Integer Programming and IP Proof Systems Part 2

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University of California, San Diego

Recap of Last Time

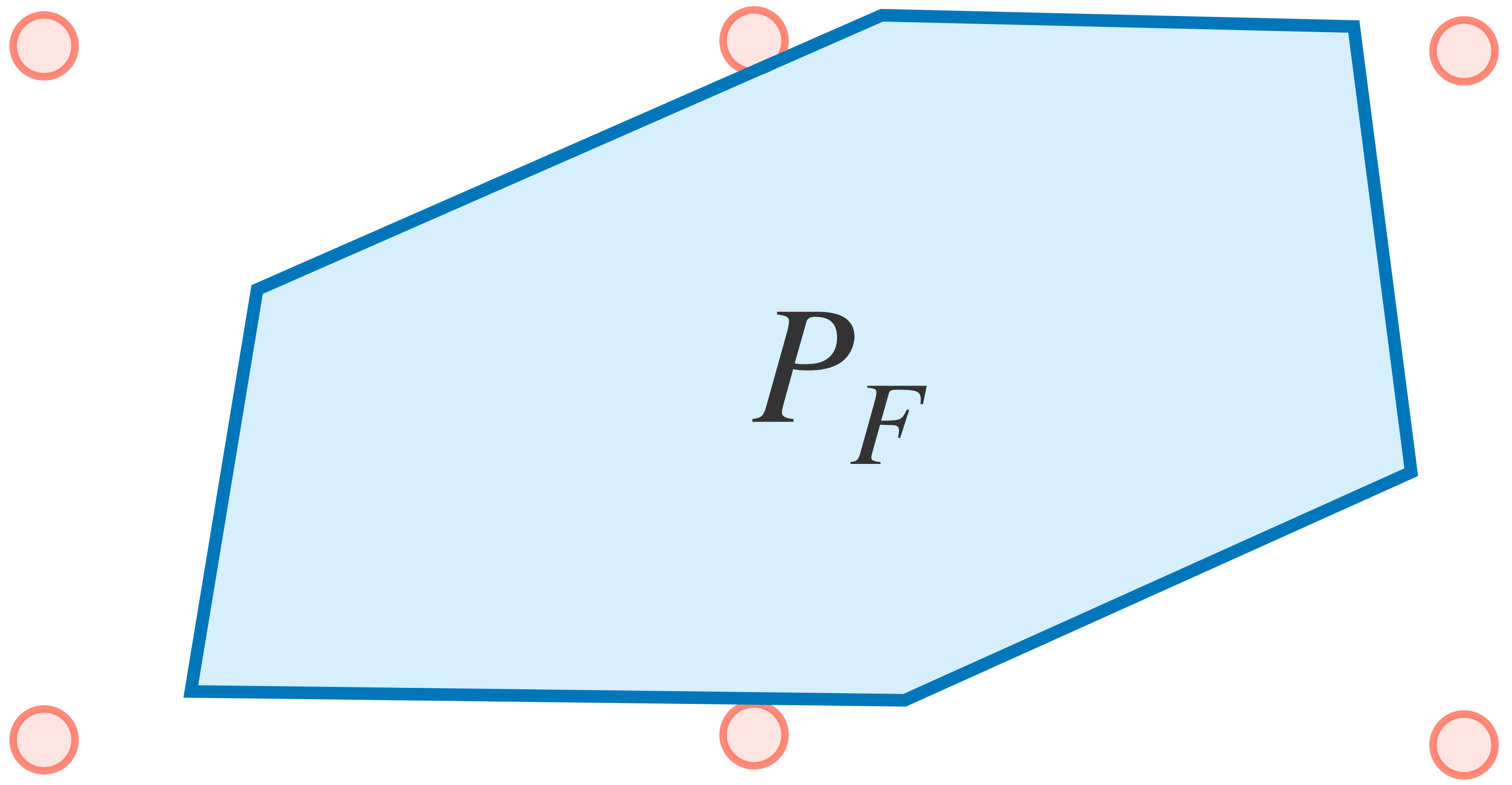
Encode unsatisfiable CNF formulas F as polytopes (systems of linear inequalities) P_F with no integer points

$$F = C_1 \wedge \ldots \wedge C_m \implies P_F = \{x : Ax \geq b\}$$

For each $C_i = \bigvee_{i \in I} x_i \vee \bigvee_{j \in J} \neg x_j$

Include in $Ax \geq b$

$$\sum_{i \in I} x_i + \sum_{j \in J} (1 - x_j) \geq 1 \quad \text{and} \quad x_i \geq 0, -x_i \geq -1$$



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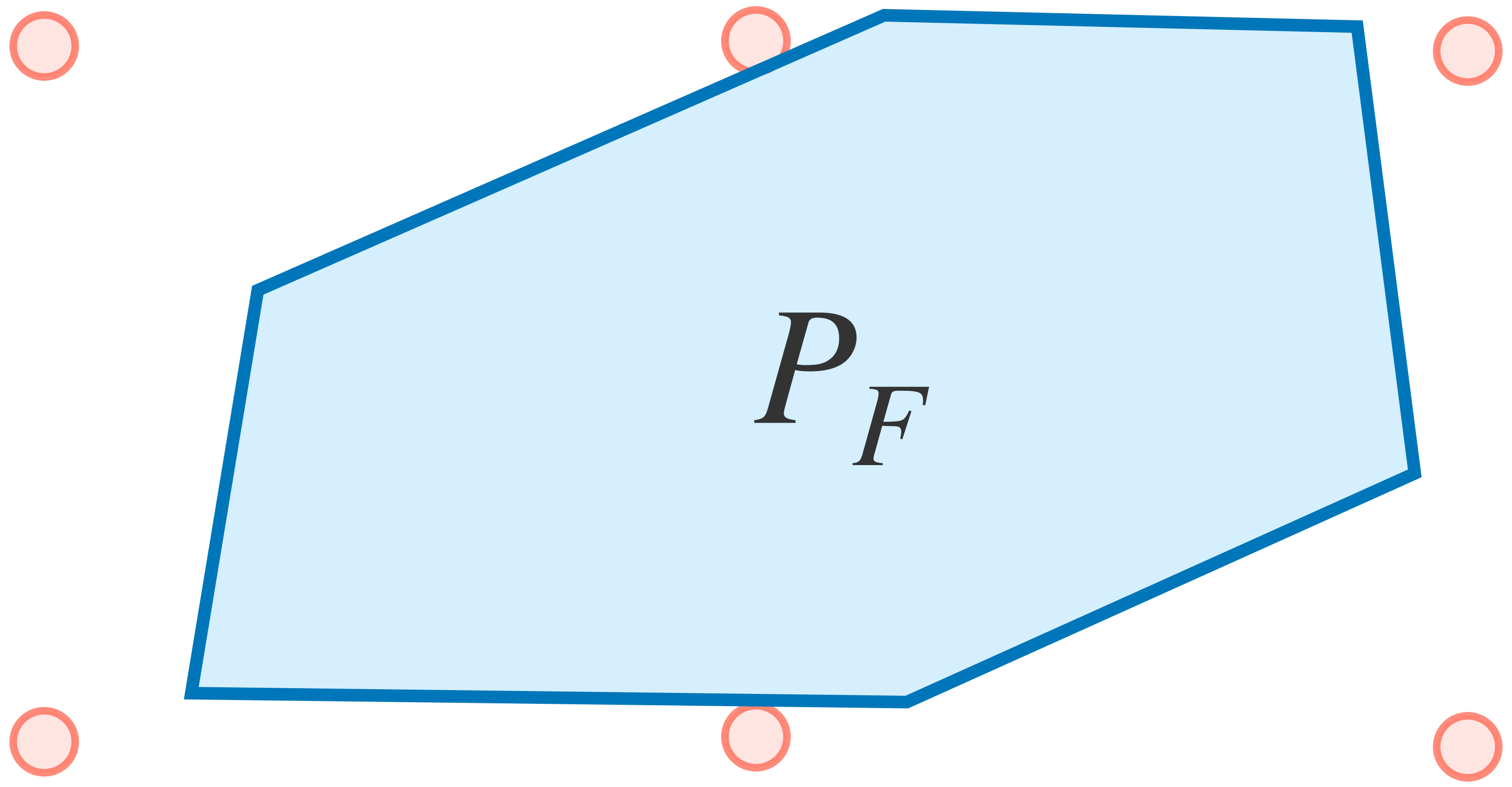

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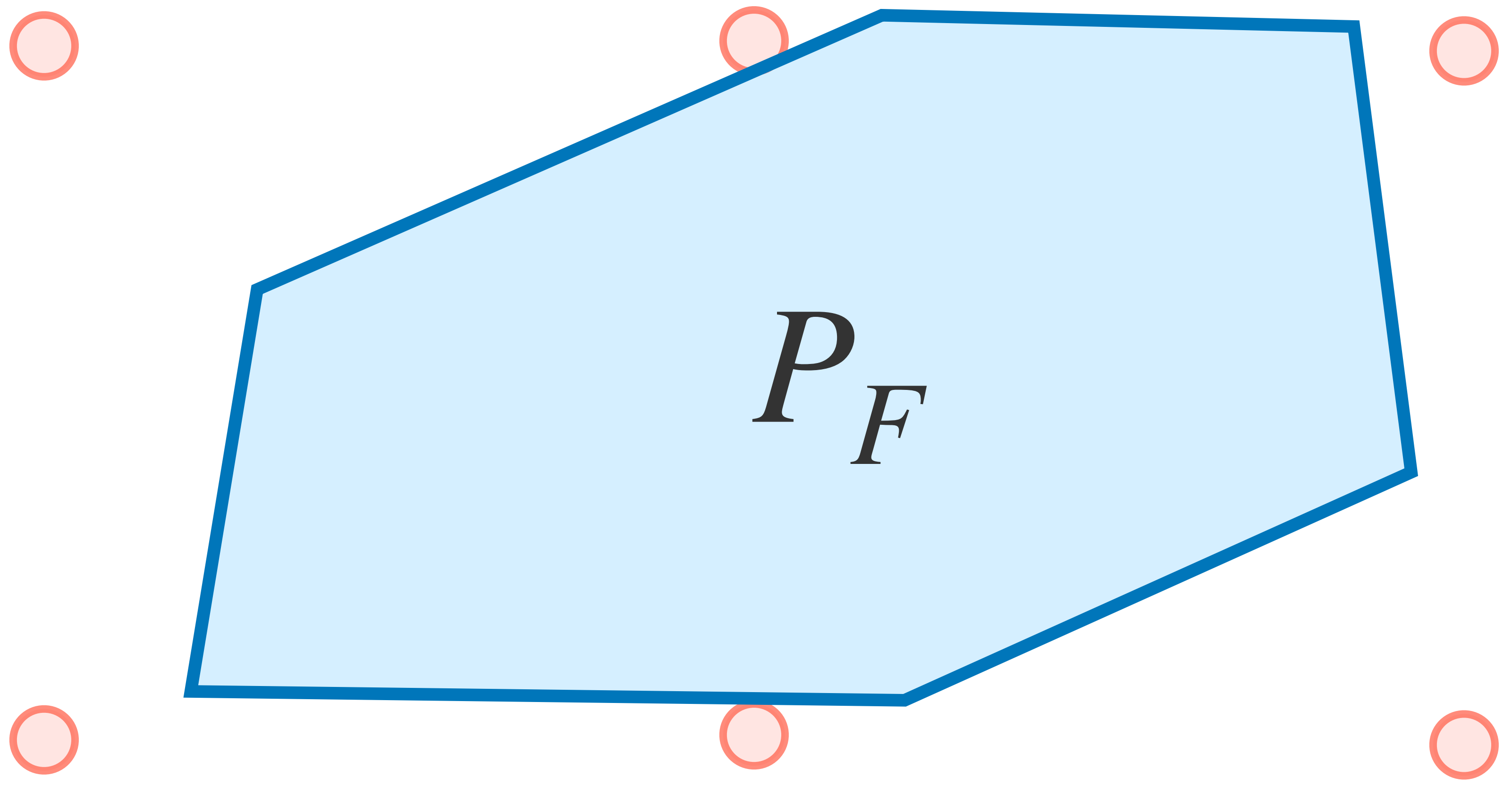

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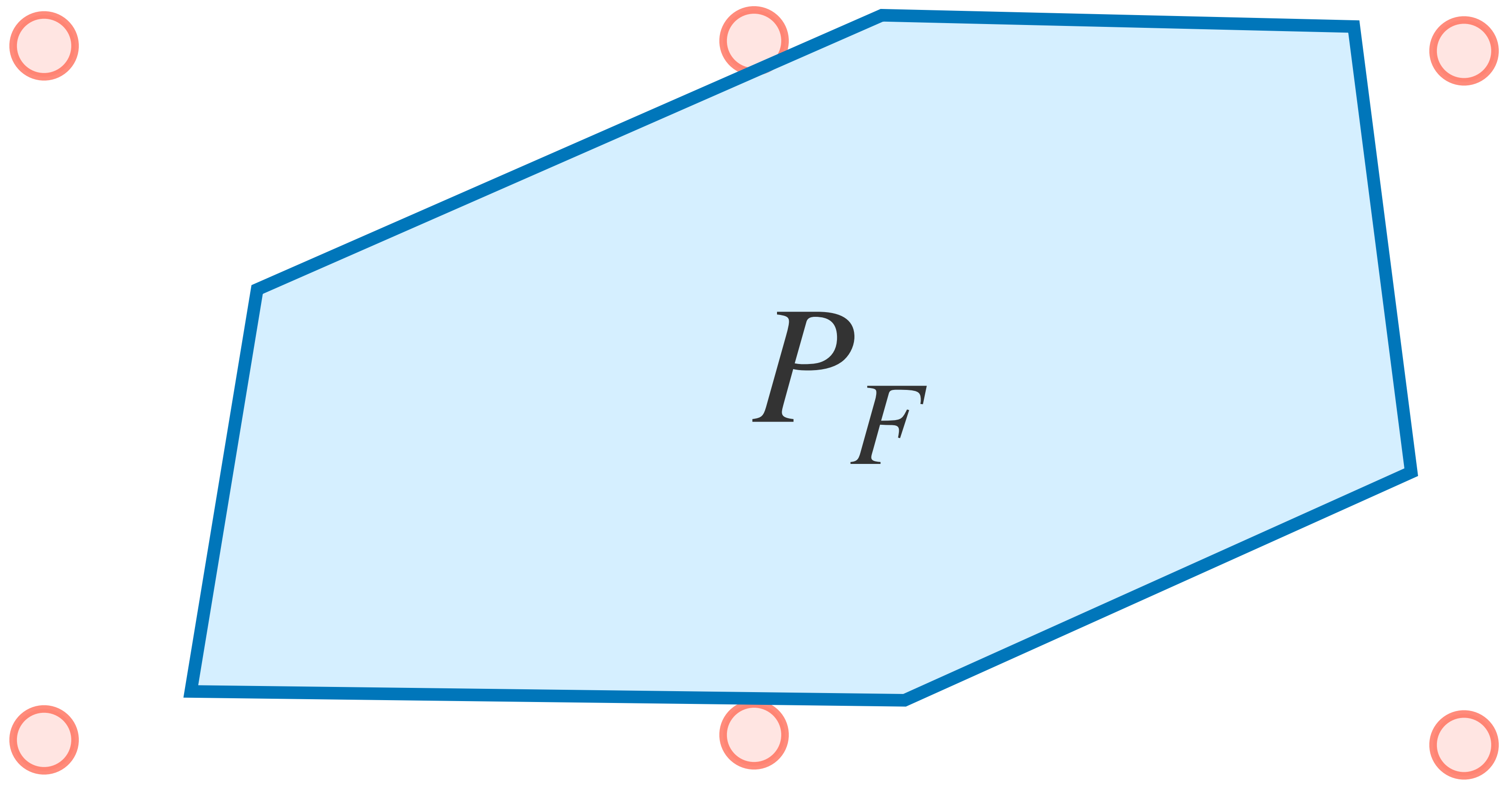
$\rightarrow$ Stabbing Planes — captures branch-and-cut



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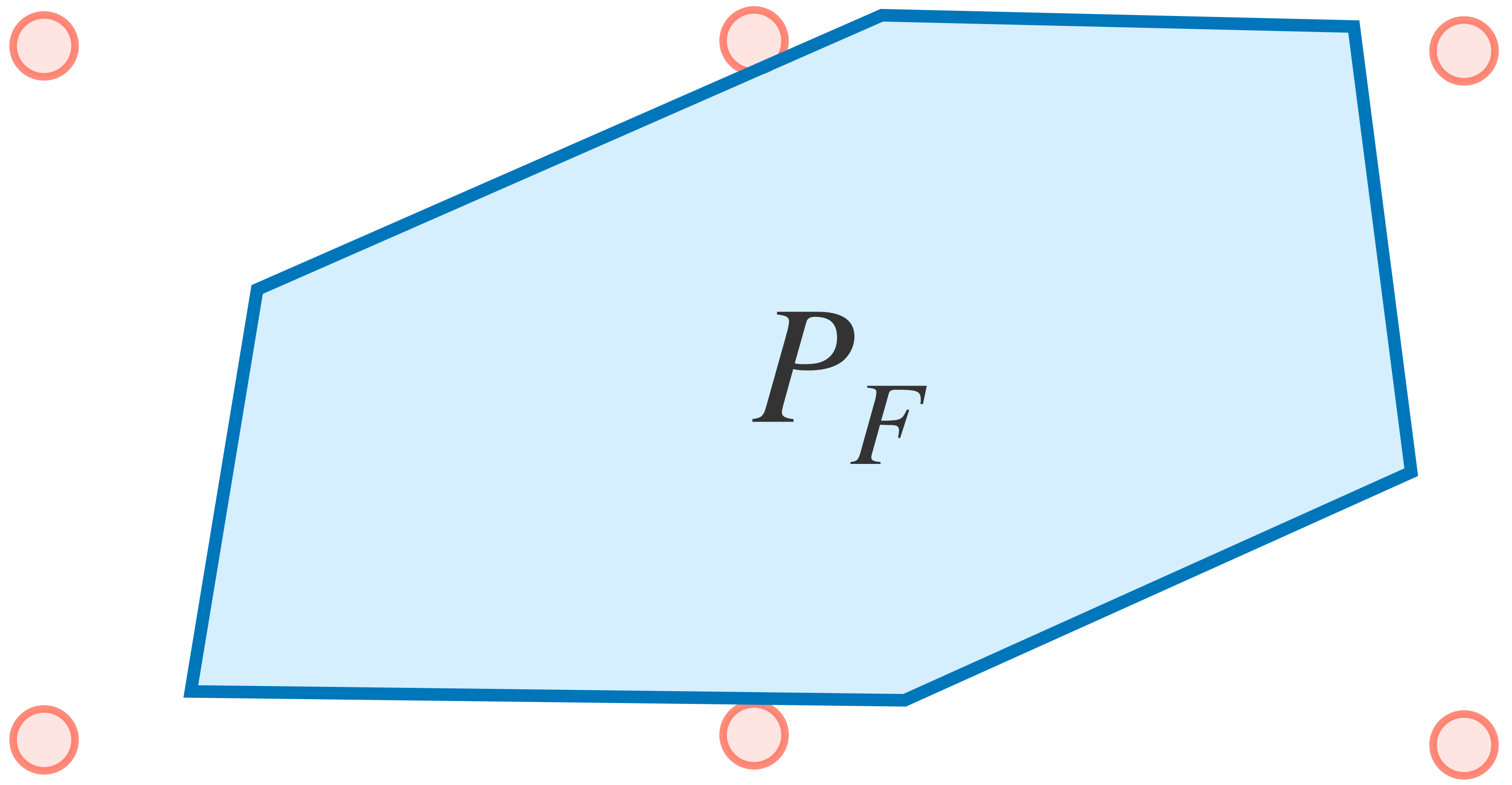

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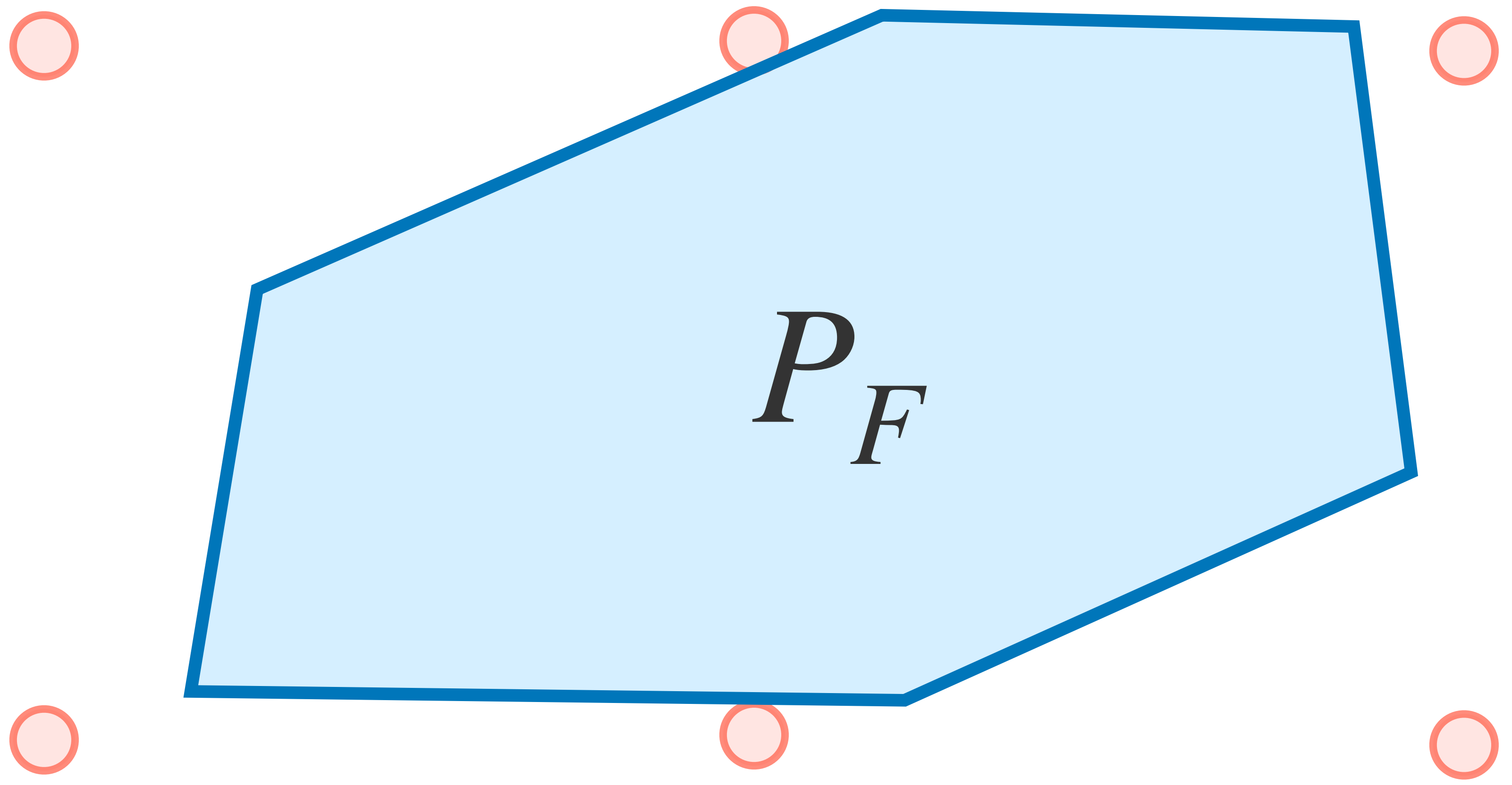

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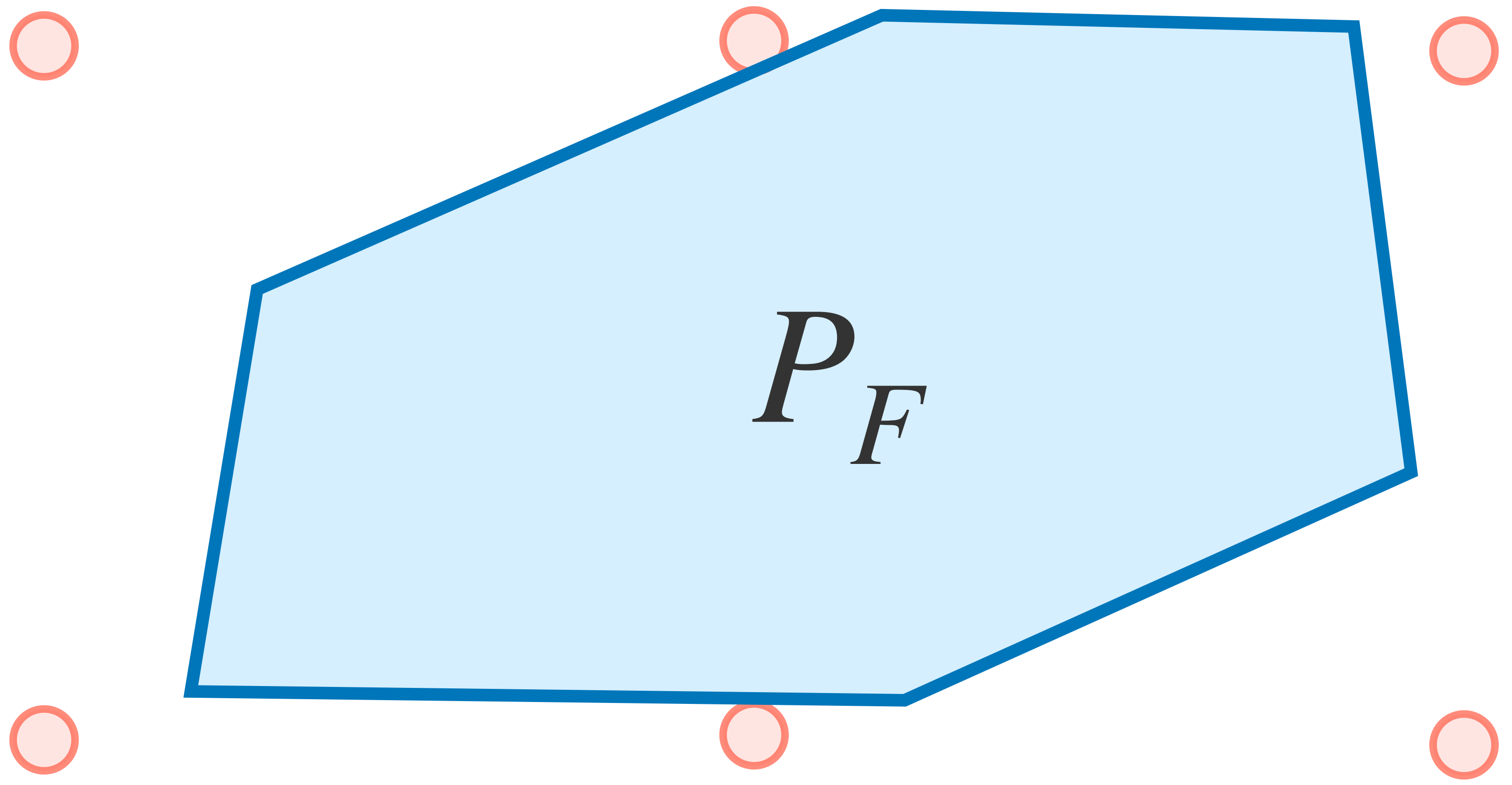

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Any Stabbing Planes proof with coefficients at most $2^{\text{polylog } n}$ (SP*) can be translated into Cutting Planes with a quasi-polynomial blow-up in the size.

$\implies$ Can prove bounds on branch-and-cut by proving bounds on Cutting Planes

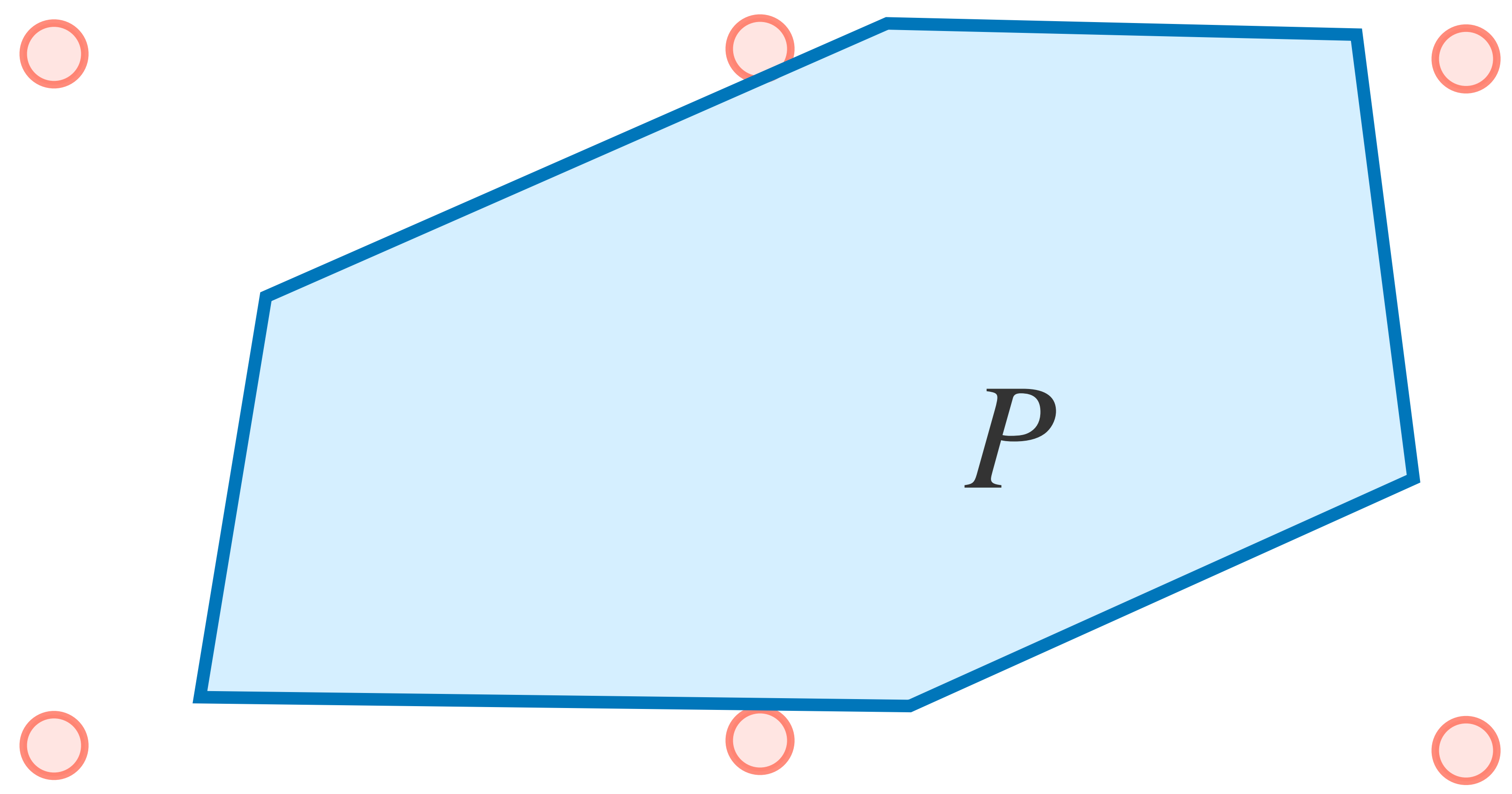
Today

Lower bounds on the size of Cutting Planes proofs!

Let’s recall Cutting Planes…

Cutting Planes Proofs

Suppose $Ax \geq b$ has **no integer solutions**



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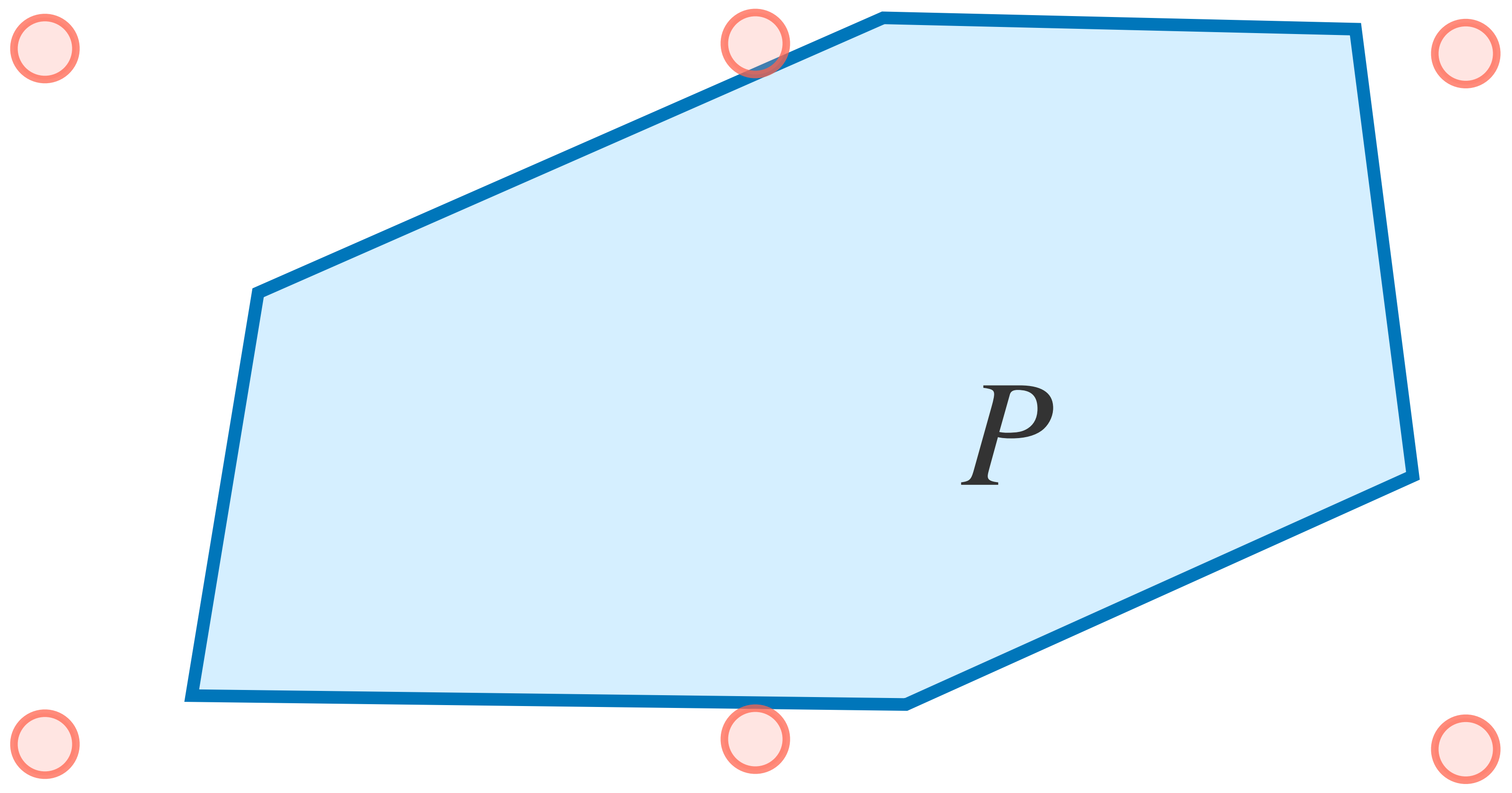

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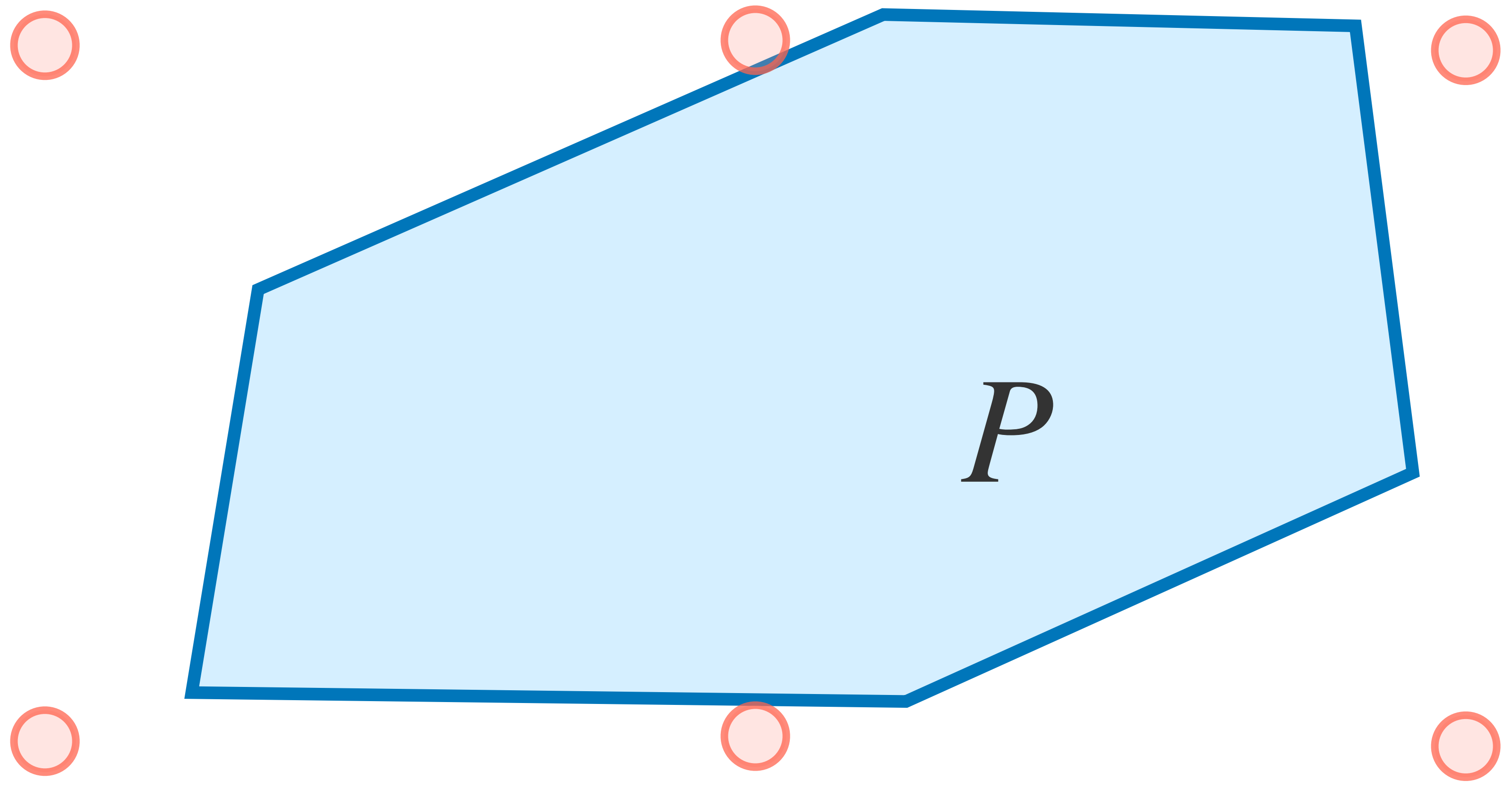

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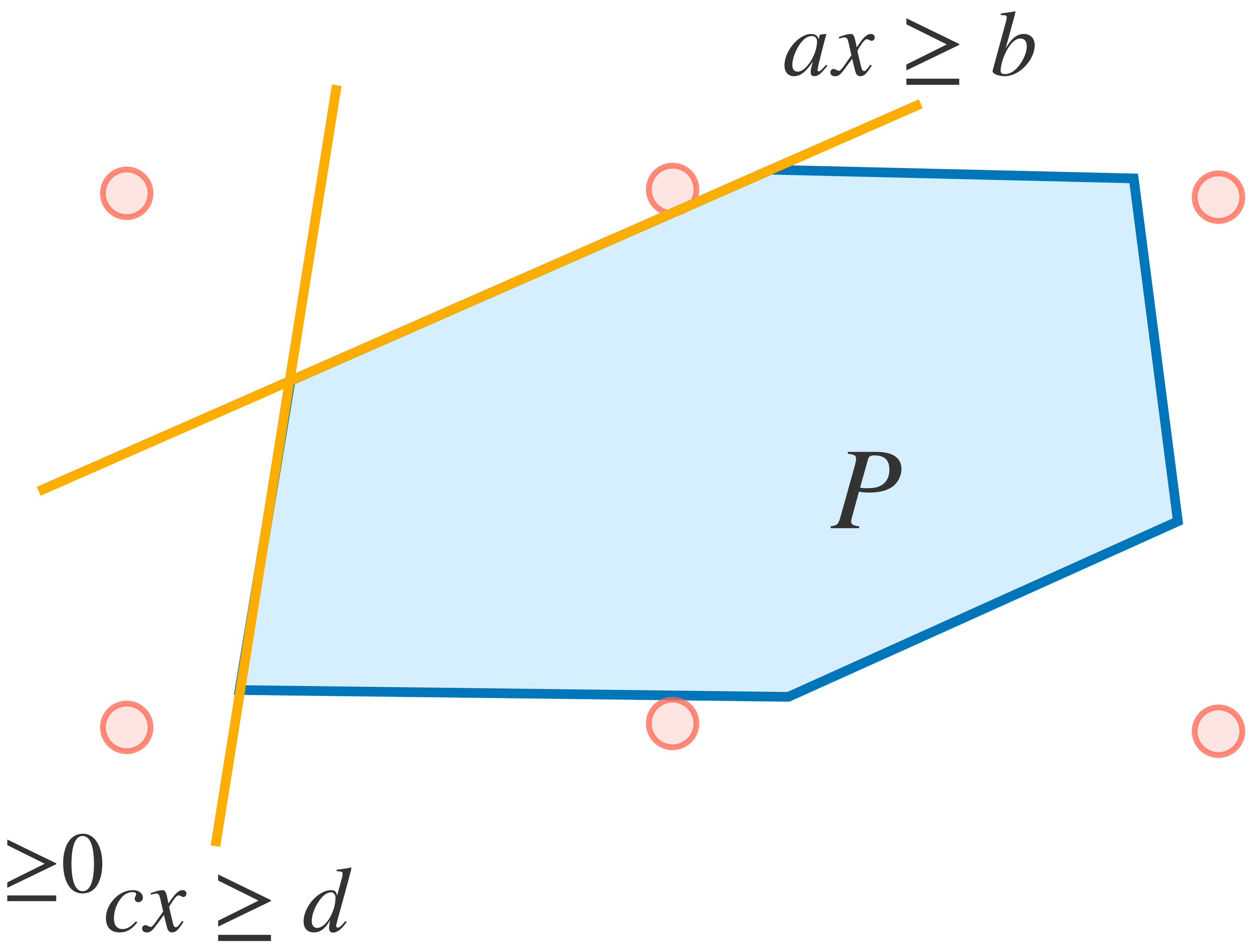

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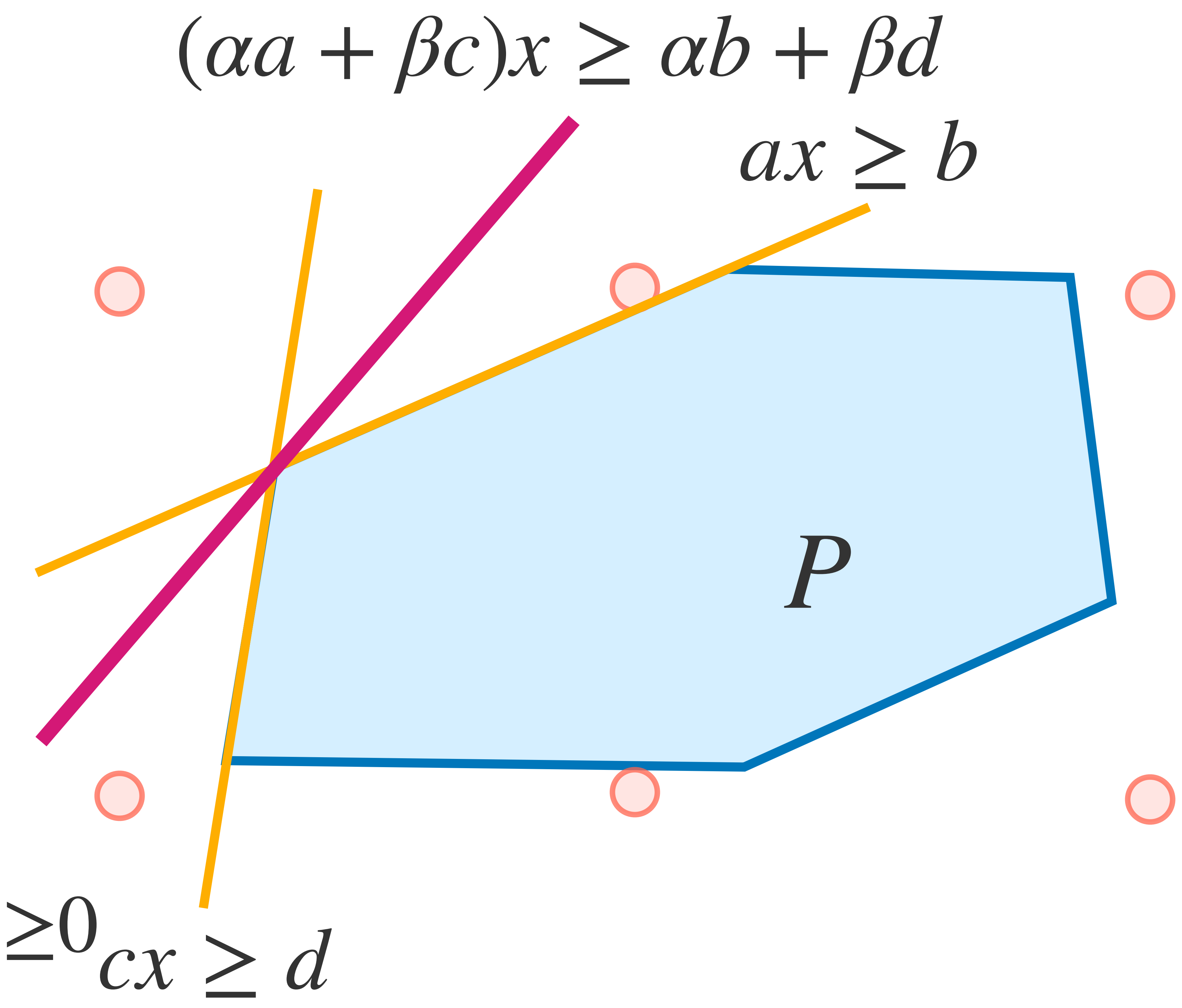

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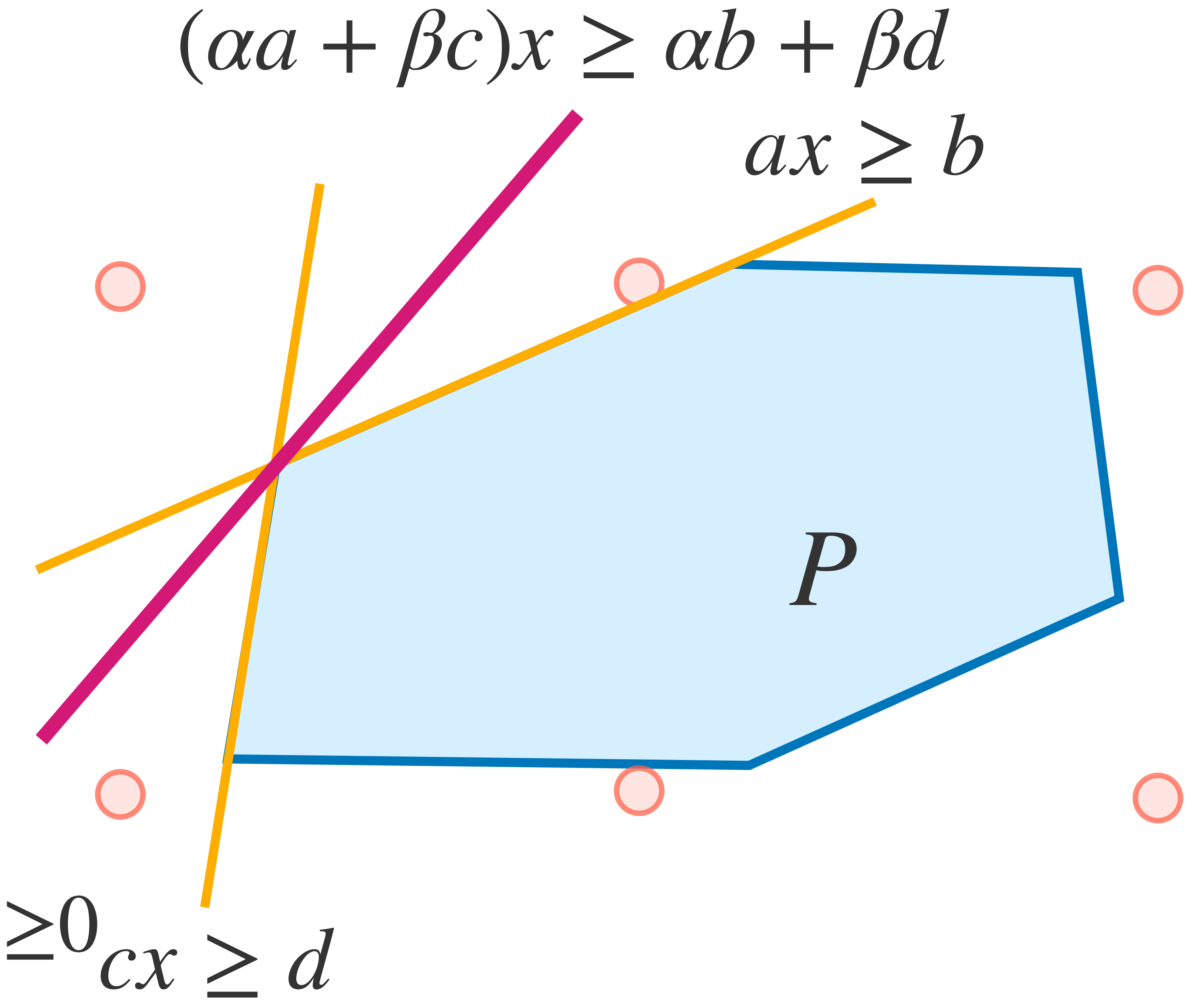

Preserves **all** points in P

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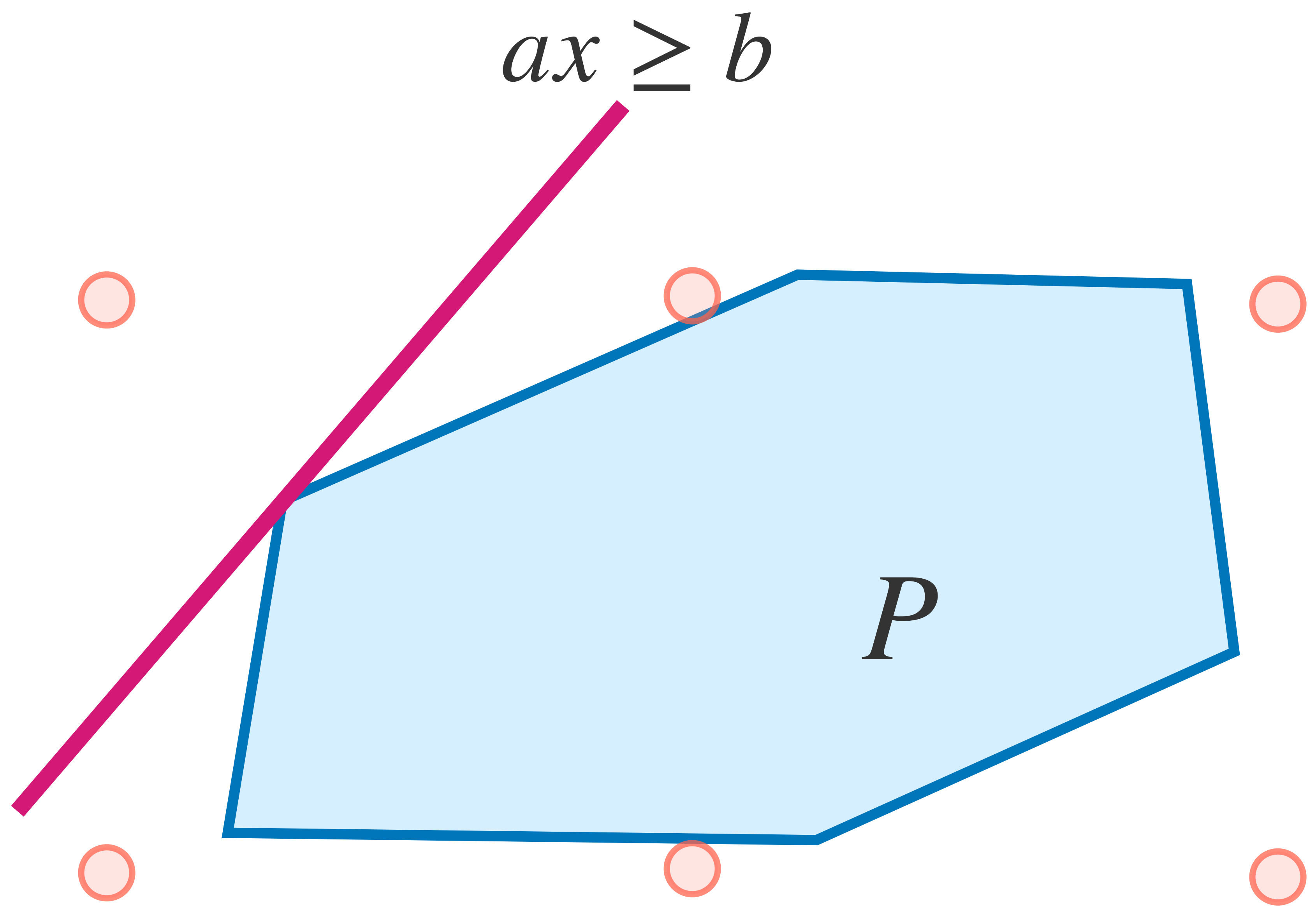

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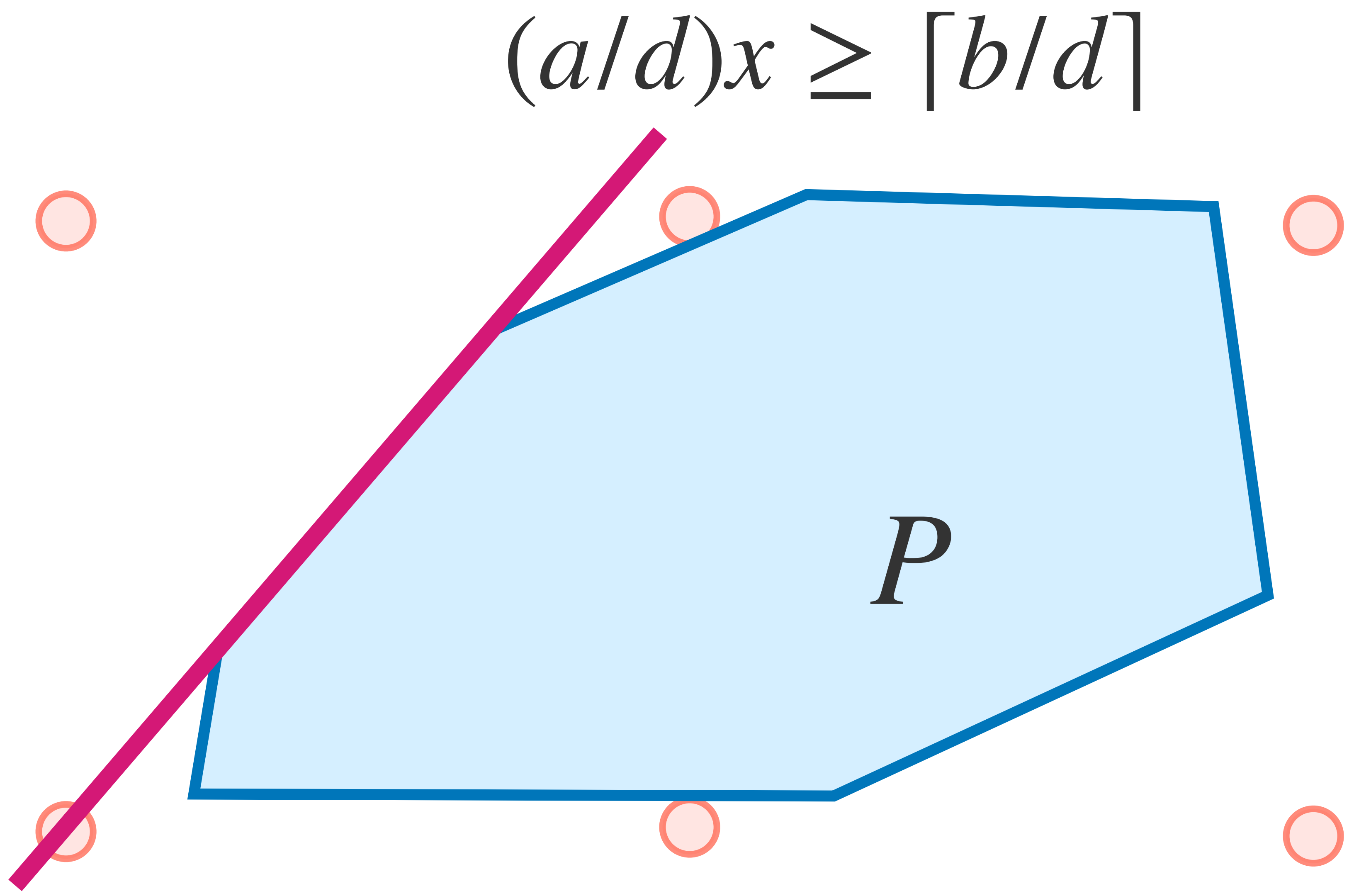

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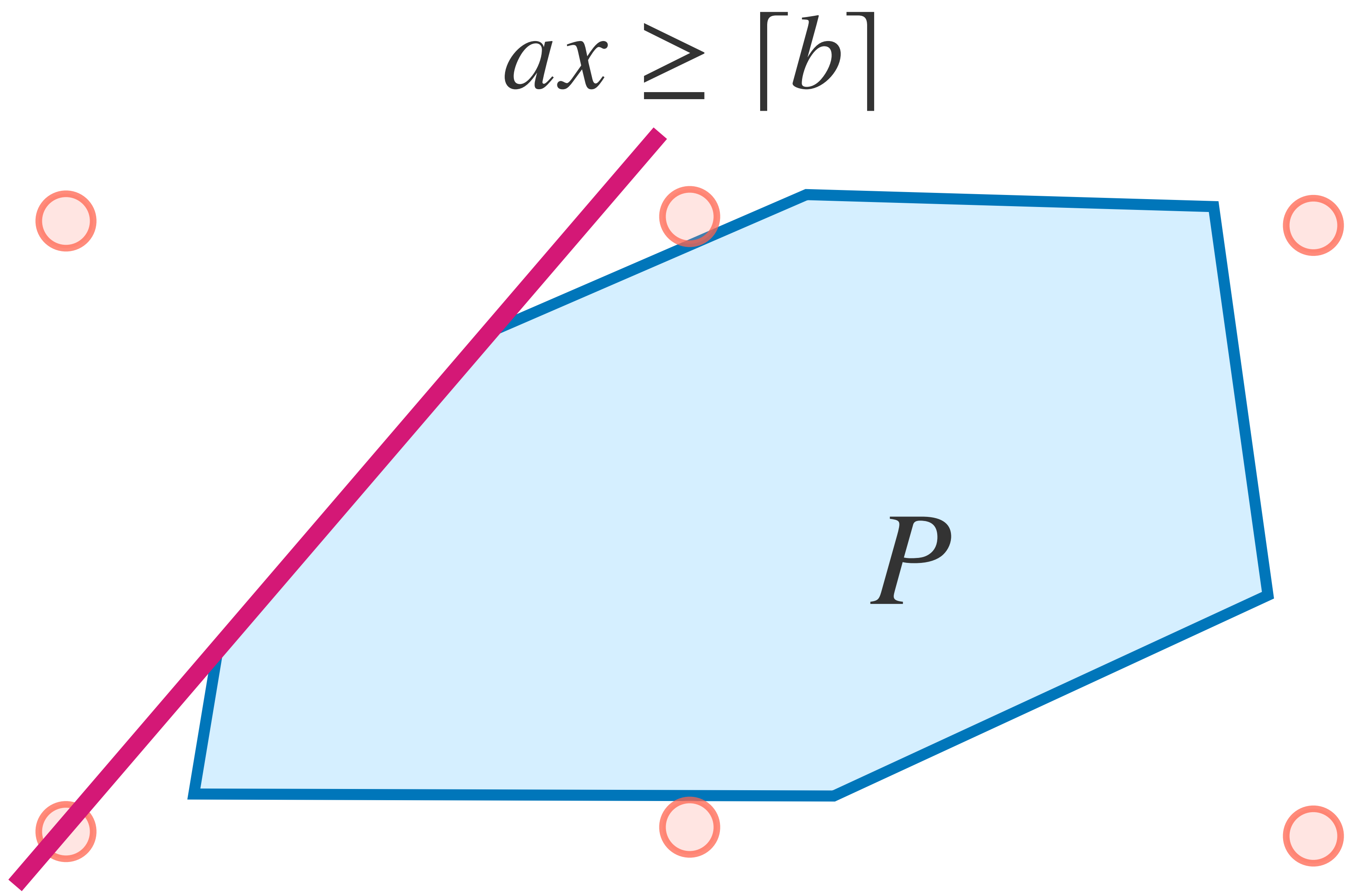

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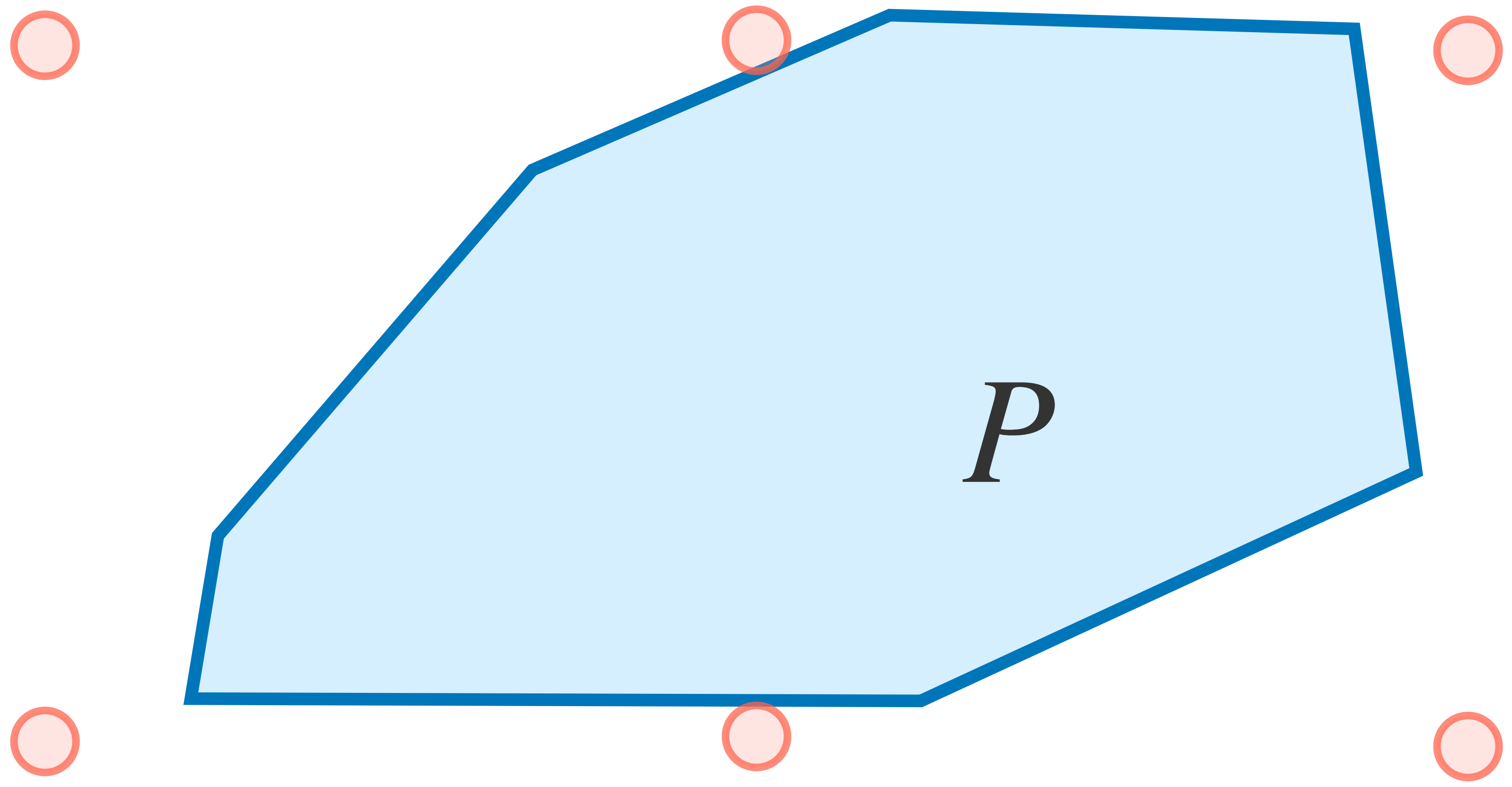

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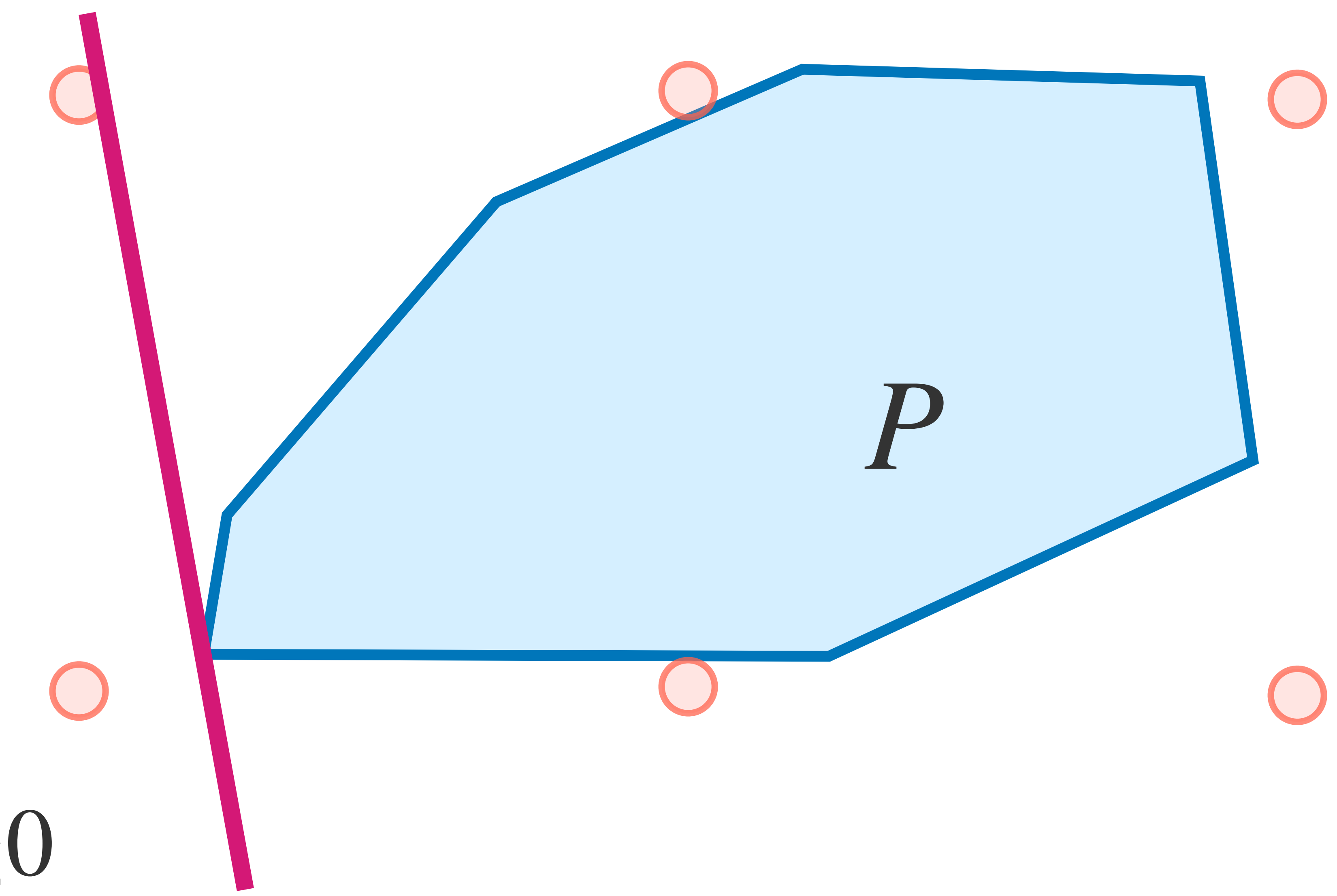

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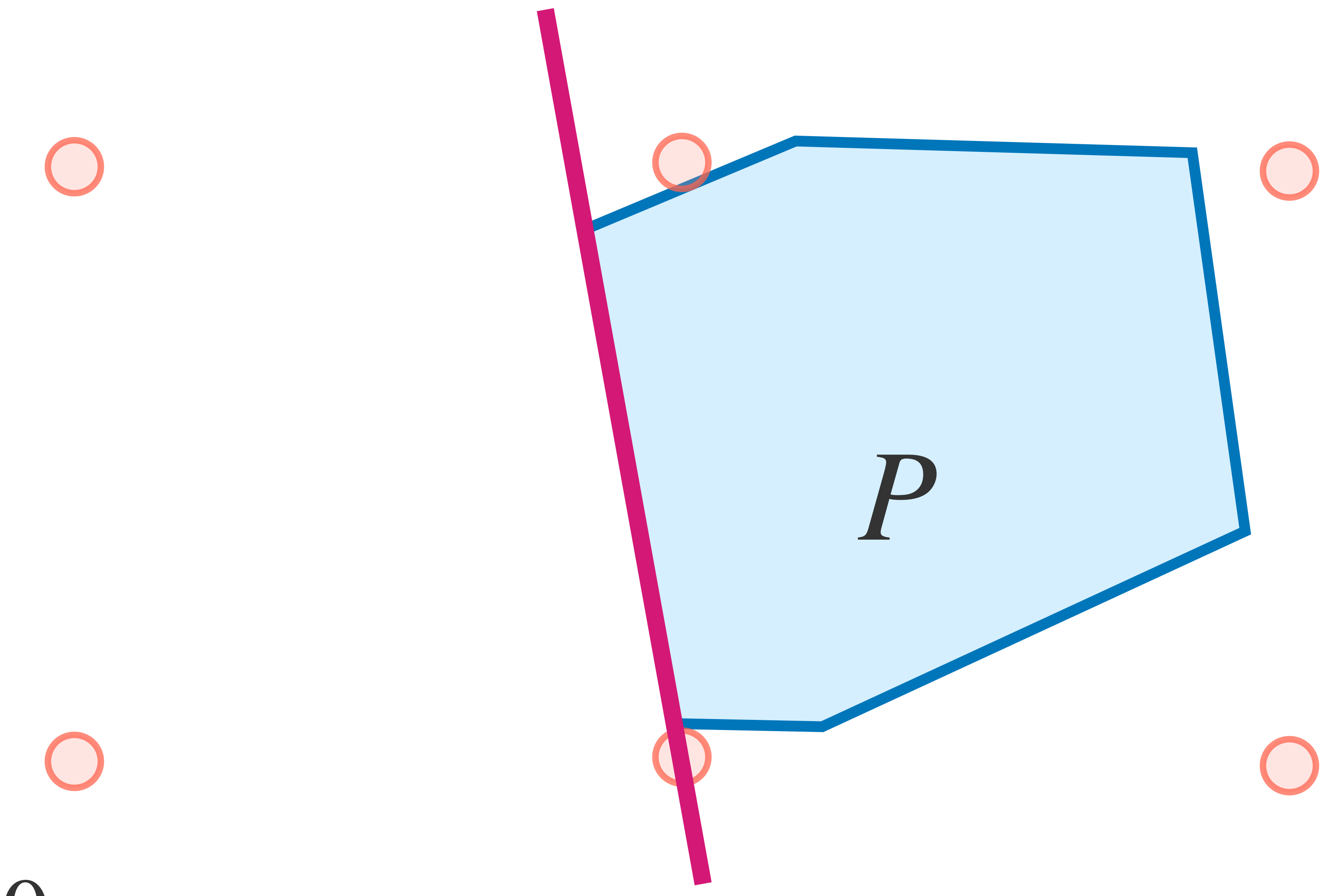

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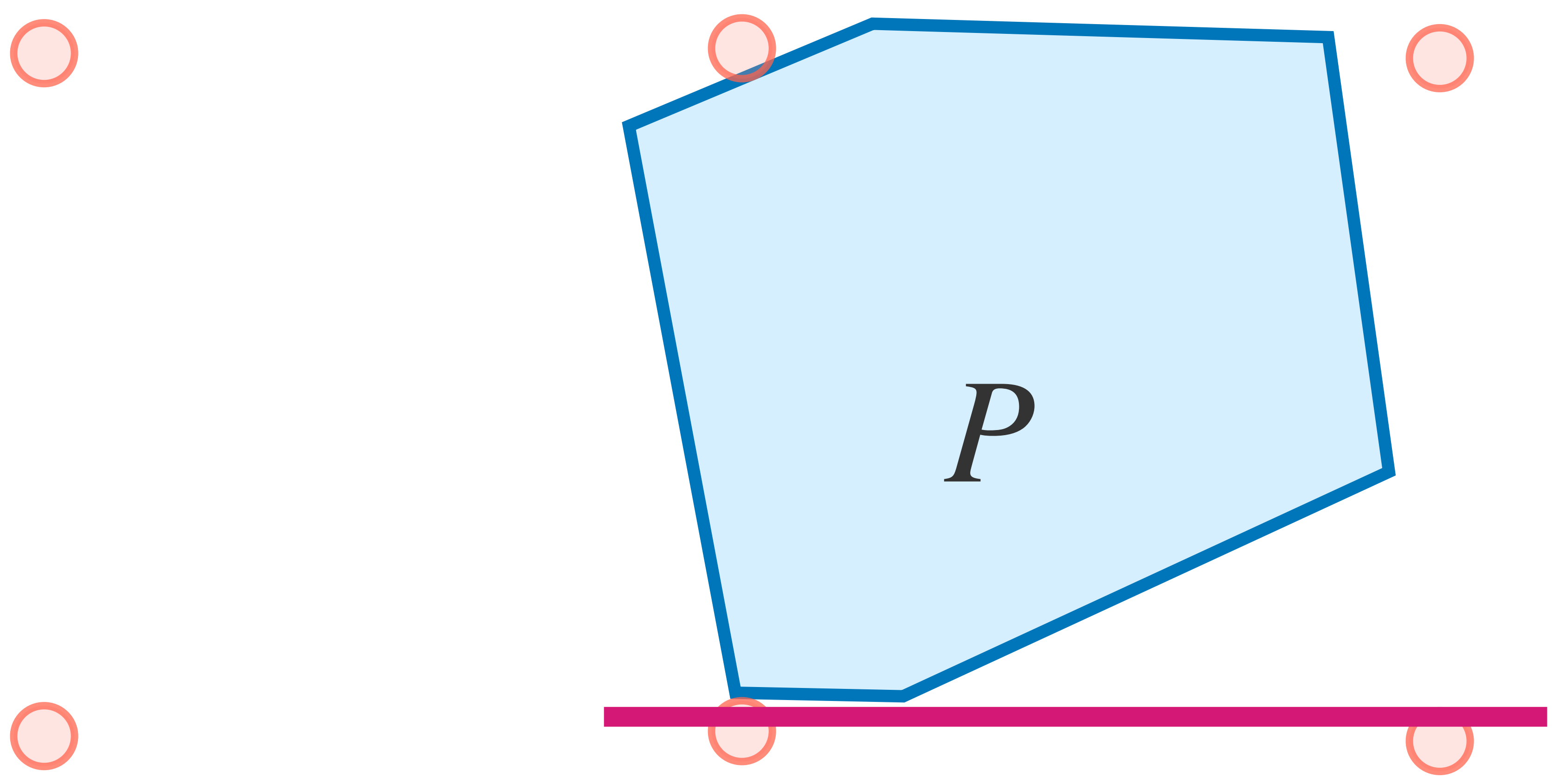

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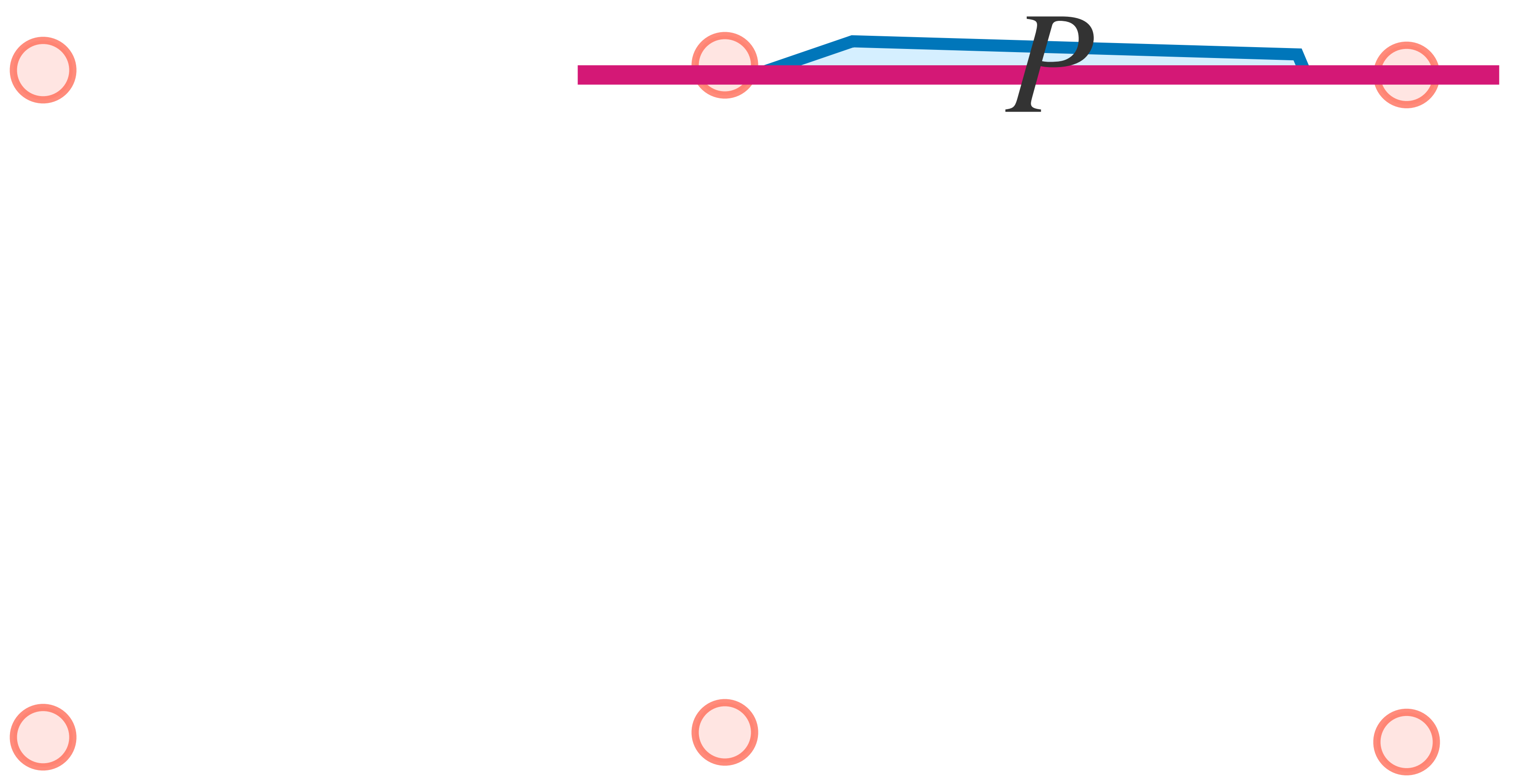

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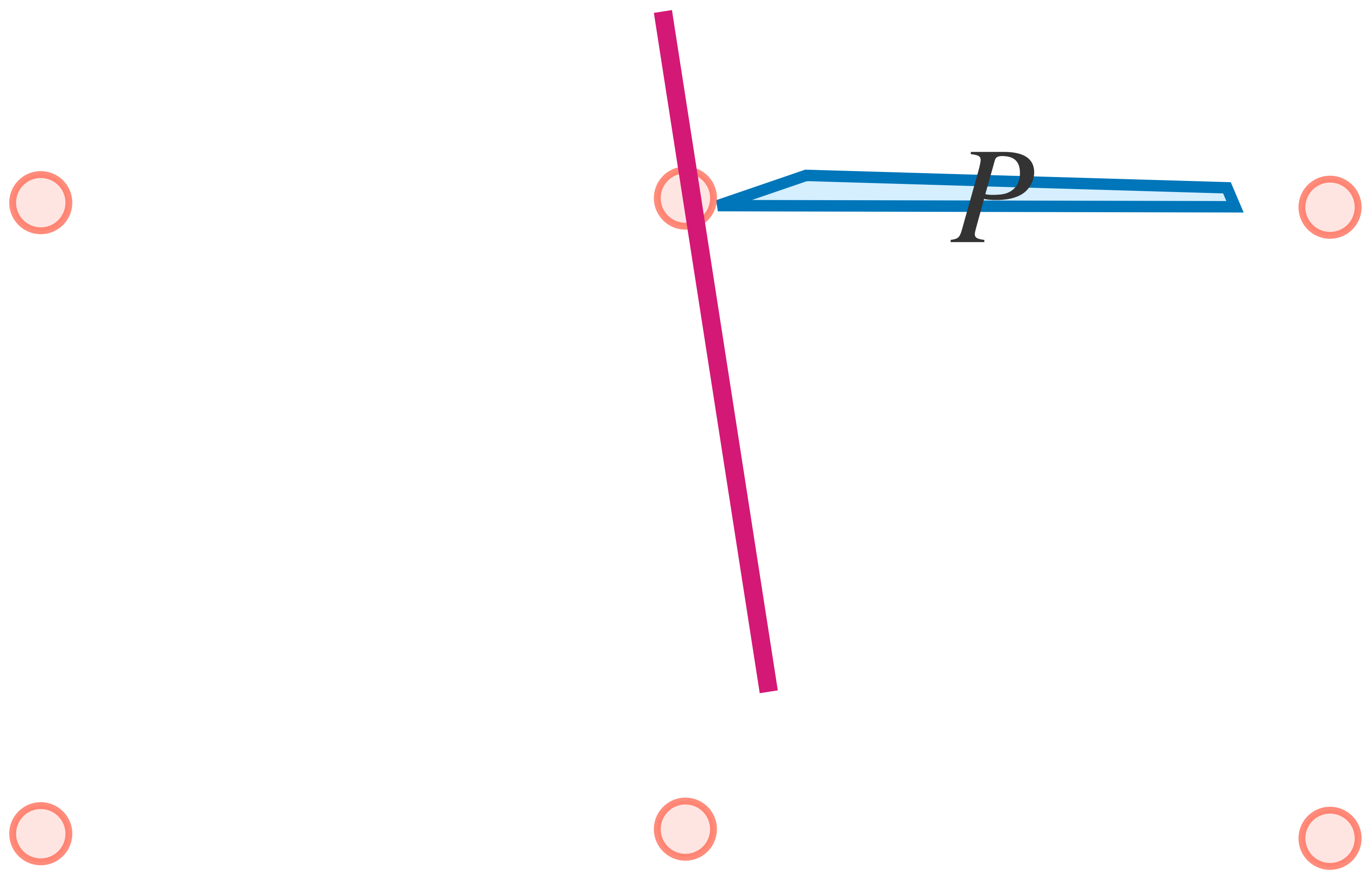

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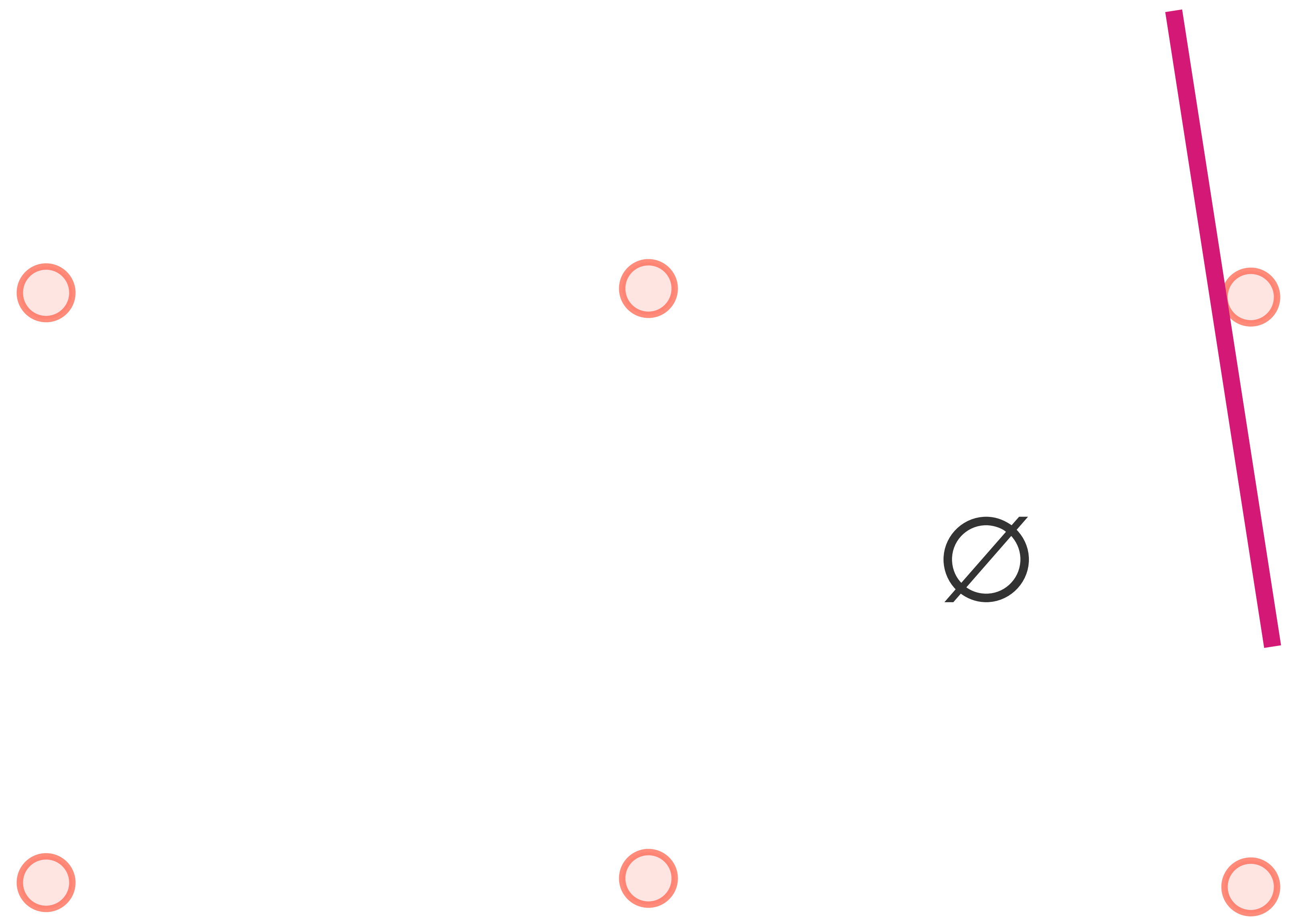

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How? An exciting connection between proofs and circuits!

Monotone Feasible Interpolation

For many (all?) proof systems P it is possible to relate their complexity to the complexity of circuits in some associated model C_P of monotone computation

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Where $f_F : \{0,1\}^m \to \{0,1\}$ is an associated monotone function (defined later)

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In many cases, a converse is possible as well!

Split Formulas

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For simplicity, we'll restrict to the case of **split** formulas:

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Where A, B are CNF and all y variables occur positively in A

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- $z \in \{0,1\}^{n(k-1)}$ defines a $(k-1)$-coloring of $G(y)$:

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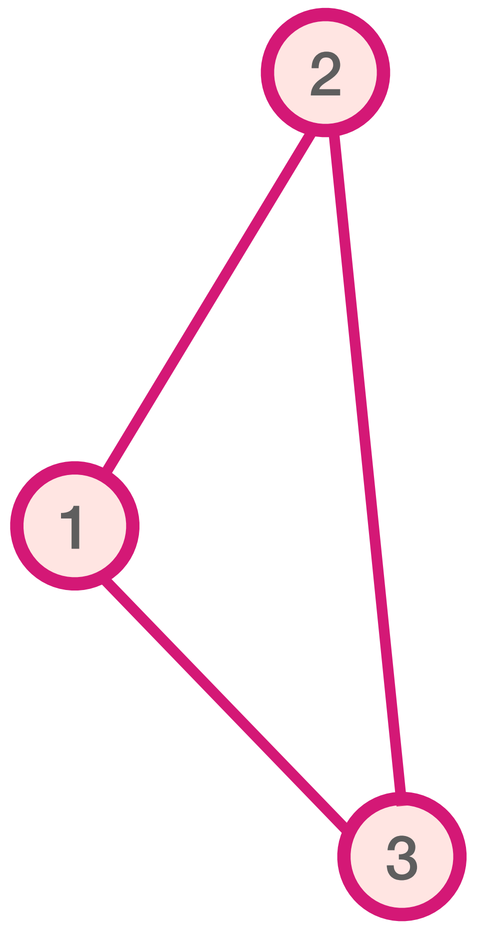

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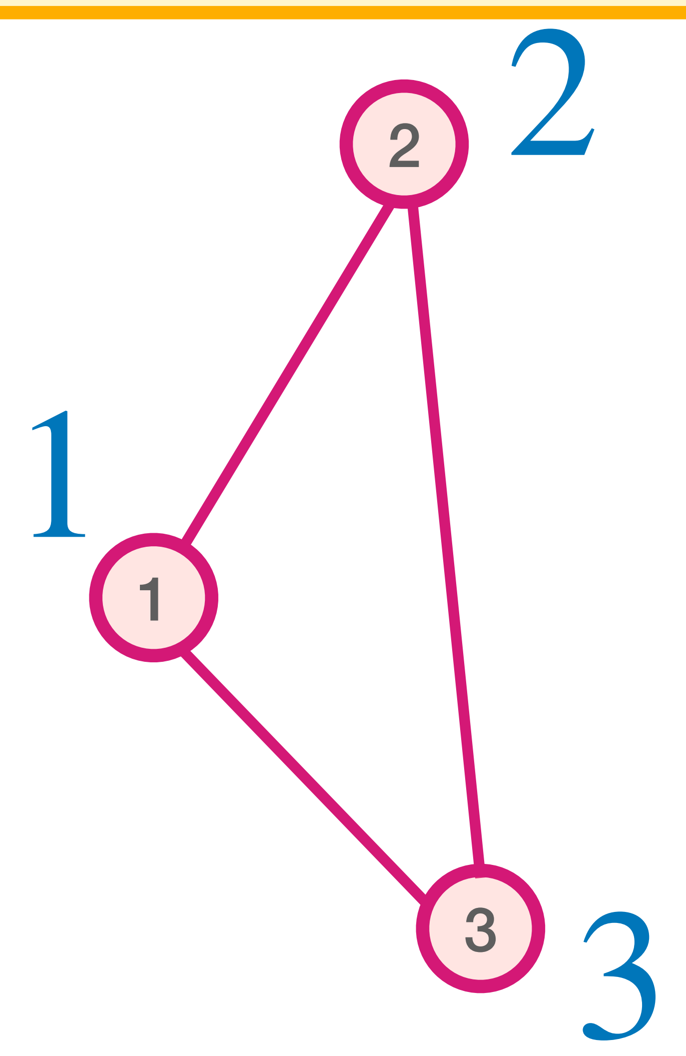
$\rightarrow$ $x = [1,0,0,0,1,0,0,0,1]$ satisfies the $Clique(x, y)$ constraints



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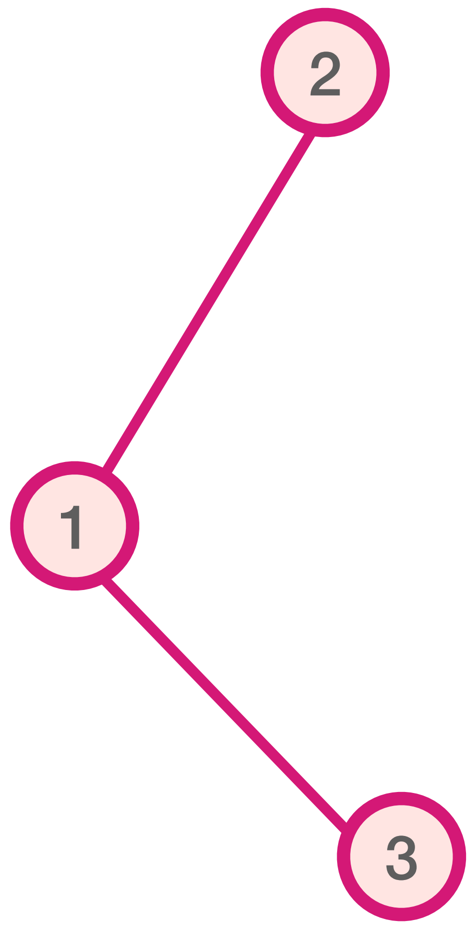

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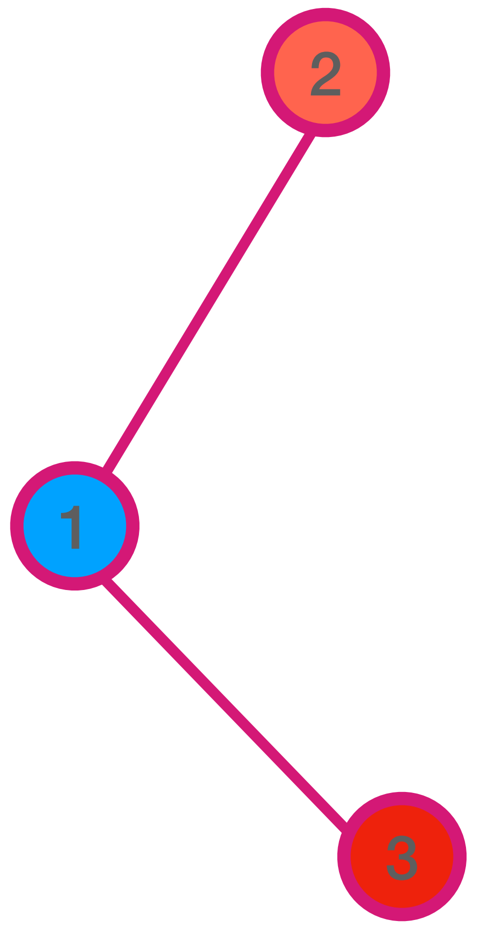

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Interpolation theorem for P implies P-proof of $Clique(x, y) \wedge Color(y, z)_{n,k} \Longrightarrow$ C_P-computation separating graphs with k-cliques from $(k-1)$-colorable graphs

Feasible Interpolation Theorems

[R95]

$S^2_1(\alpha) \Longrightarrow$ Boolean circuits

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Only worked for split formulas!

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Feasible Interpolation For CP

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Remainder of today:

1. Prove this theorem

Feasible Interpolation For CP

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Remainder of today:

1. Prove this theorem
2. Use known lower bounds on monotone real circuits computing clique to obtain Cutting Planes lower bounds for $Clique - Color$

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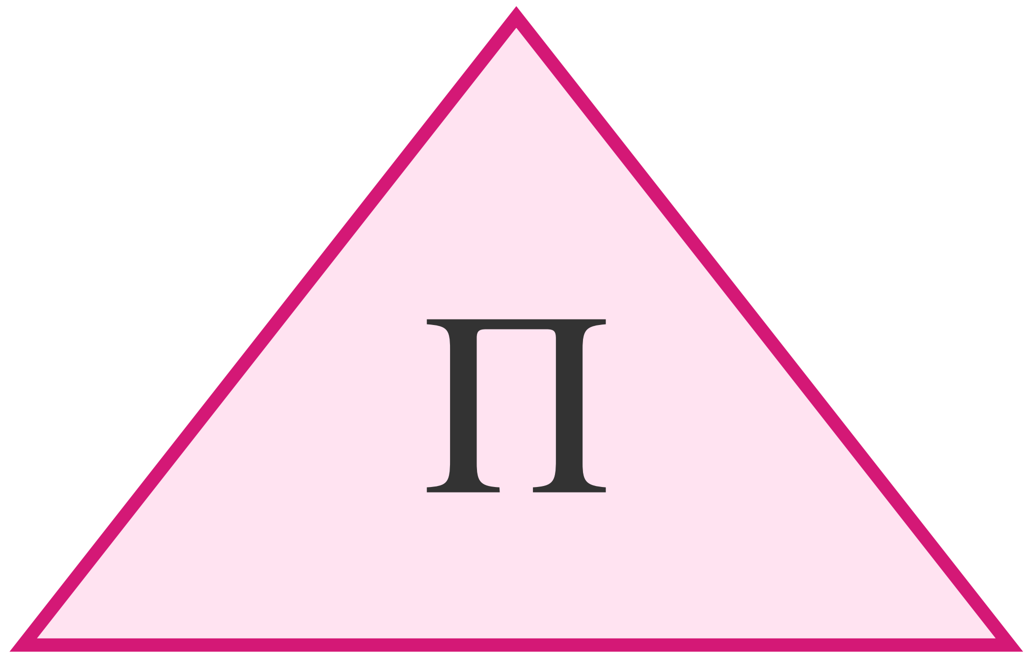
$0 \geq 1$



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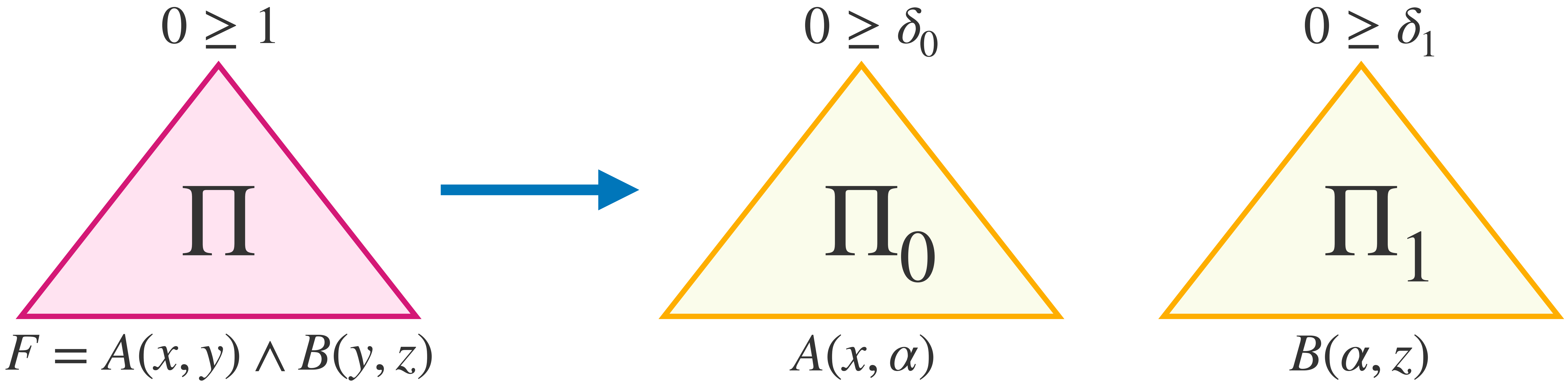

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Proof of Lemma: Claim allows us to extract from Π a proof of

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1. There are CP derivations of $ax \geq \delta_0$ from $A(x, \alpha)$ and $cz \geq \delta_1$ from $B(\alpha, z)$
2. δ_0, δ_1 are constructible in poly(s) time from Π and α
3. $\delta_0 + \delta_1 \geq d - b\alpha$

Proof of Lemma: Claim allows us to extract from Π a proof of

$\rightarrow$ $A(x, \alpha)$ if $A(x, \alpha)$ is unsatisfiable

$\rightarrow$ $B(\alpha, z)$ if $B(\alpha, z)$ is unsatisfiable

Indeed, …

Feasible Interpolation For CP

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With $\delta_0 + \delta_1 \geq 1$

Either $\delta_0 > 0$ and so $A(x,\alpha)$ is unsatisfiable

or $\delta_1 > 0$ and so $B(\alpha,z)$ is unsatisfiable

Feasible Interpolation For CP

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Proof of Lemma: The poly-time algorithm:

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Proof of Lemma: The poly-time algorithm:

on input $\alpha \in \{0,1\}^y$

1. Constructs δ_0 and δ_1 in time poly(s)

Feasible Interpolation For CP

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Proof of Lemma: The poly-time algorithm:

on input $\alpha \in \{0,1\}^y$

1. Constructs δ_0 and δ_1 in time poly(s)
2. If $\delta_0 > 0$ then $A(x, \alpha)$ is unsatisfiable and we output 0

Feasible Interpolation For CP

Lemma: There is a time poly(s) algorithm which given a split formula $F = A(x, y) \wedge B(y, z)$, a size s CP proof of Π of F, and $\alpha \in \{0,1\}^y$ outputs $I_F(\alpha)$

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Proof of Lemma: The poly-time algorithm:

on input $\alpha \in \{0,1\}^y$

1. Constructs δ_0 and δ_1 in time poly(s)
2. If $\delta_0 > 0$ then $A(x, \alpha)$ is unsatisfiable and we output 0
3. Otherwise, $\delta_1 > 0$ and $B(\alpha, z)$ is unsatisfiable, so output 1

Feasible Interpolation For CP

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2. δ_0, δ_1 are constructible in poly(s) time from Π and α
3. $\delta_0 + \delta_1 \geq d - b\alpha$

Proof: by induction. **Base case:**

- If $ax + by + cz \geq d$ belongs to $A(x, y)$ then $c = 0$
  - → Let $\delta_0 = d - b\alpha$ and the proof Π_0 be the axiom $ax \geq d - b\alpha$ of $A(x, \alpha)$
  - → Let $\delta_1 = 0$ and the proof Π_1 be the trivial axiom $0 \geq 0$
- If $ax + by + cz \geq d$ is an axiom of $B(y, z)$ then $a = 0$
  - → Let $\delta_0 = 0$ and Π_0 be $0 \geq 0$
  - → Let $\delta_1 = d - b\alpha$ and Π_1 be the axiom $cz \geq d - b\alpha$ of $B(\alpha, z)$

Feasible Interpolation For CP

Claim: For each inequality $ax + by + cz \geq d$ in Π there are constants δ_0, δ_1 s.t.

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3. $\delta_0 + \delta_1 \geq d - b\alpha$

Proof: by induction. **Base case:**

- If $ax + by + cz \geq d$ belongs to $A(x, y)$ then $c = 0$

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Feasible Interpolation For CP

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Feasible Interpolation For CP

Claim: For each inequality $ax + by + cz \geq d$ in Π there are constants δ_0, δ_1 s.t.

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  → Let $\delta_1 = 0$ and the proof Π_1 be the trivial axiom $0 \geq 0$
- If $ax + by + cz \geq d$ is an axiom of $B(y, z)$ then $a = 0$

Feasible Interpolation For CP

Claim: For each inequality $ax + by + cz \geq d$ in Π there are constants δ_0, δ_1 s.t.

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3. $\delta_0 + \delta_1 \geq d - b\alpha$

Proof: **Cut:** Suppose that $ax + by + cz \geq d$ is deduced by cut in Π

$$\frac{a'x + b'y + c'z \geq d'}{(a'/t)x + (b'/t)y + (c'/t)z \geq \lceil d'/t \rceil}$$

For t dividing a', b', c'

$ax + by + cz \geq d$

Feasible Interpolation For CP

Claim: For each inequality $ax + by + cz \geq d$ in Π there are constants δ_0, δ_1 s.t.

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For t dividing a', b', c'

And by induction we have derived

From $A(x, \alpha)$: $\boxed{a'x \geq \delta_0'}$

From $B(\alpha, z)$: $\boxed{c'z \geq \delta_1'}$

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From $B(\alpha, z)$: $\boxed{c'z \geq \delta_1'}$

With $\delta_0' + \delta_1' \geq d' - b'\alpha$

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$$\frac{a'x + b'y + c'z \geq d'}{(a'/t)x + (b'/t)y + (c'/t)z \geq \lceil d'/t \rceil}$$ For t dividing a', b', c'

And by induction we have derived

From $A(x, \alpha)$: $a'x \geq \delta_0'$ → Cut → $(a'/t)x \geq \lceil \delta_0'/t \rceil$

From $B(\alpha, z)$: $c'z \geq \delta_1'$ → Cut → $(c'/t)z \geq \lceil \delta_1'/t \rceil$

With $\delta_0' + \delta_1' \geq d' - b'\alpha$

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For t dividing a', b', c'

And by induction we have derived

From $A(x, \alpha)$: $a'x \geq \delta_0'$ $\rightarrow$ Cut $\rightarrow$ $(a'/t)x \geq \lceil \delta_0'/t \rceil = \delta_0$

From $B(\alpha, z)$: $c'z \geq \delta_1'$ $\rightarrow$ Cut $\rightarrow$ $(c'/t)z \geq \lceil \delta_1'/t \rceil = \delta_1$

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For t dividing a', b', c'

And by induction we have derived

$a'x \geq \delta_0'$ → Cut → $(a'/t)x \geq \lceil \delta_0'/t \rceil = \delta_0$

$c'z \geq \delta_1'$ → Cut → $(c'/t)z \geq \lceil \delta_1'/t \rceil = \delta_1$

Invariant:

$\delta_0 + \delta_1$

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Invariant:

$$\delta_0 + \delta_1 = \lceil \delta_0'/t \rceil + \lceil \delta_1'/t \rceil$$

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$c'z \geq \delta_1'$ → Cut → $(c'/t)z \geq \lceil \delta_1'/t \rceil = \delta_1$

Invariant:

$$\delta_0 + \delta_1 = \lceil \delta_0'/t \rceil + \lceil \delta_1'/t \rceil \geq \lceil (\delta_0' + \delta_1')/t \rceil$$

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$a'x \geq \delta_0'$ $\rightarrow$ Cut $\rightarrow$ $(a'/t)x \geq \lceil \delta_0'/t \rceil = \delta_0$

$c'z \geq \delta_1'$ $\rightarrow$ Cut $\rightarrow$ $(c'/t)z \geq \lceil \delta_1'/t \rceil = \delta_1$

Invariant:

$$\delta_0 + \delta_1 = \lceil \delta_0'/t \rceil + \lceil \delta_1'/t \rceil \geq \lceil (\delta_0' + \delta_1')/t \rceil \geq \lceil (d - b\alpha)/t \rceil$$

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For t dividing a', b', c'

And by induction we have derived

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Invariant:

$$\delta_0 + \delta_1 = \lceil \delta_0'/t \rceil + \lceil \delta_1'/t \rceil \geq \lceil (\delta_0' + \delta_1')/t \rceil \geq \lceil (d - b\alpha)/t \rceil = \lceil d/t \rceil - b\alpha/t$$

Feasible Interpolation For CP

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1. There are CP derivations of $ax \geq \delta_0$ from $A(x, \alpha)$ and $cz \geq \delta_1$ from $B(\alpha, z)$
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Proof: Non-negative Linear Combination:

Feasible Interpolation For CP

Claim: For each inequality $ax + by + cz \geq d$ in Π there are constants δ_0, δ_1 s.t.

1. There are CP derivations of $ax \geq \delta_0$ from $A(x, \alpha)$ and $cz \geq \delta_1$ from $B(\alpha, z)$
2. δ_0, δ_1 are constructible in poly(s) time from Π and α
3. $\delta_0 + \delta_1 \geq d - b\alpha$

Proof: Non-negative Linear Combination:

$$\frac{a'x + b'y + c'z \geq d', \qquad a''x + b''y + c''z \geq d''}{(\gamma'a' + \gamma''a'')x + (\gamma'b' + \gamma''b'')y + (\gamma'c' + \gamma''c'')z \geq \gamma'd' + \gamma''d''} \quad \text{For } \gamma', \gamma'' \geq 0$$

$$ax + by + cz \geq d$$

Feasible Interpolation For CP

Claim: For each inequality $ax + by + cz \geq d$ in Π there are constants δ_0, δ_1 s.t.

1. There are CP derivations of $ax \geq \delta_0$ from $A(x, \alpha)$ and $cz \geq \delta_1$ from $B(\alpha, z)$
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Proof: **Non-negative Linear Combination:**

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Feasible Interpolation For CP

Claim: For each inequality $ax + by + cz \geq d$ in Π there are constants δ_0, δ_1 s.t.

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Feasible Interpolation by Real Circuits

Lemma: There is a time poly(s) algorithm which given a split formula $F = A(x, y) \wedge B(y, z)$, a size s CP proof of Π of F, and $\alpha \in \{0,1\}^y$ outputs $I_F(\alpha)$

This lemma is overkill!

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We will define a computational model can do all of this but is still weak enough to prove lower bounds on!

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Monotone Circuits: boolean circuits using only $\wedge$ and $\vee$ gates — no $\neg$

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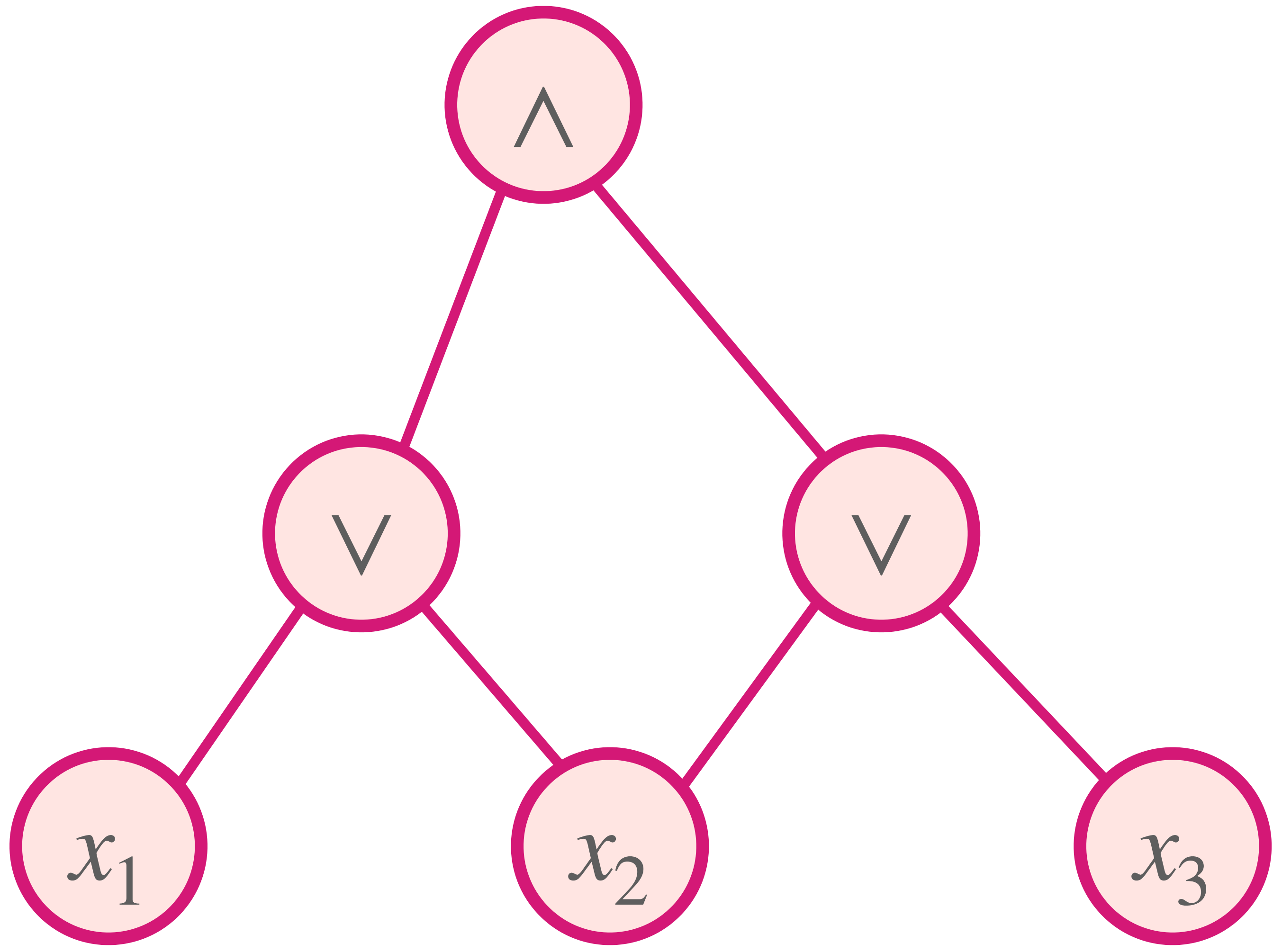

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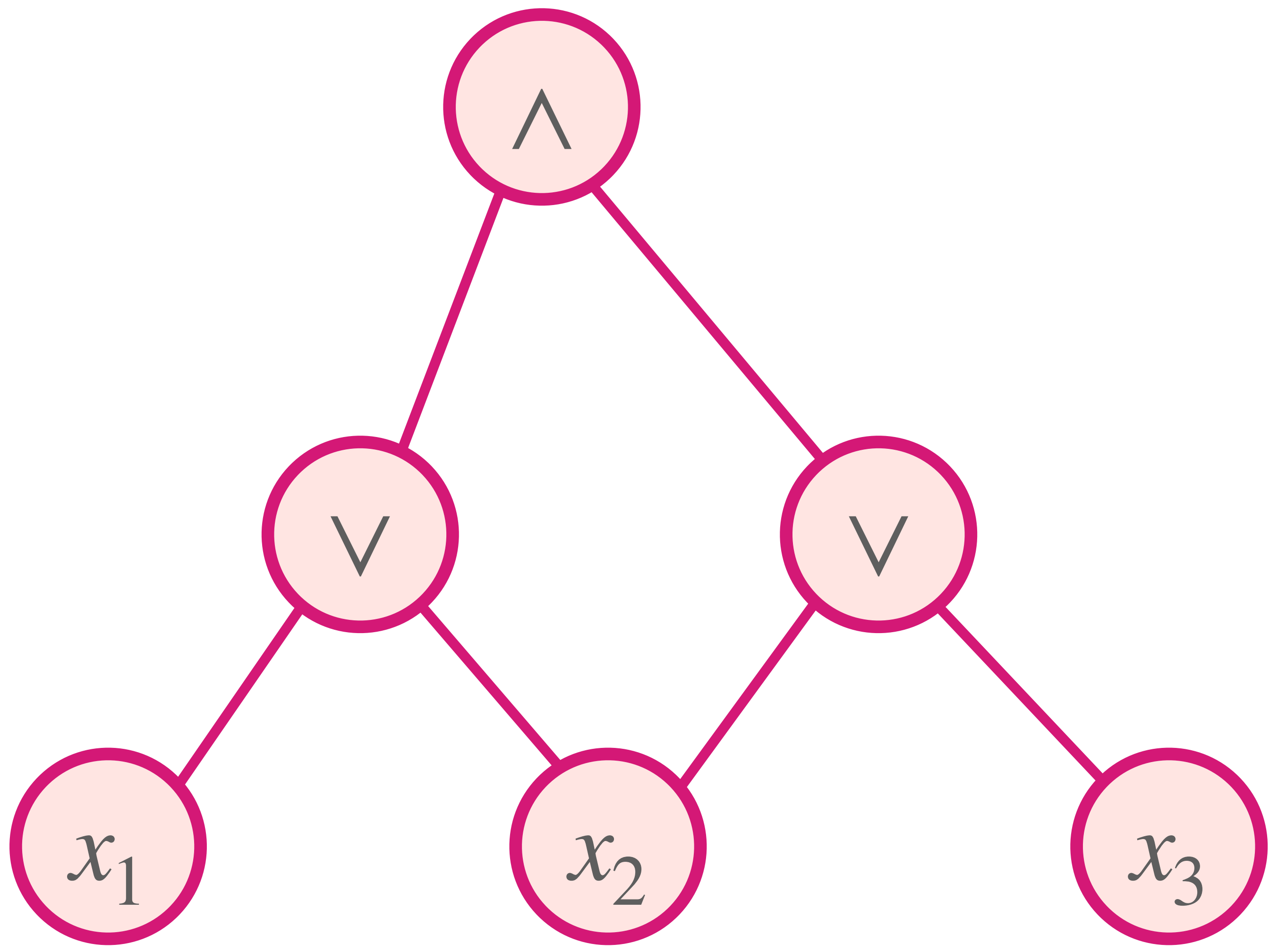
Monotone Real Circuits [P97]: A monotone real circuit computing $f : \{0,1\}^n \to \{0,1\}$ is a circuit in which gates are any monotone real-valued function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ on (at most) two inputs!



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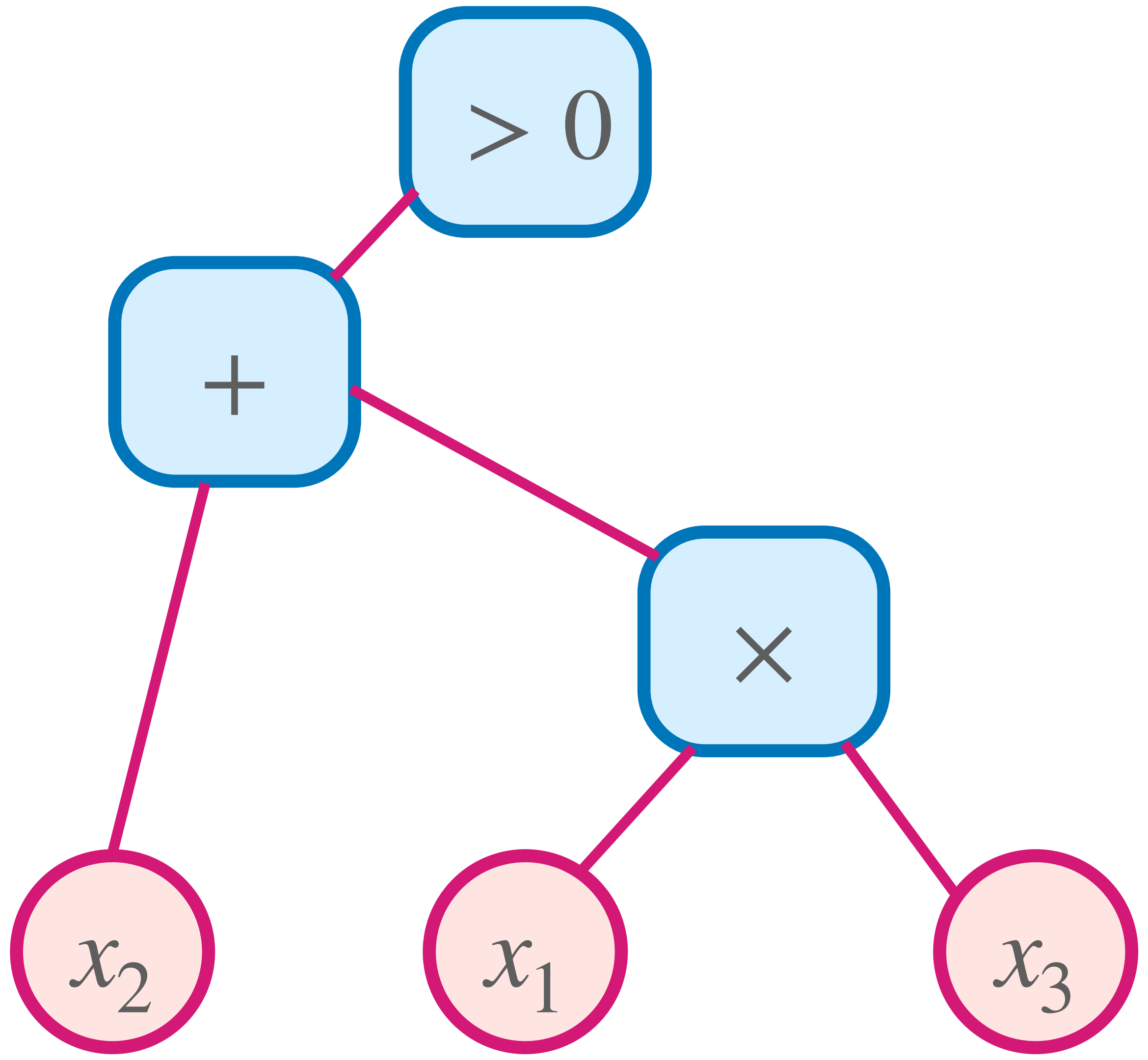

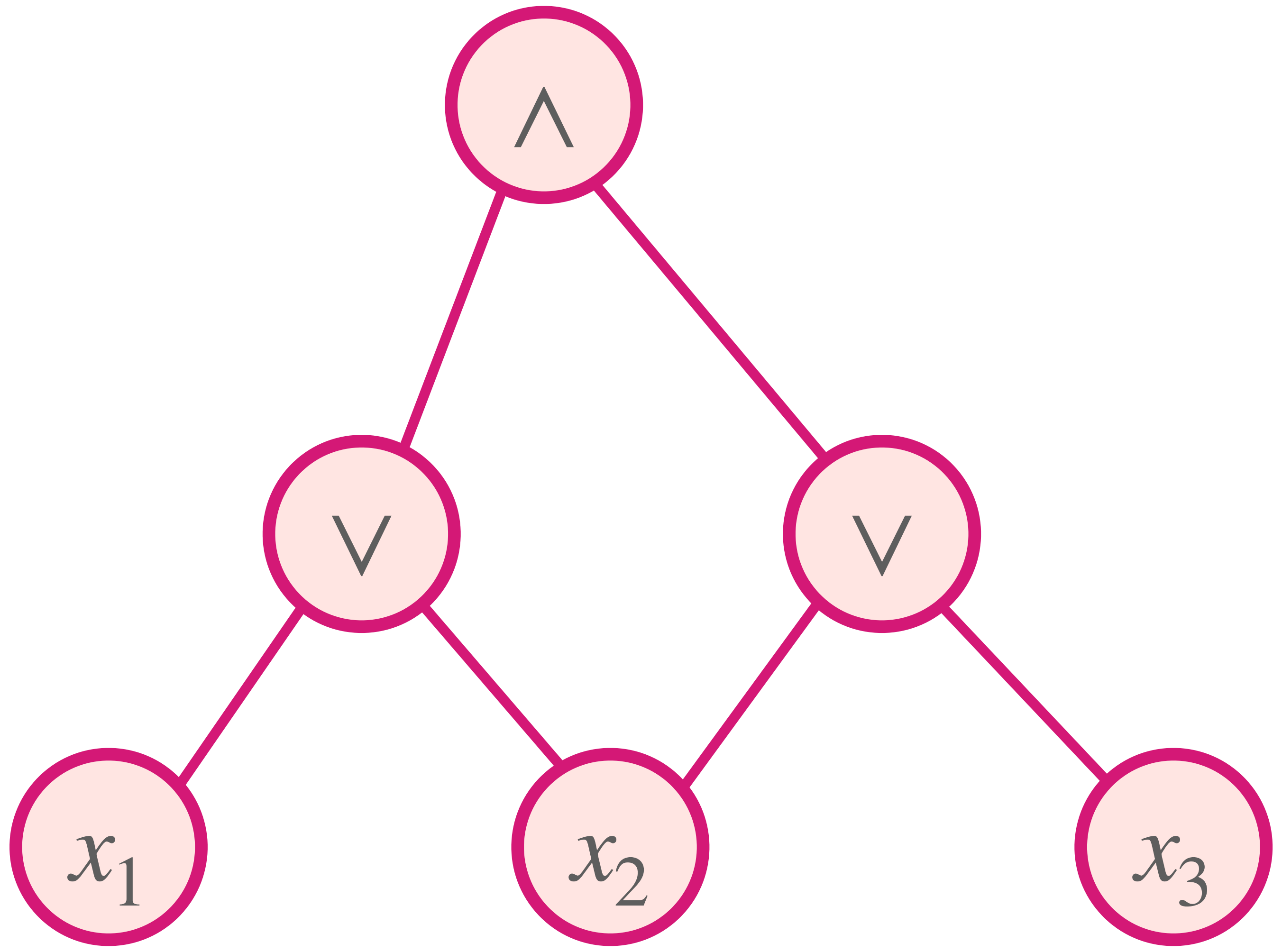

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Thm: If there is a size s CP proof Π of $F = A(x, y) \wedge B(y, z)$ then there is a size poly(s) monotone real circuit computing $I_F(y)$

Proof: Recall that y-variables occurs only positively in $A(x, y)$.

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→ Cut: From $-\delta_0'$ derive $\lceil -\delta_0'/t \rceil$

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$\Longrightarrow$ Let the output gate of the circuit be $-\delta_0 \geq 0$.

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Recall $Clique - Color$ formula

Interpolant function: $$I_F(y) = \begin{cases} 0 & \text{if } Clique(x, \alpha) \text{ is unsatisfiable} \\ 1 & \text{if } Color(\alpha, z) \text{ is unsatisfiable} \end{cases}$$

Upshot: Lower bounds on **Clique** imply lower bounds on I_F

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Thm[P97]**:** Any monotone real circuit computing Clique requires exponential size

Interpolation for any Formula

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