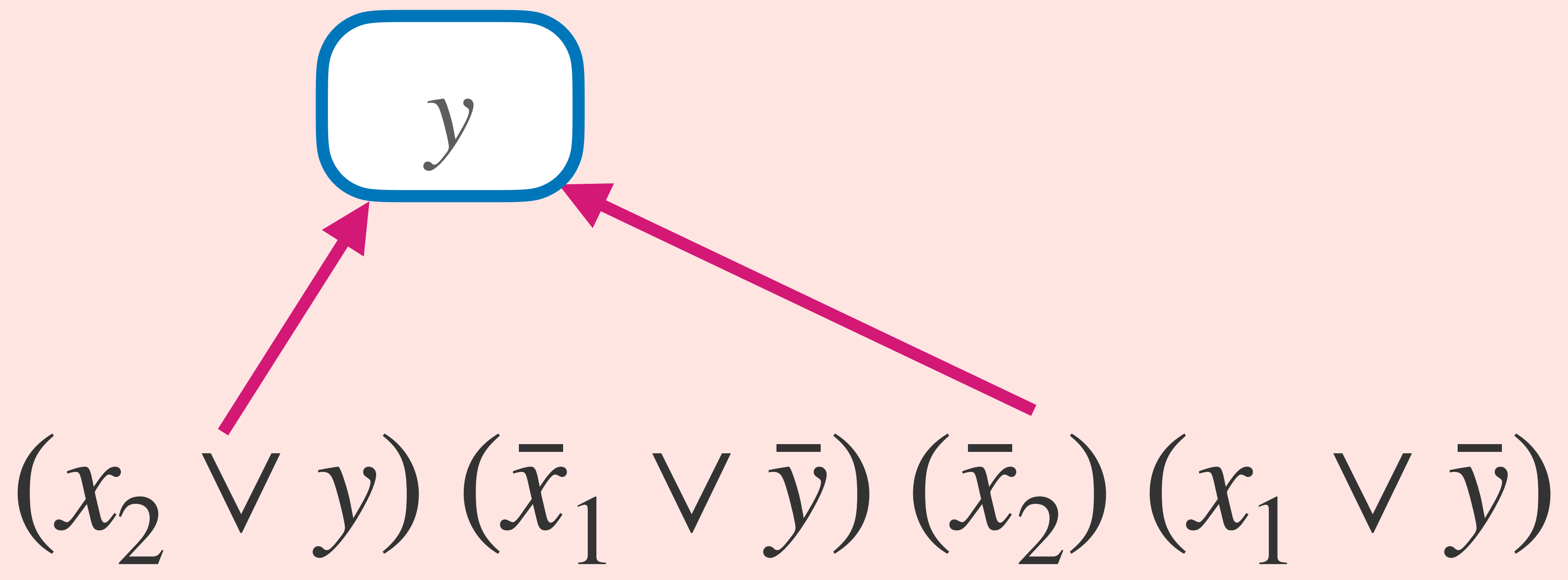

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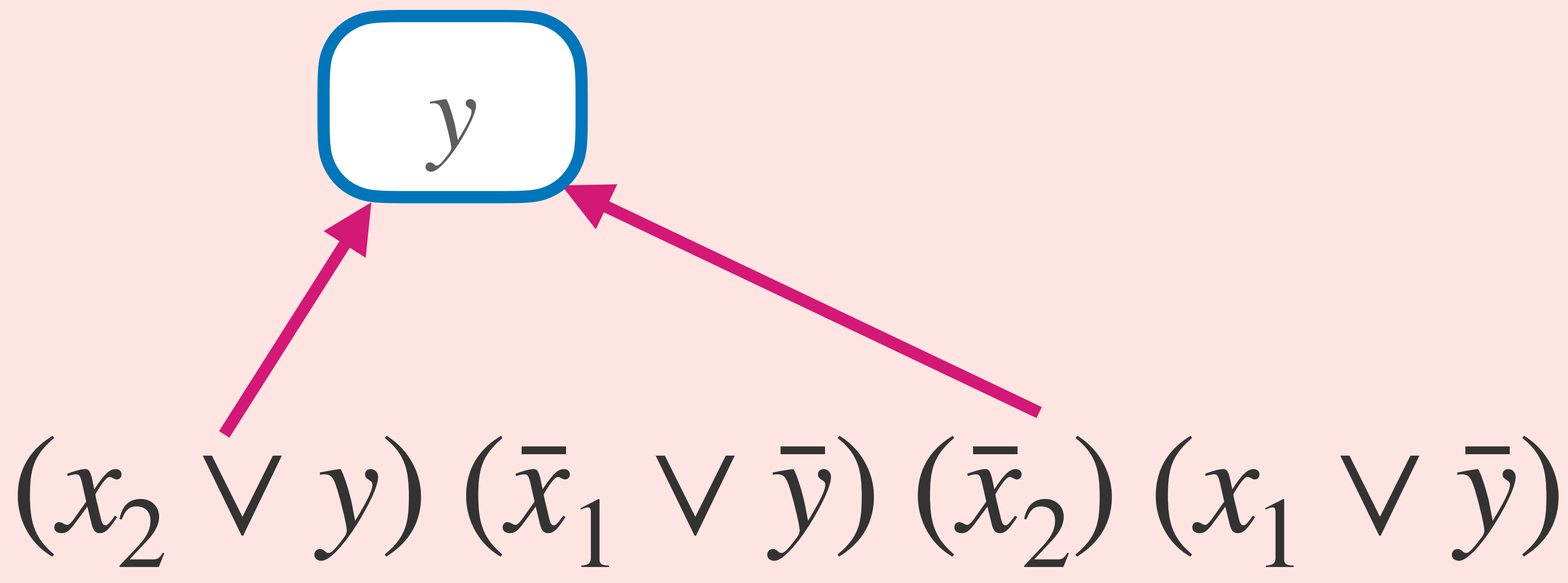

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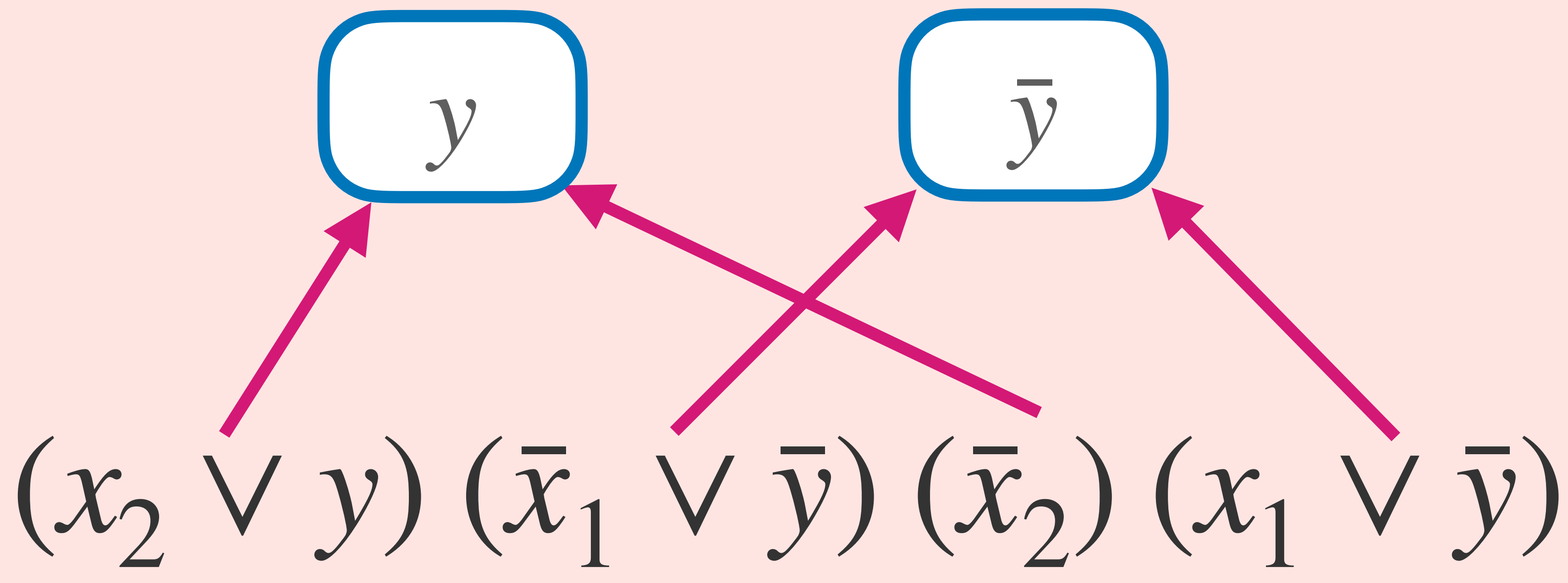

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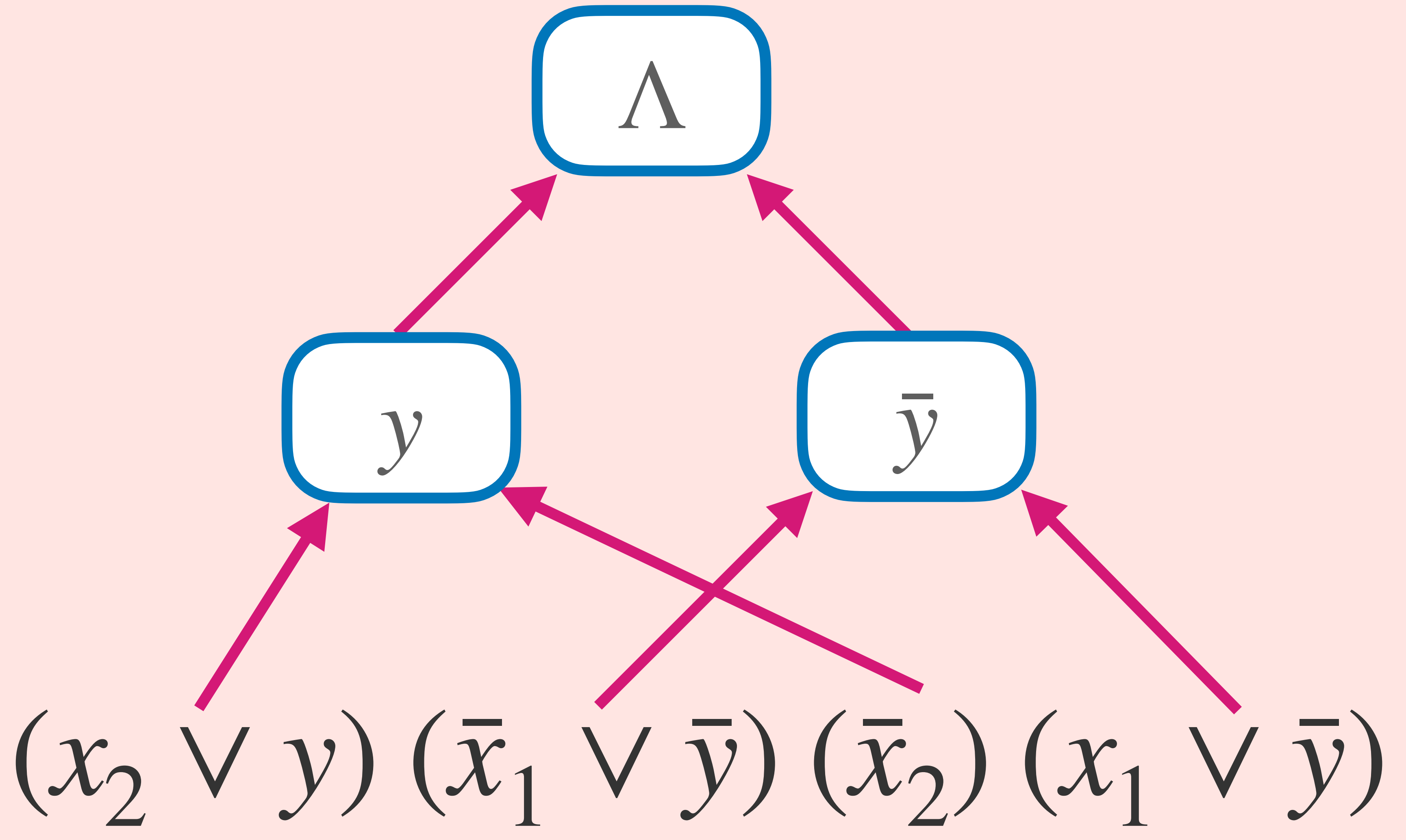

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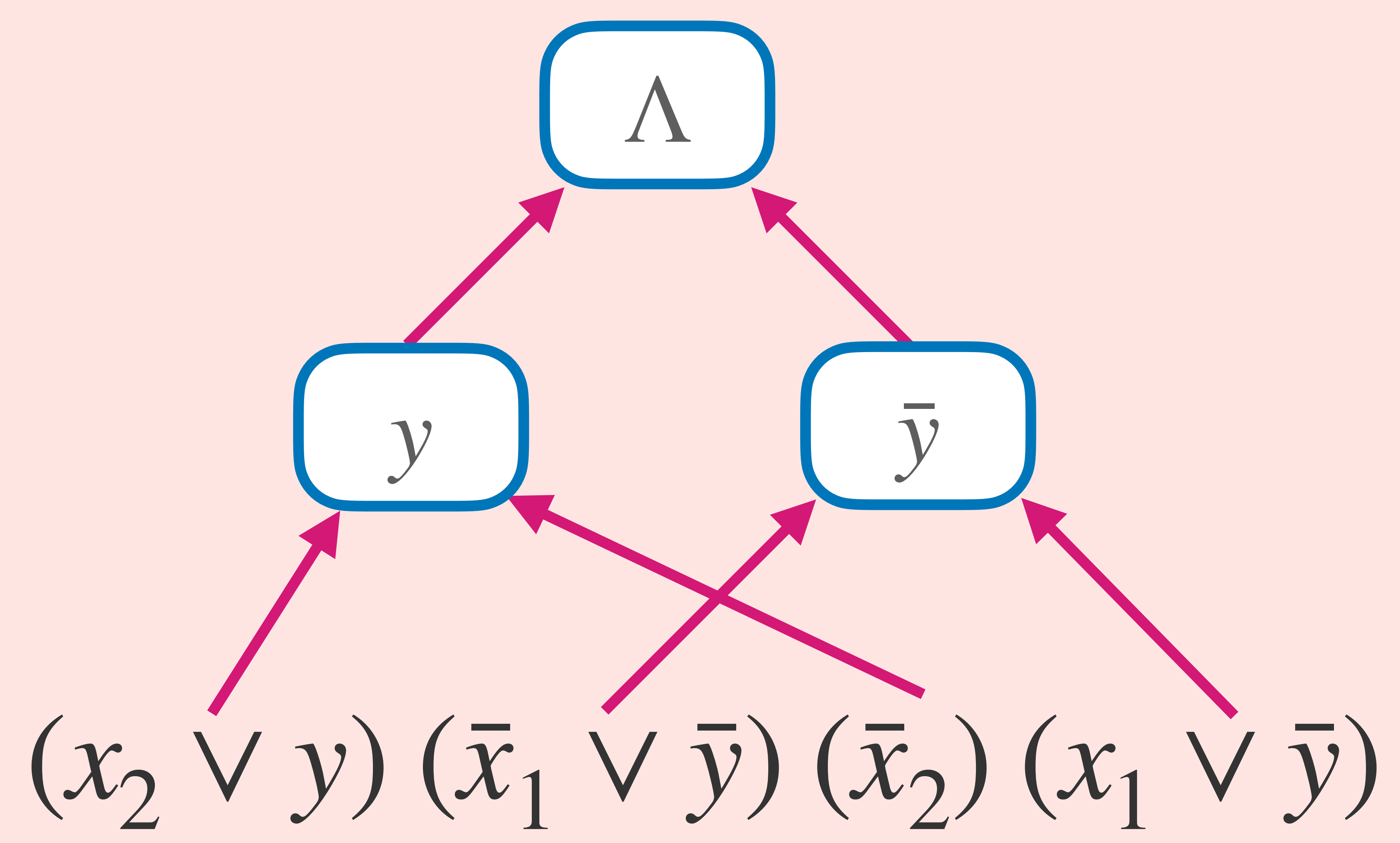

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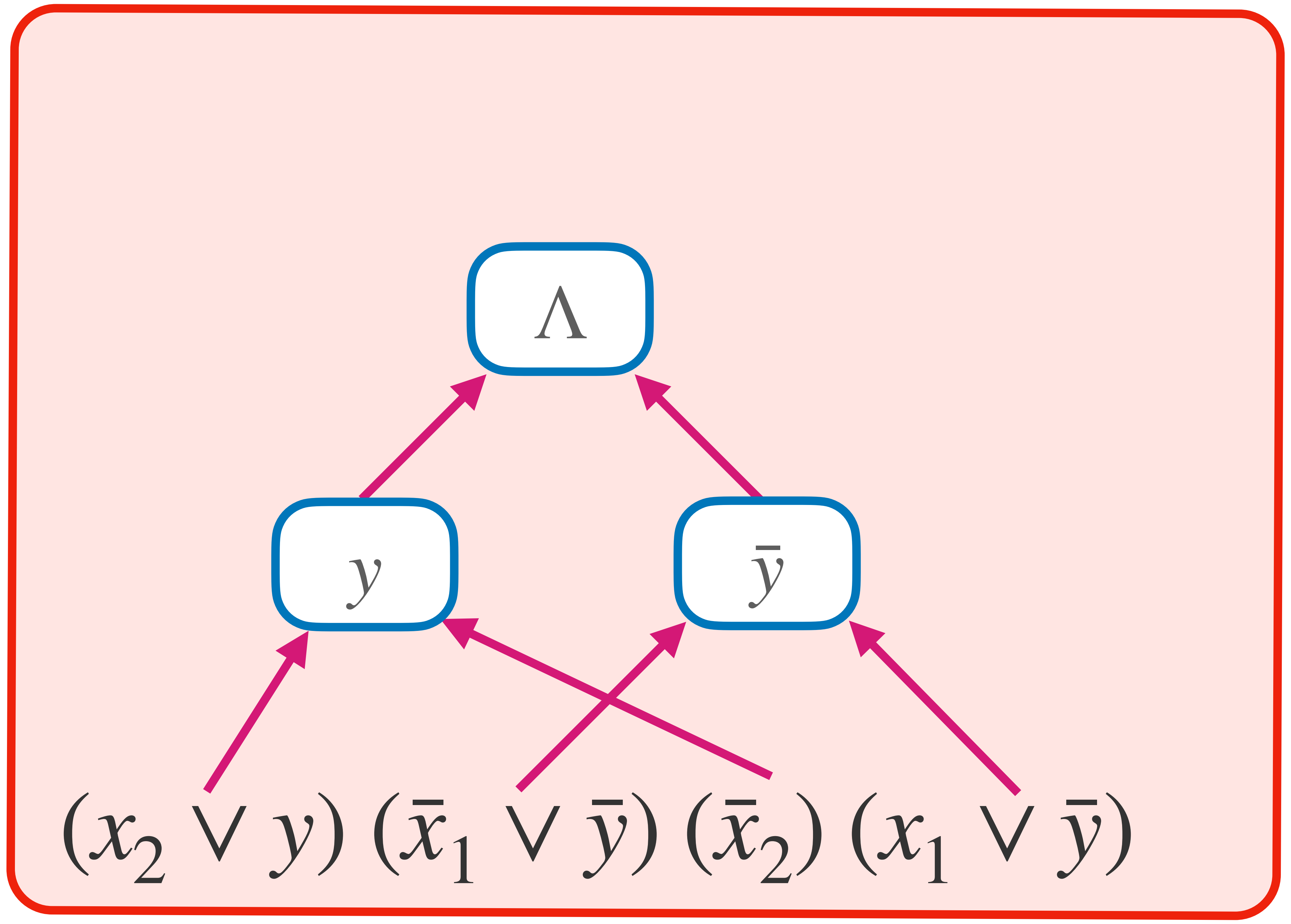

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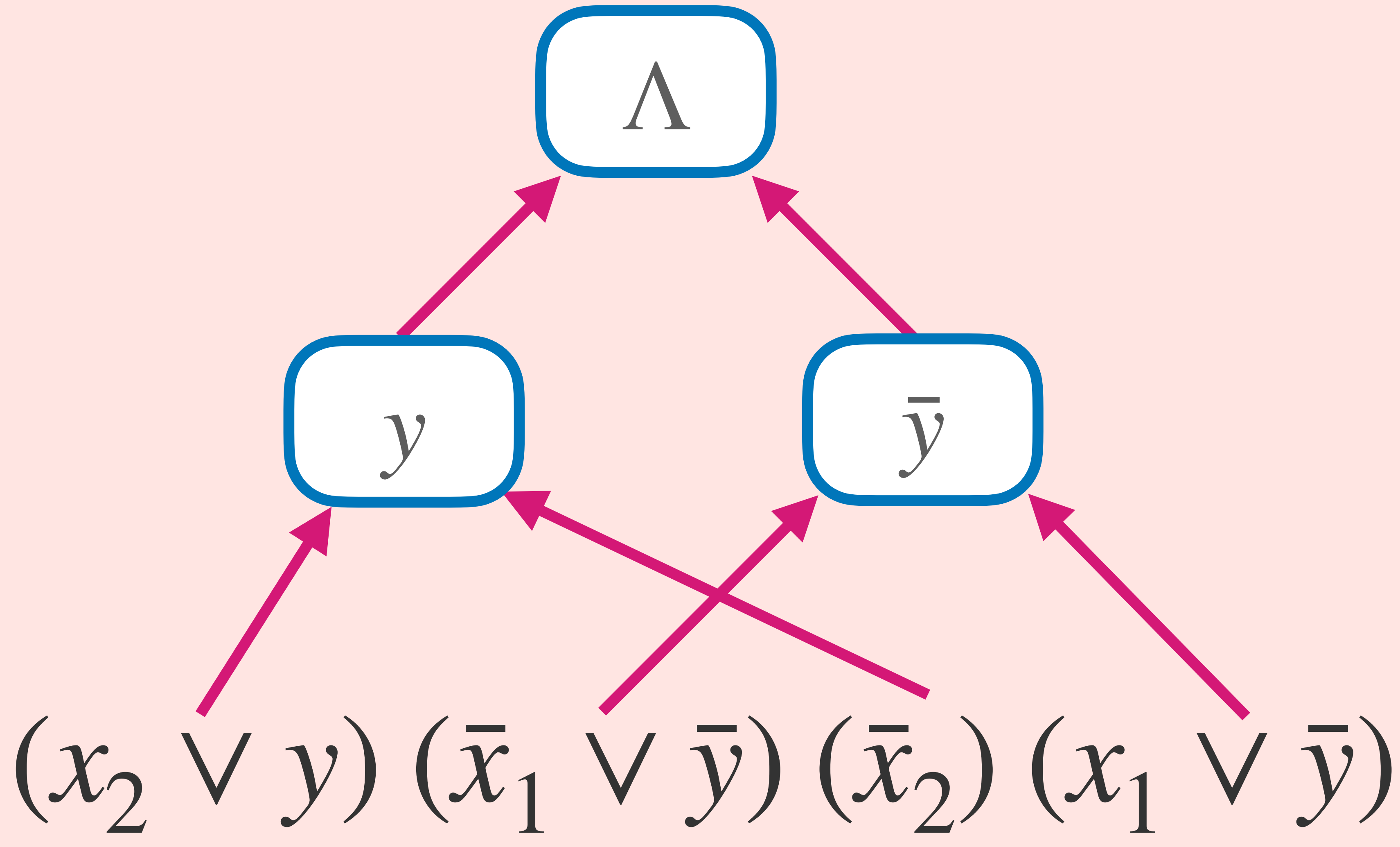

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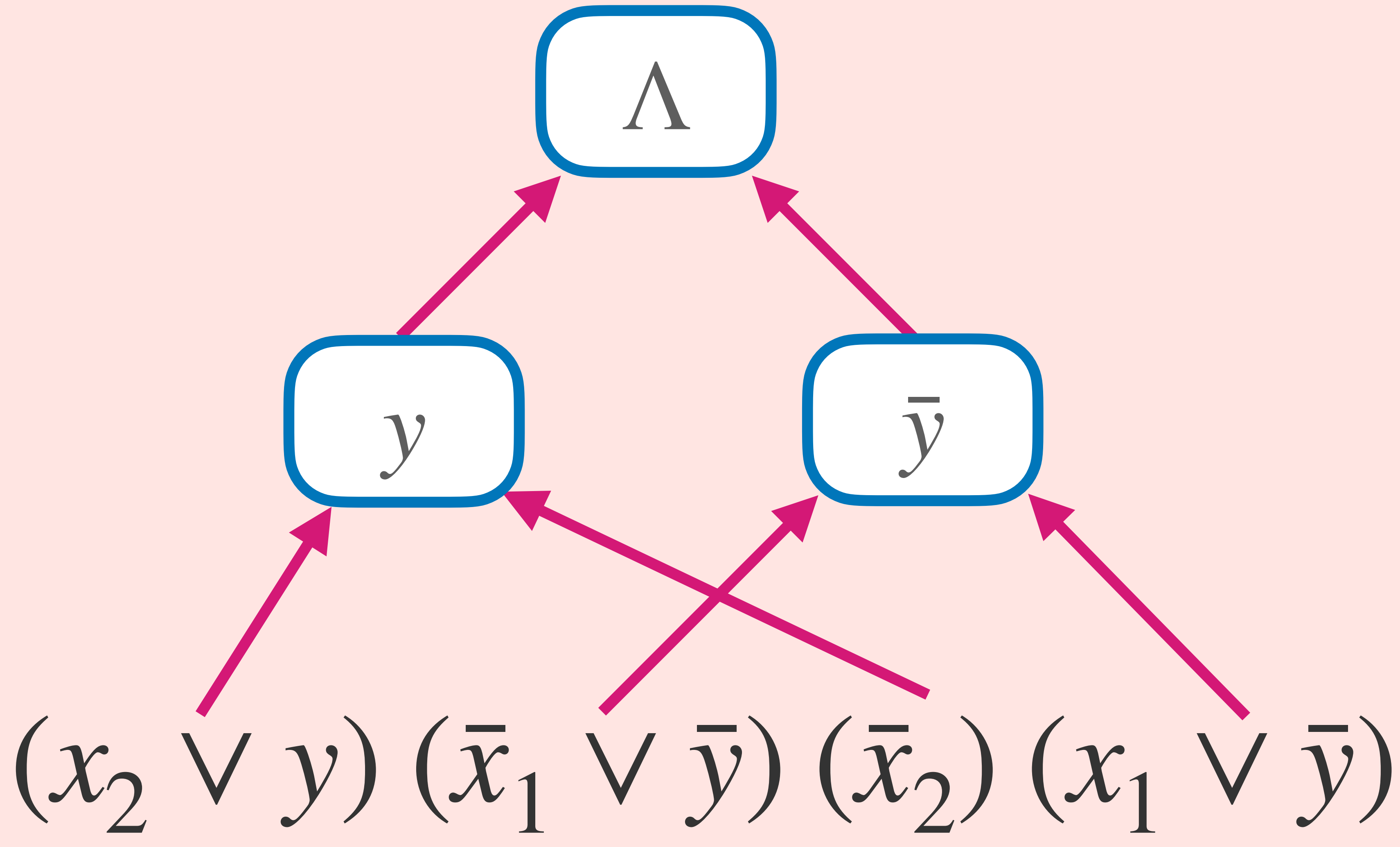

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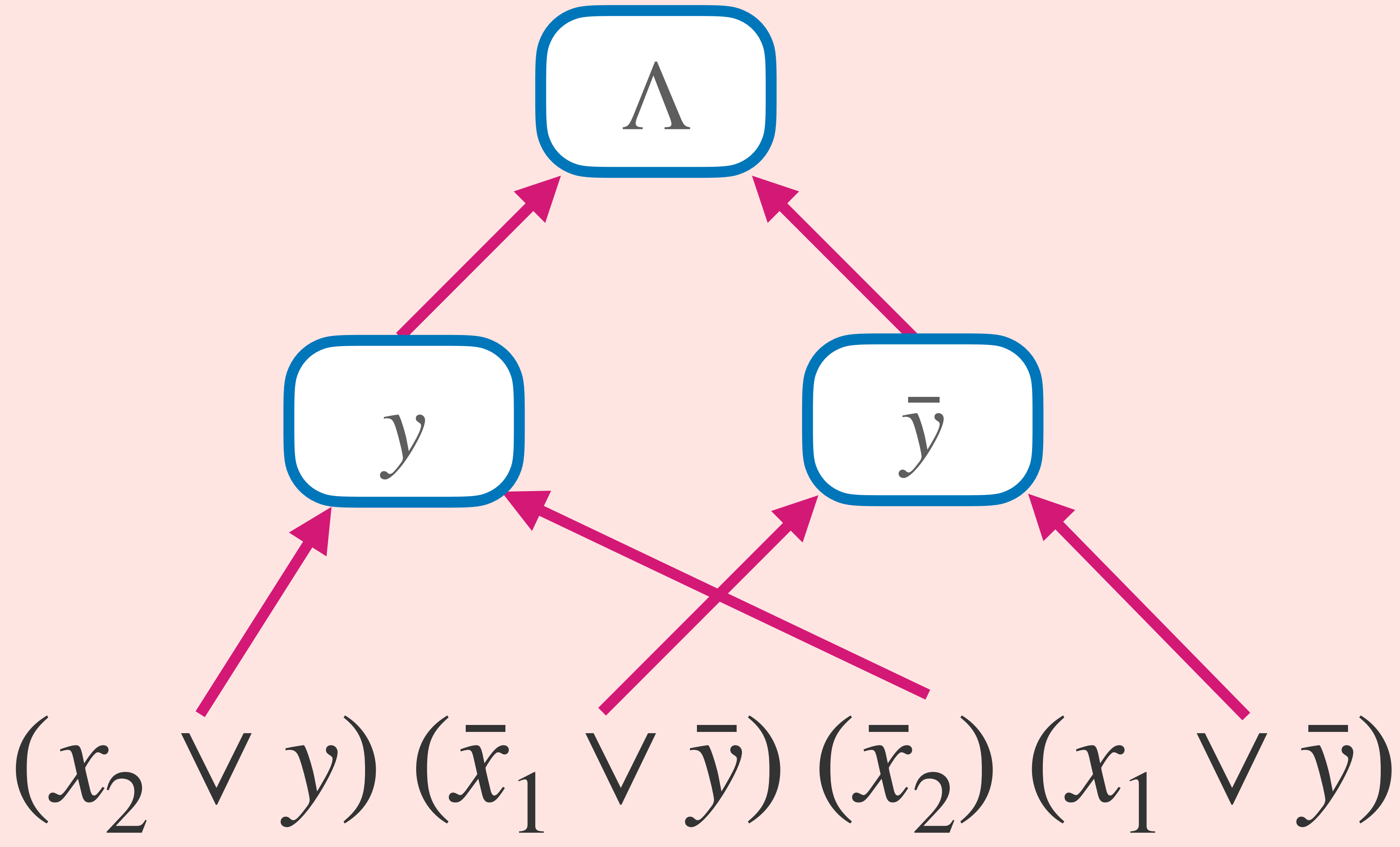

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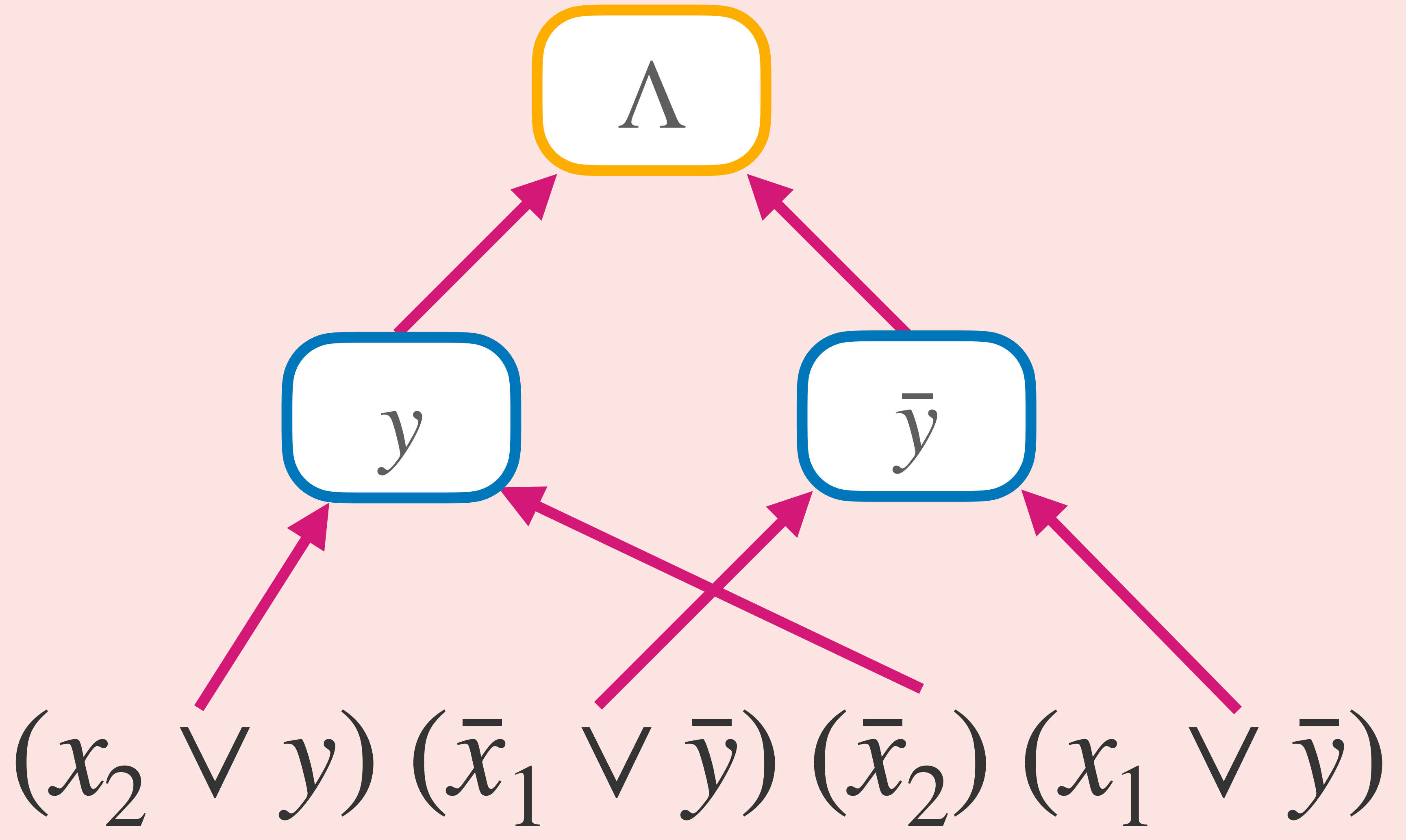

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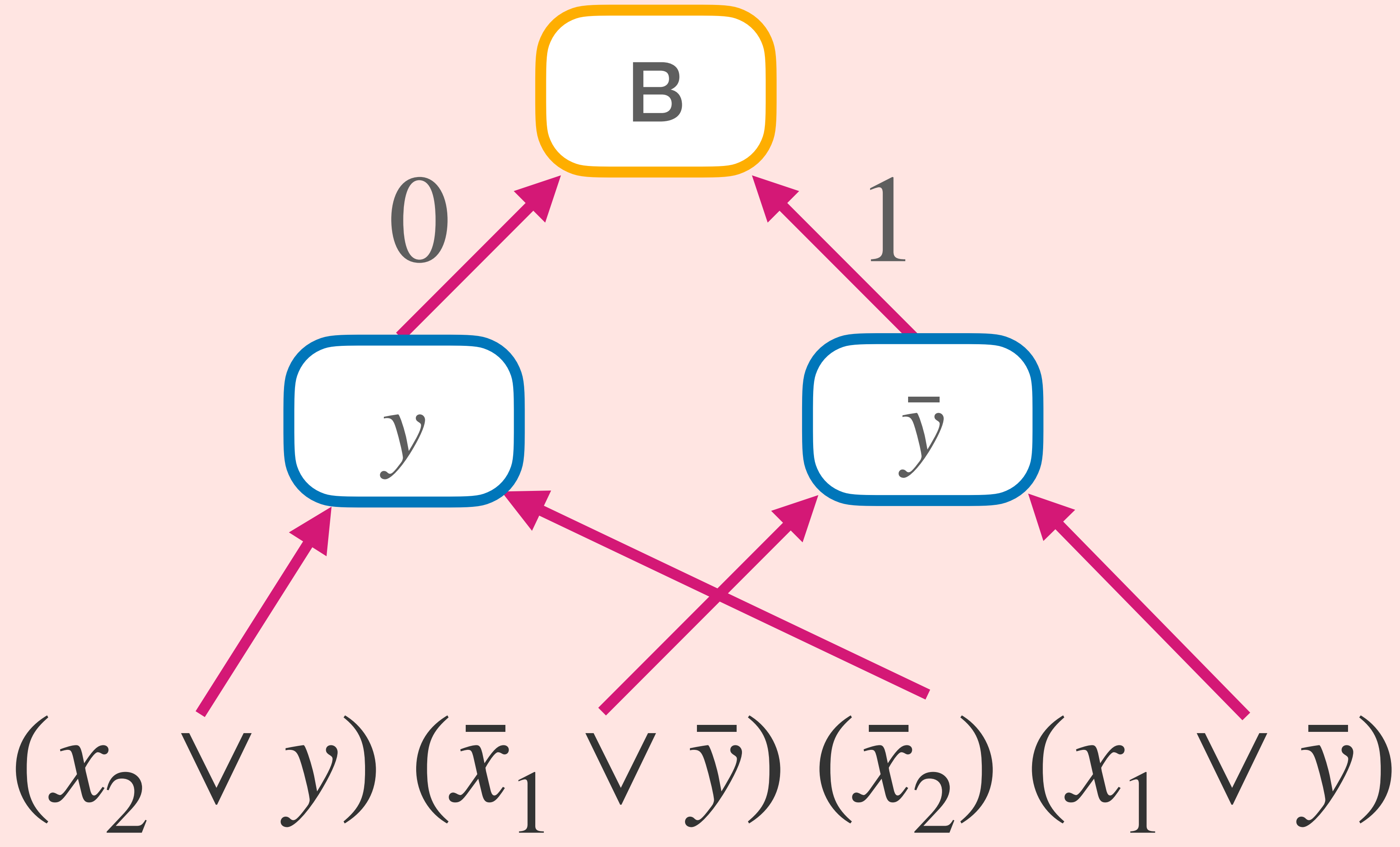

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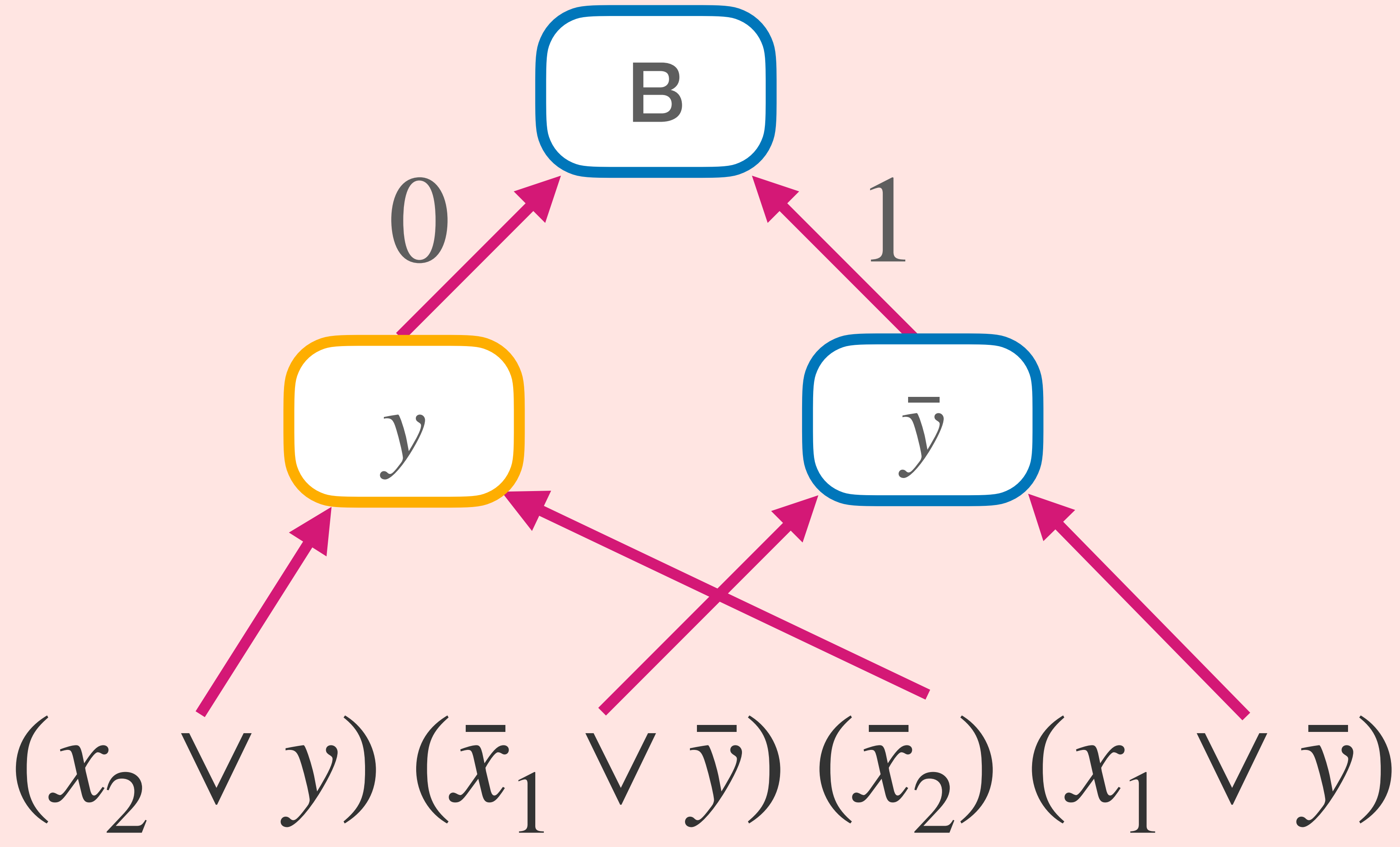

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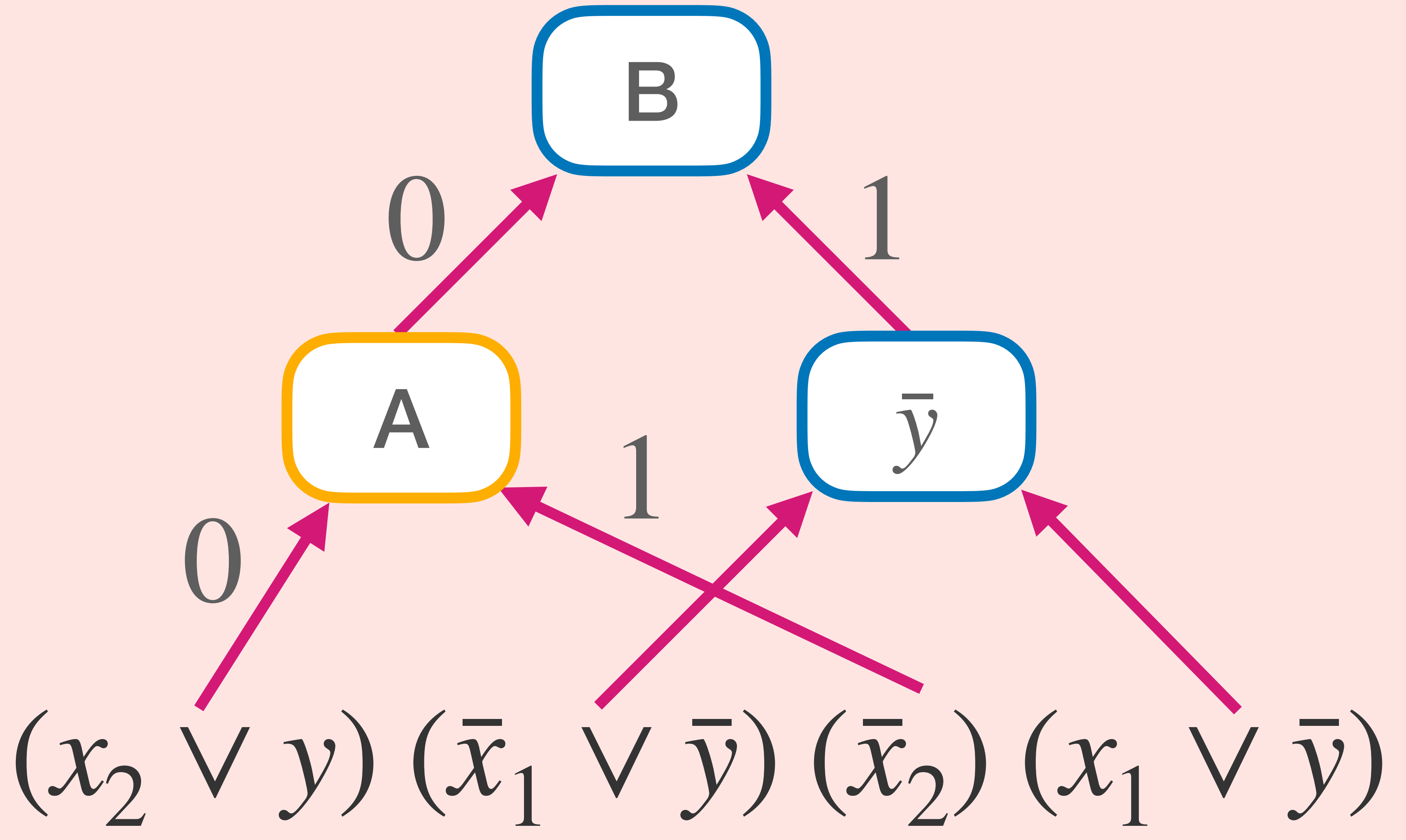

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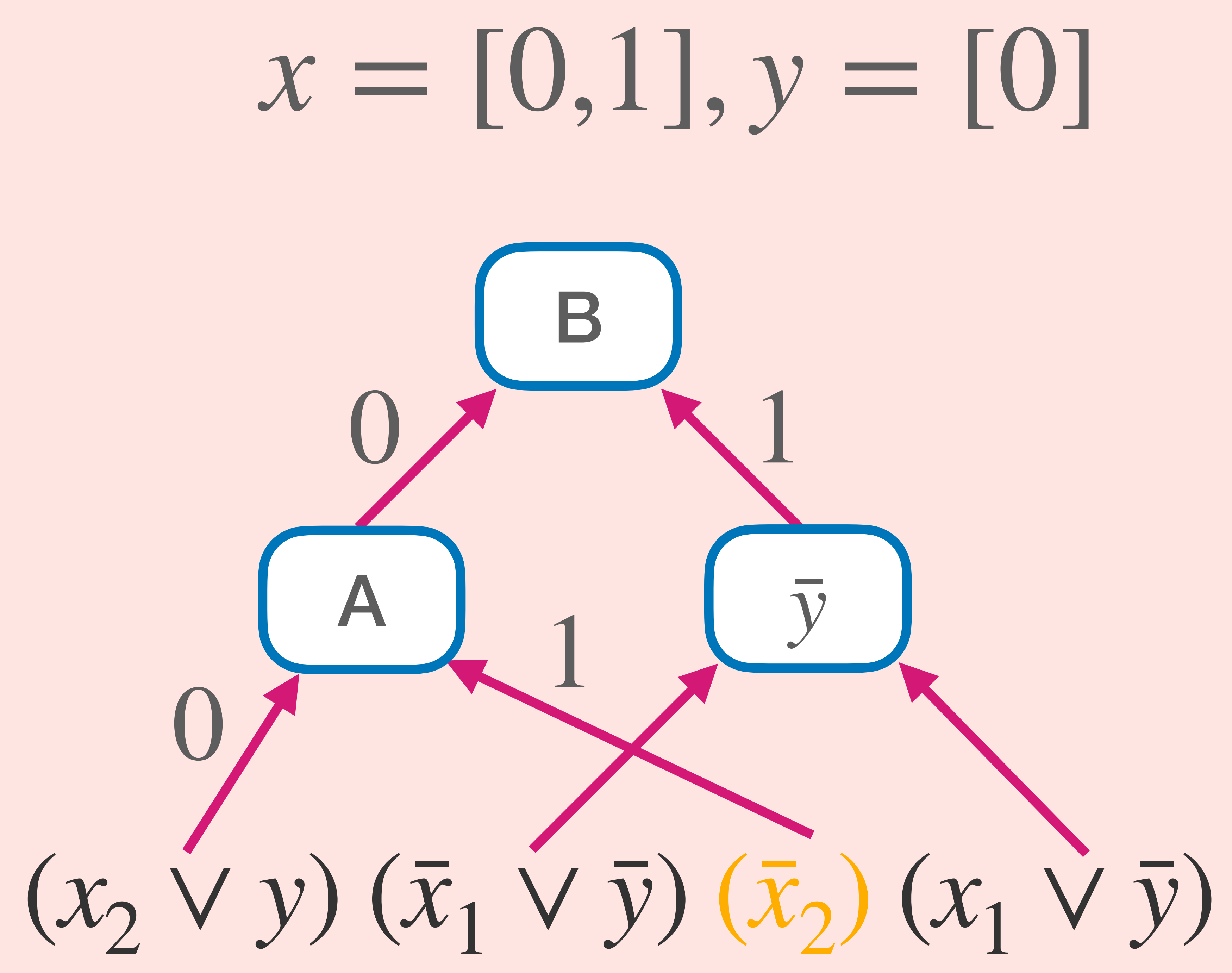

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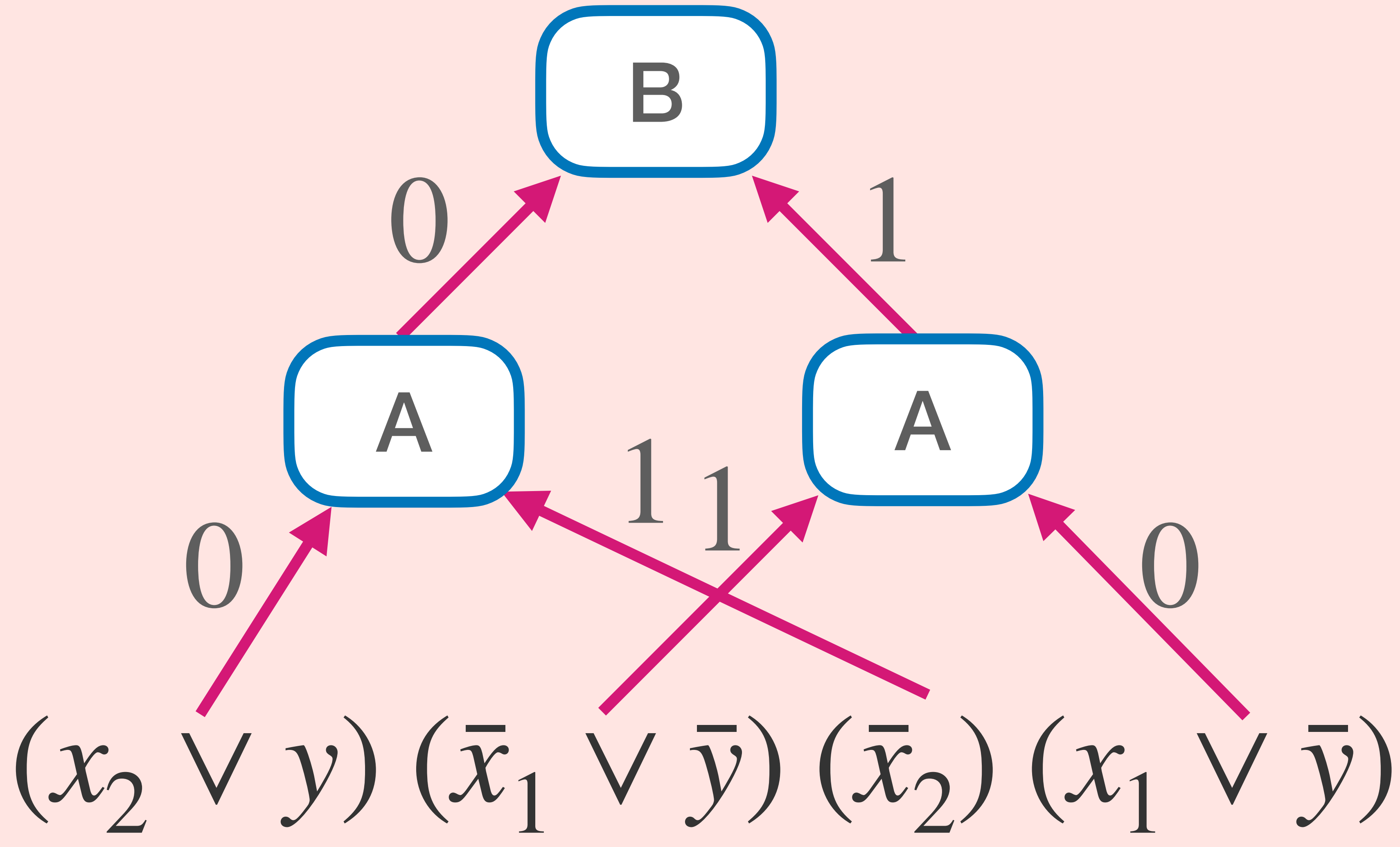

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So far we have seen

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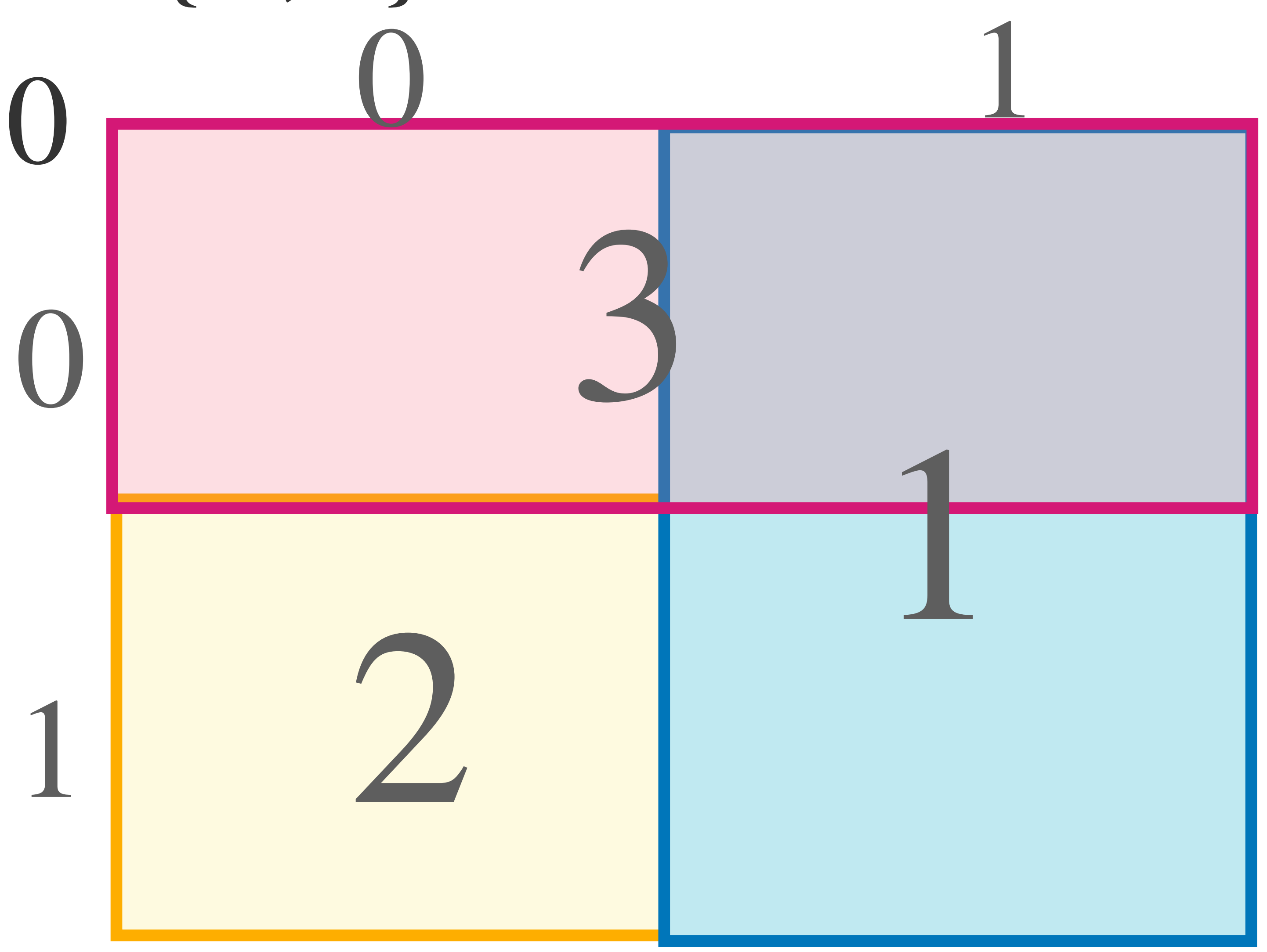

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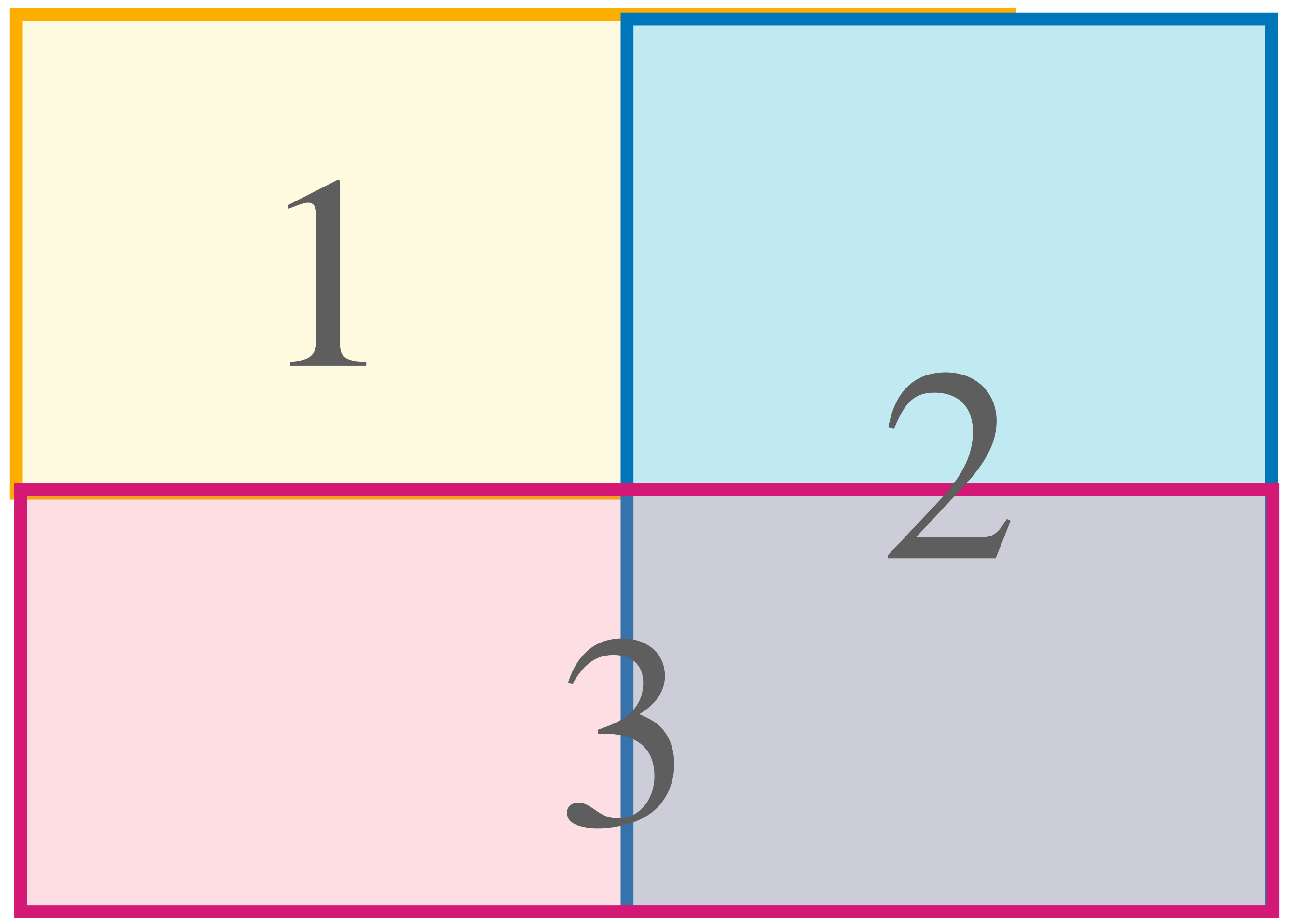

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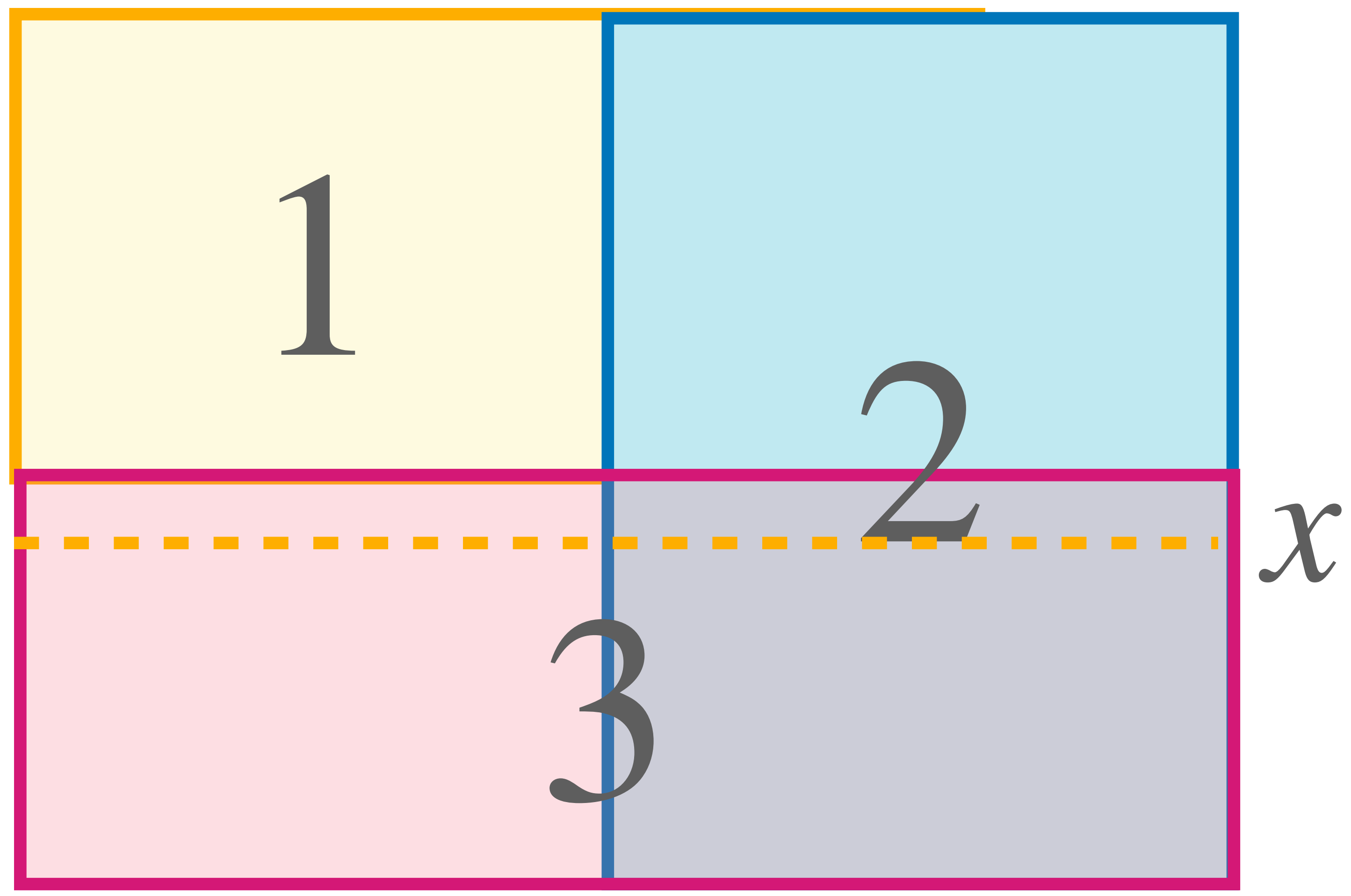

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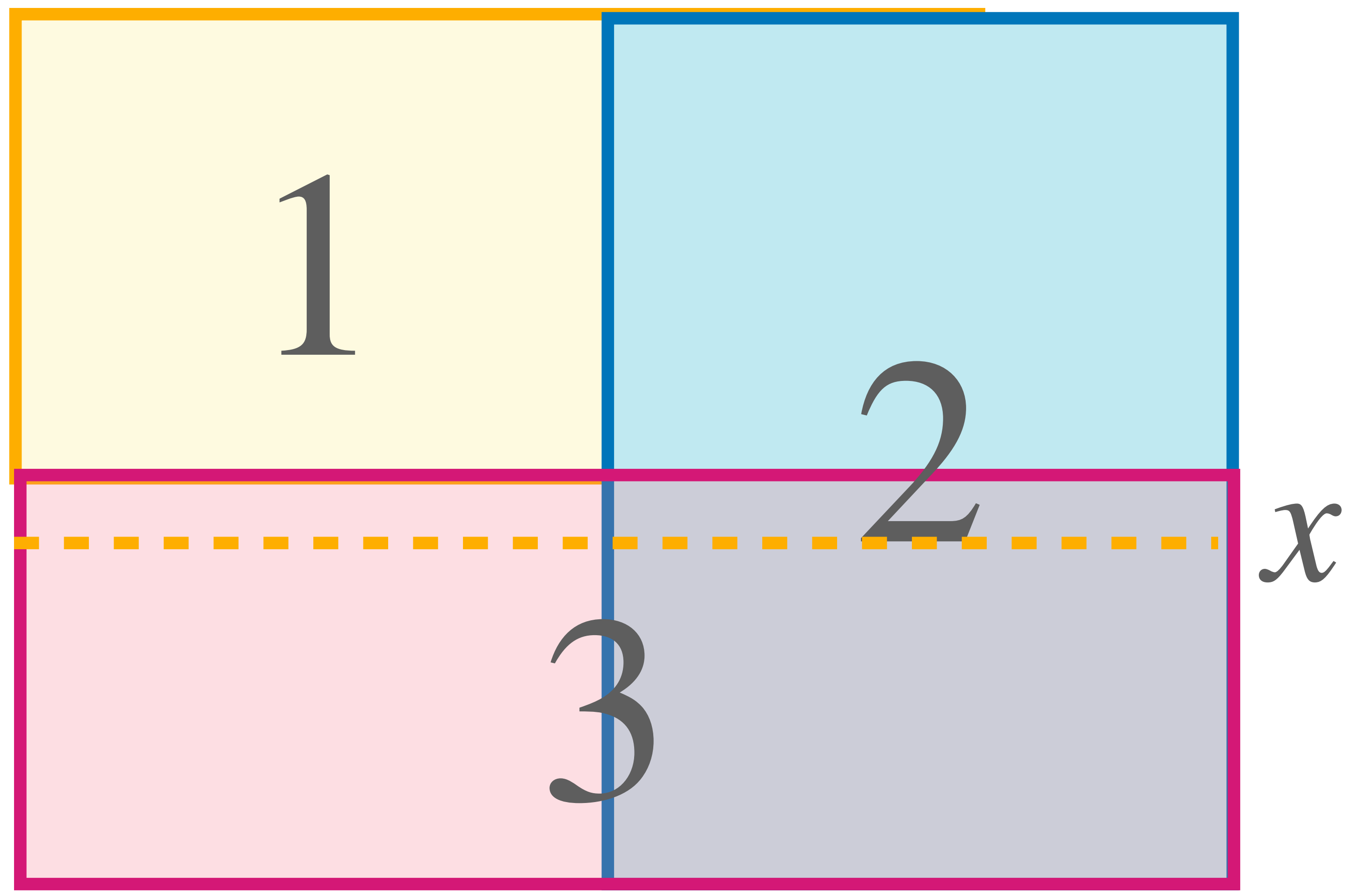

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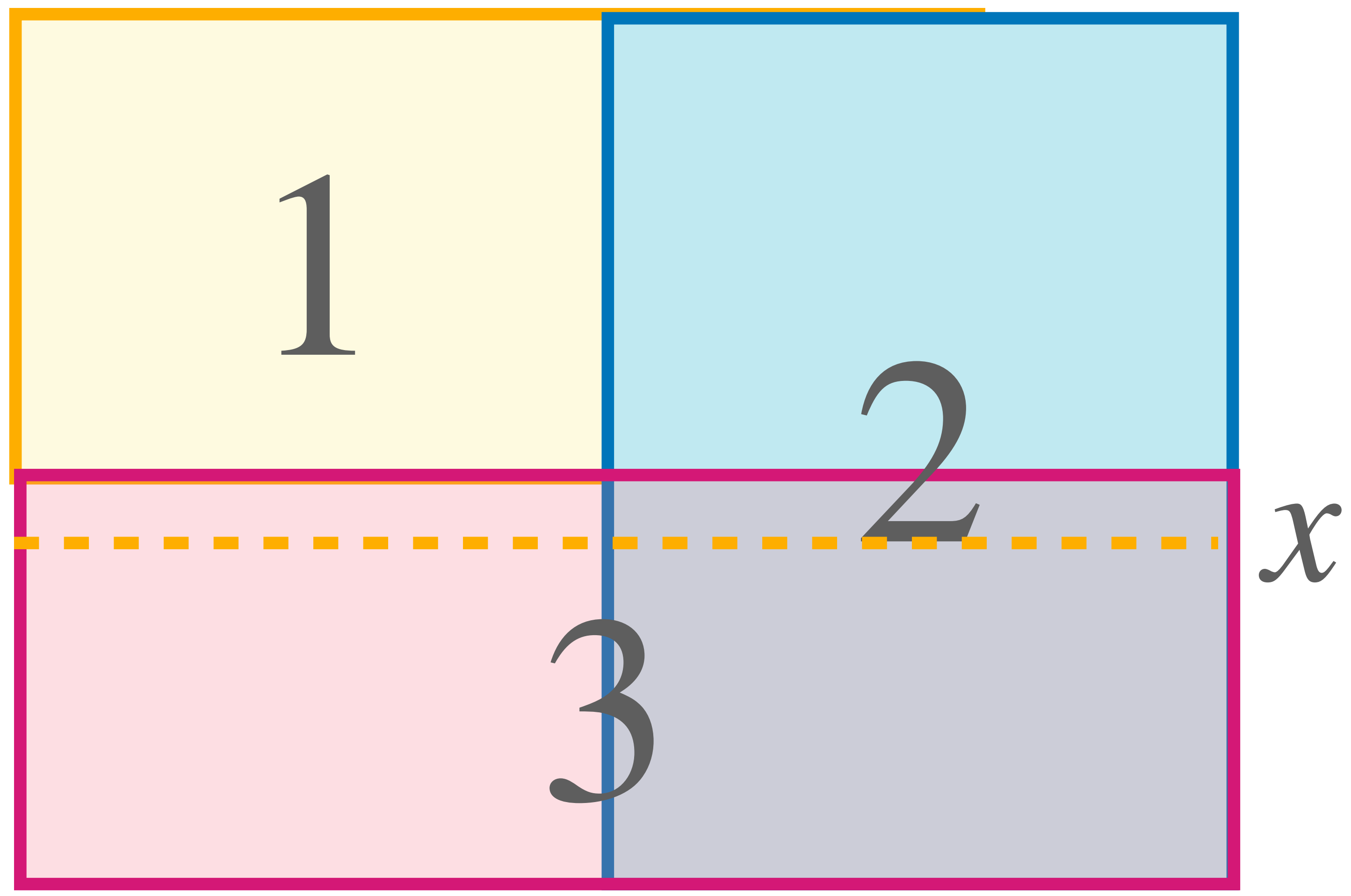

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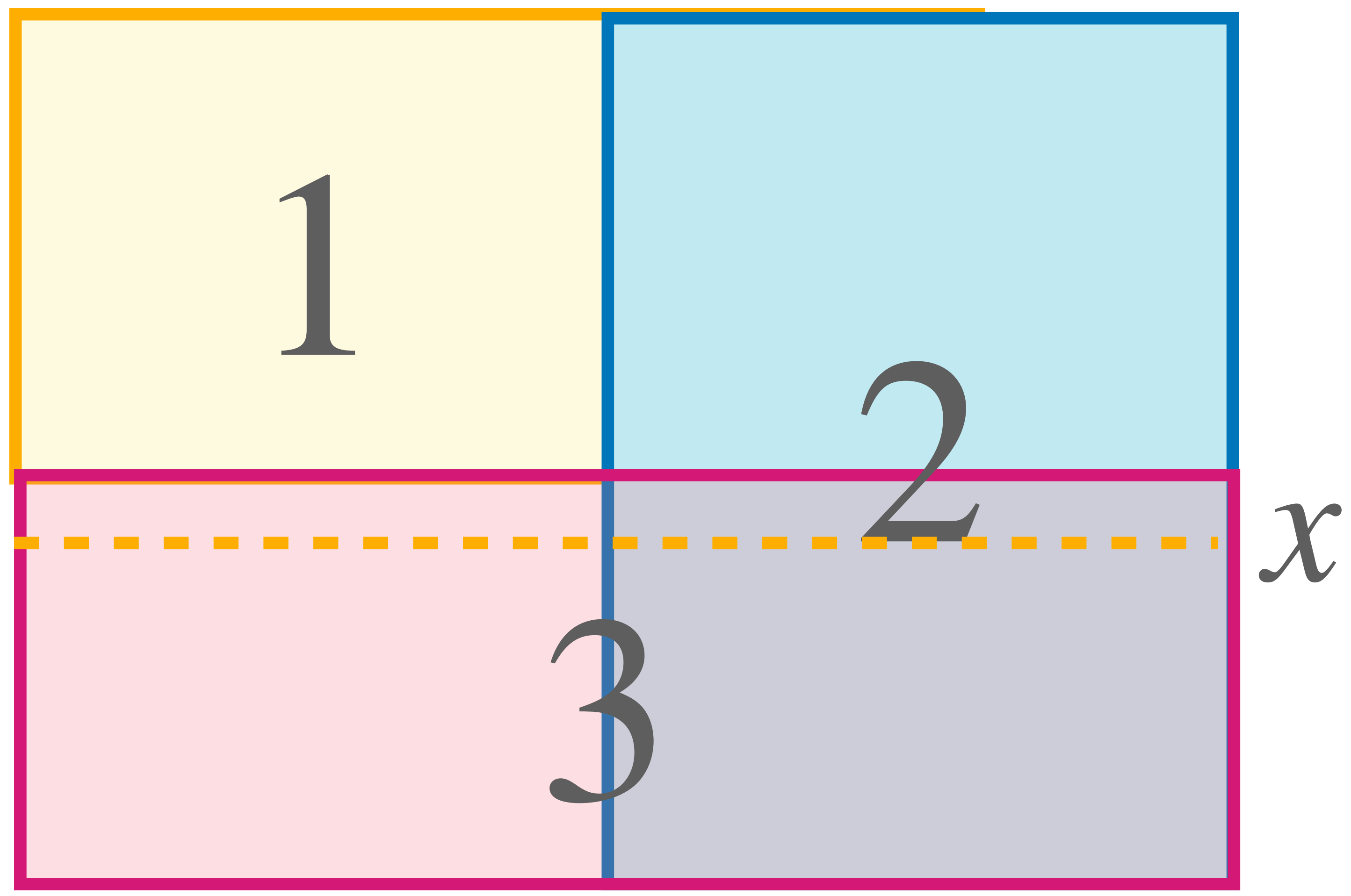

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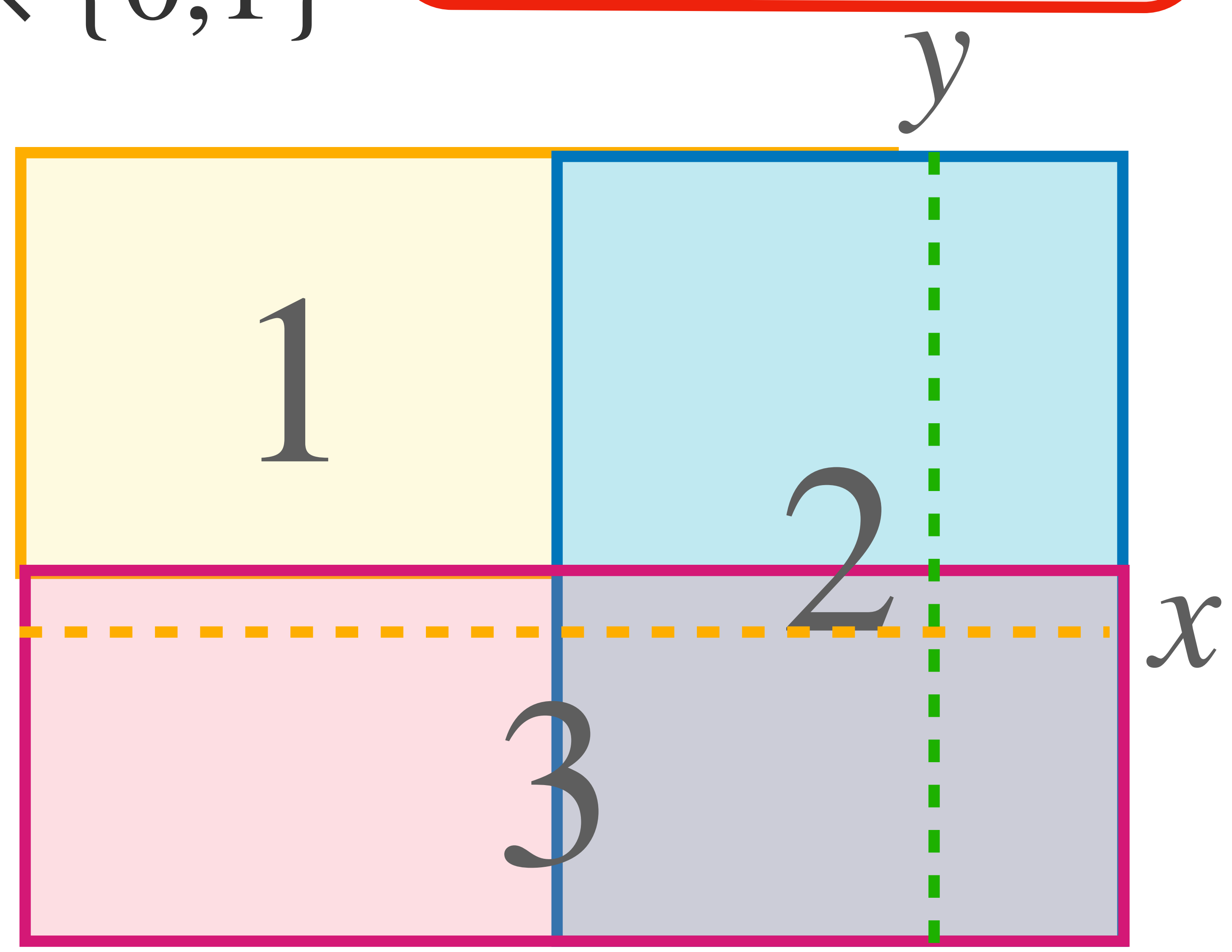

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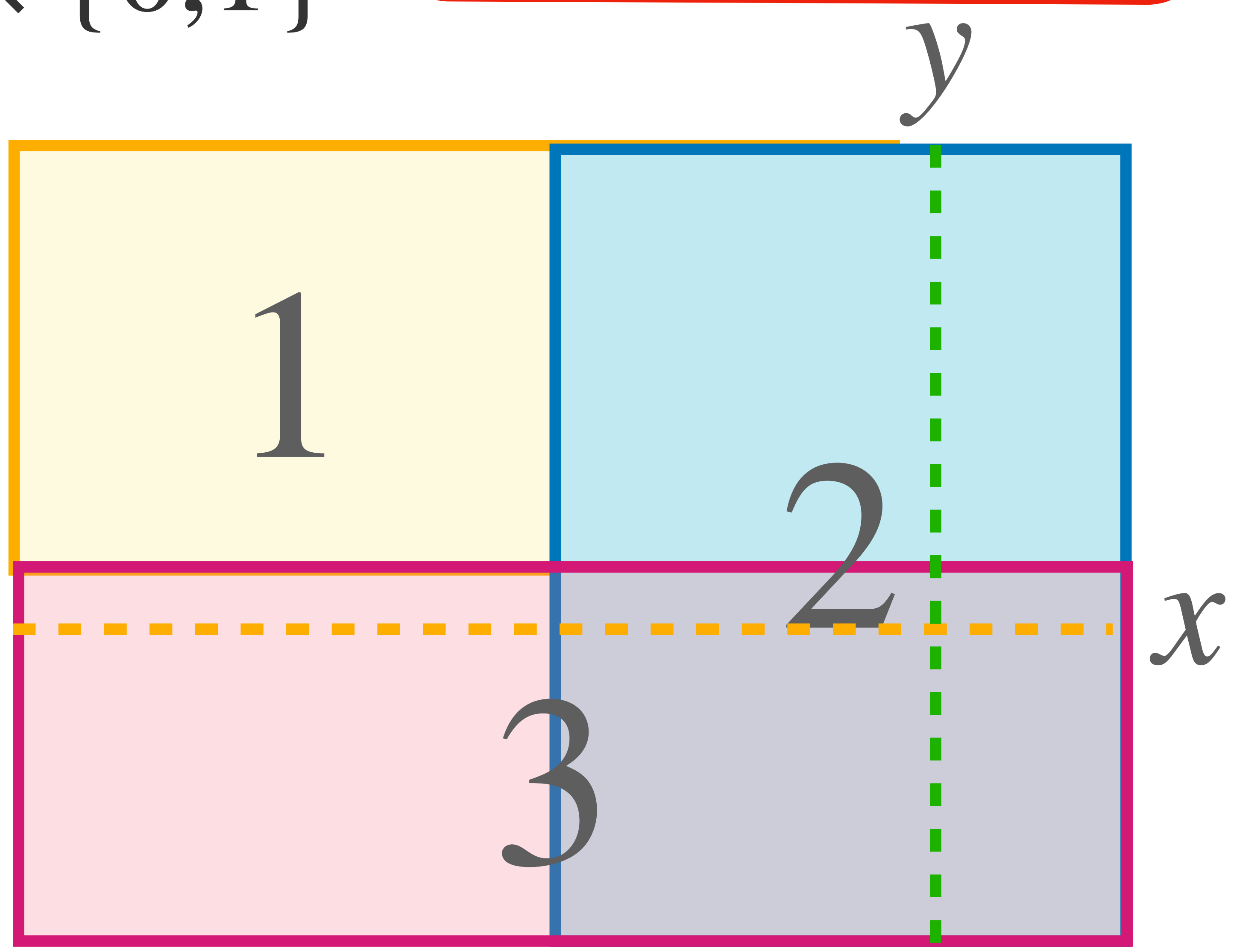

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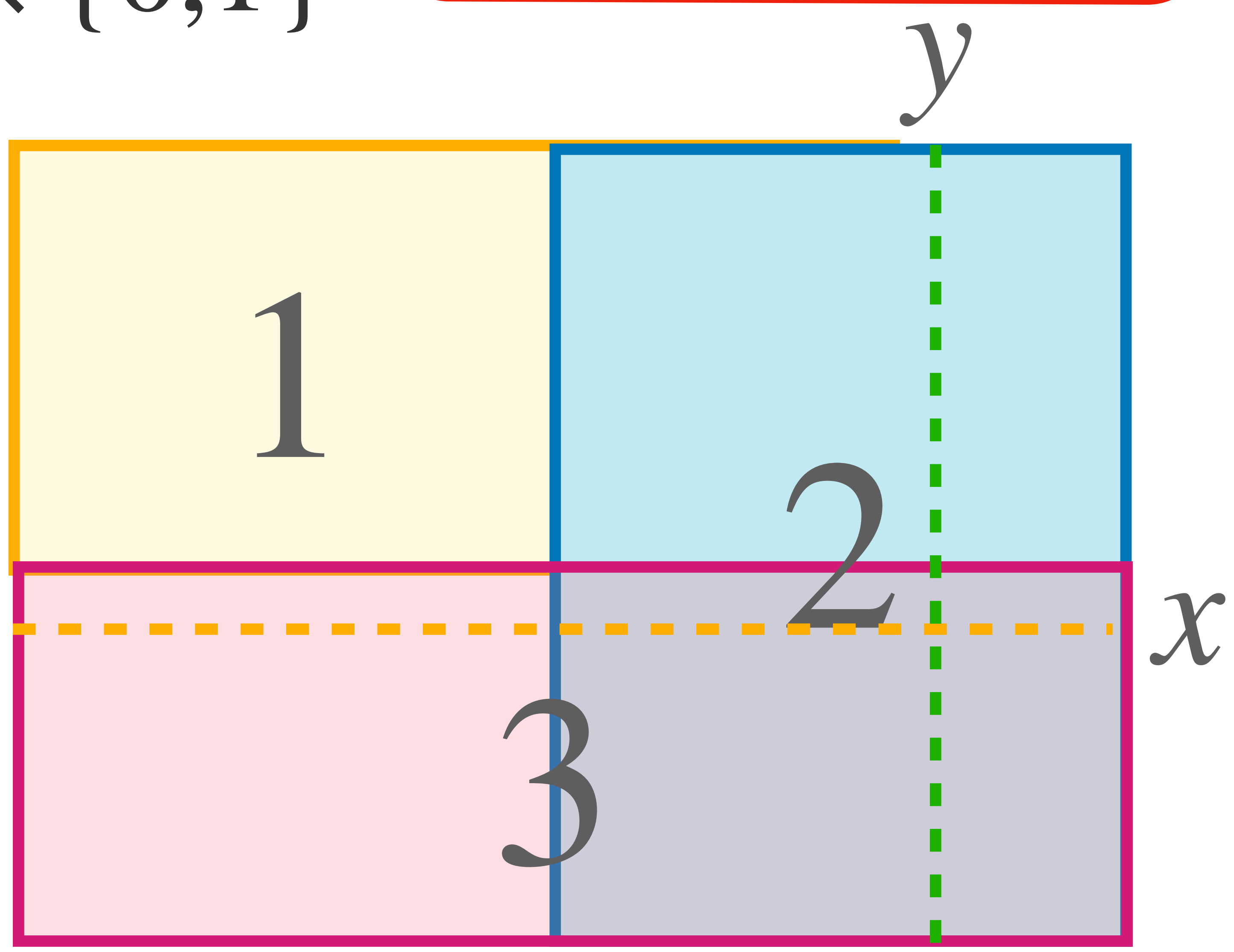

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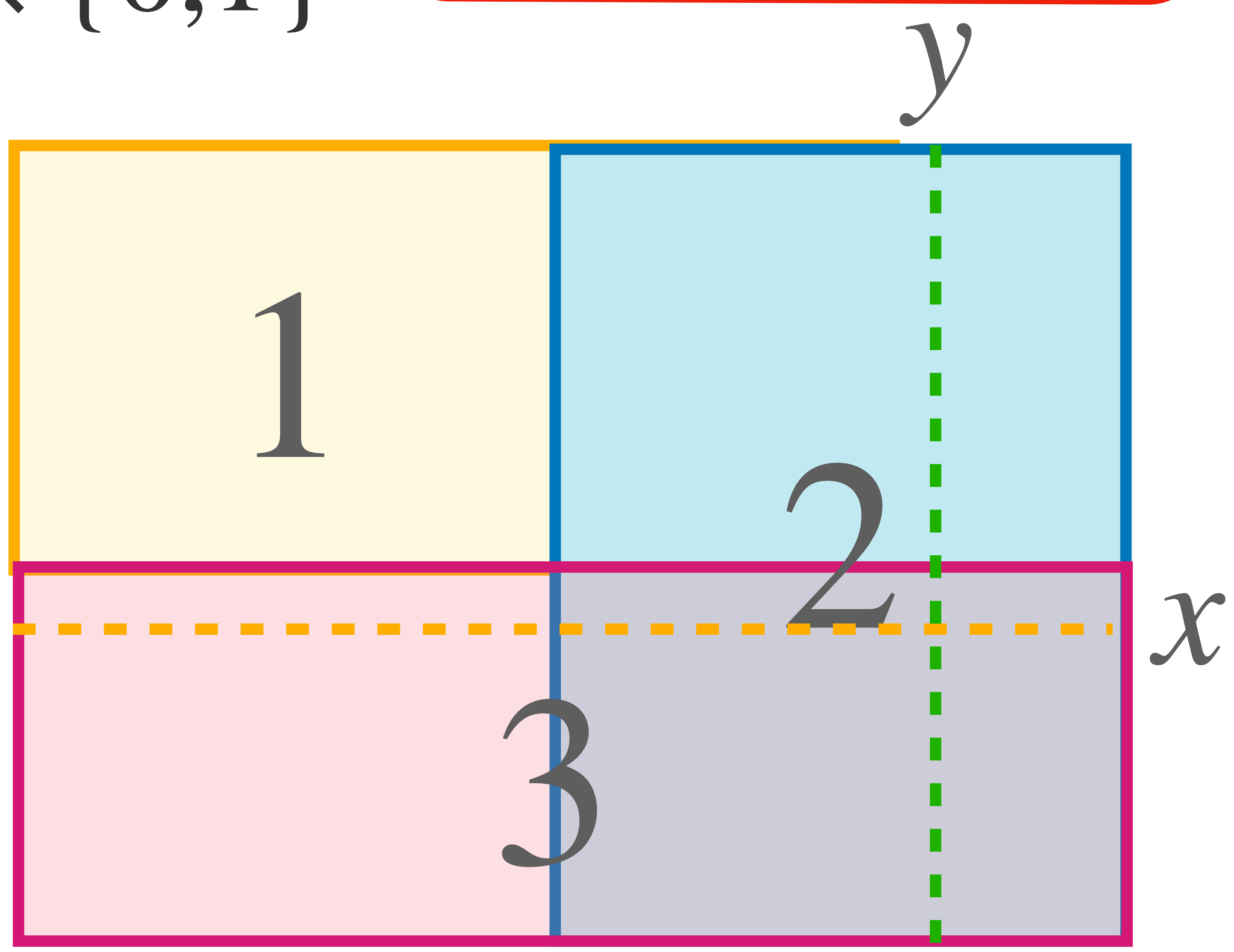

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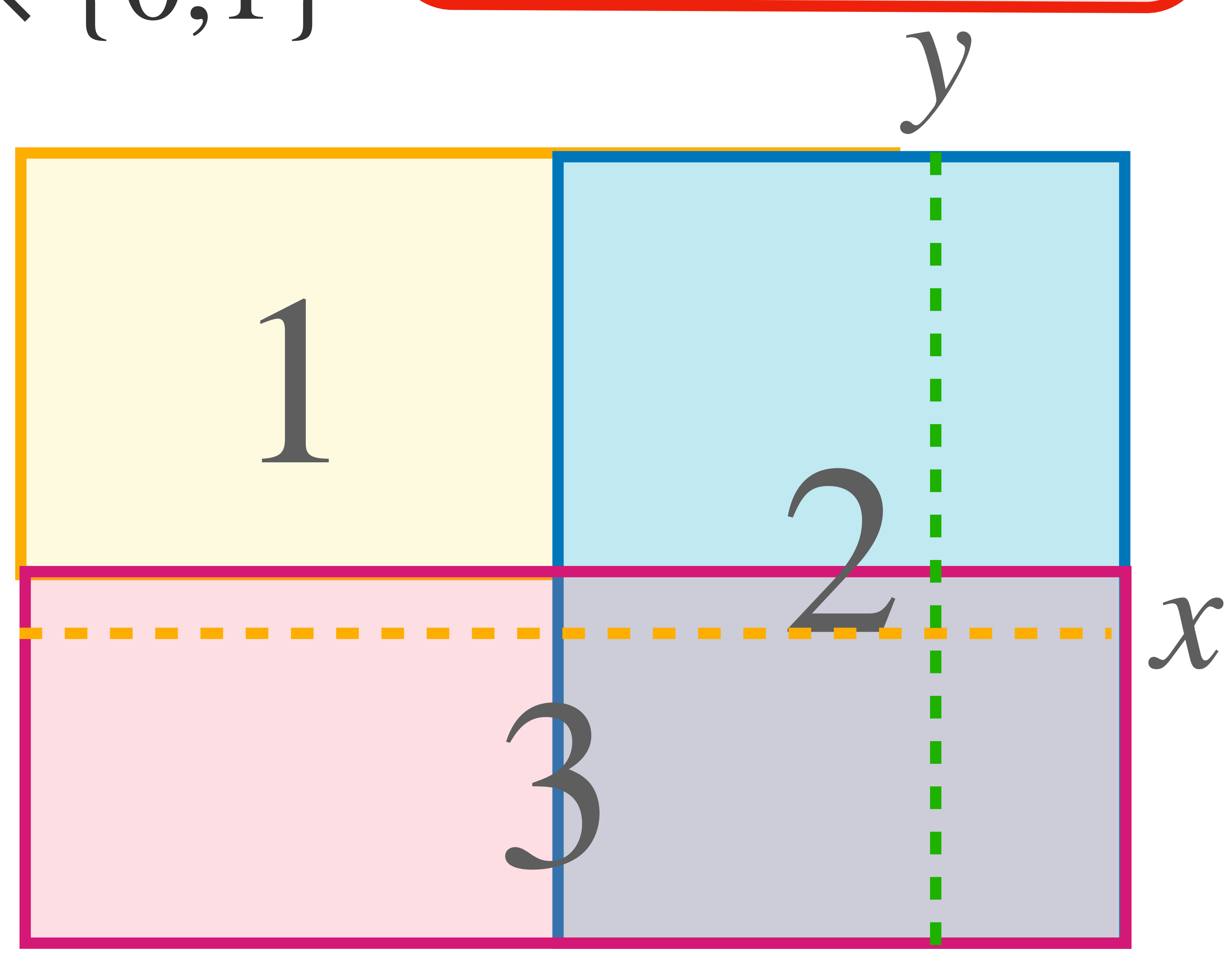

Tale of Two Search Problems

Thm: For any unsatisfiable CNF formula $F = C_1 \wedge \ldots \wedge C_m$ and any partition $X \times Y$ of $[n]$ there is a monotone function $f : \{0,1\}^m \to \{0,1\}$ and mappings $A : \{0,1\}^X \to f^{-1}(1),\ R : \{0,1\}^Y \to f^{-1}(0)$ such that

$$(x, y, i) \in \text{search}_F^{X,Y} \iff (A(x), R(y), i) \in \text{mKW}_f$$

Proof: Specify $A(x)$ of accepting inputs and $R(y)$ of rejecting inputs

The function: Let f be a monotone function which accepts $A(x)$ for all $x \in \{0,1\}^X$ and rejects $R(y)$ for all $y \in \{0,1\}^Y$

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[HP17] **unsat certificate**: $\text{cert}_F^{X,Y}(\alpha) = \begin{cases} 0 & \text{if } C_i^X : \alpha_i = 1 \text{ is satisfiable} \\ 1 & \text{if } C_i^Y : \alpha_i = 0 \text{ is satisfiable} \end{cases}$

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[FPPR17] proved same theorem using a function containing cert as a projection

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- Communication in expectation = Sherali-Adams, extended formulations

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Split formula: $F(x, y, z) = A(x, z) \wedge B(y, z)$

Where A, B are CNF and all z variables occur positively in A

Monotone Interpolant function: $I_F(\alpha) = \begin{cases} 0 & \text{if } A(x, \alpha) \text{ is satisfiable} \\ 1 & \text{if } B(\alpha, y) \text{ is satisfiable} \end{cases}$

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Then $I_{F_{\text{split}}}$ is $\text{cert}_F^{X,Y}$!

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CC_1 proofs are equivalent to cc-protocols computing $\text{search}_F^{X,Y}$