Integer Programming and IP Proof Systems

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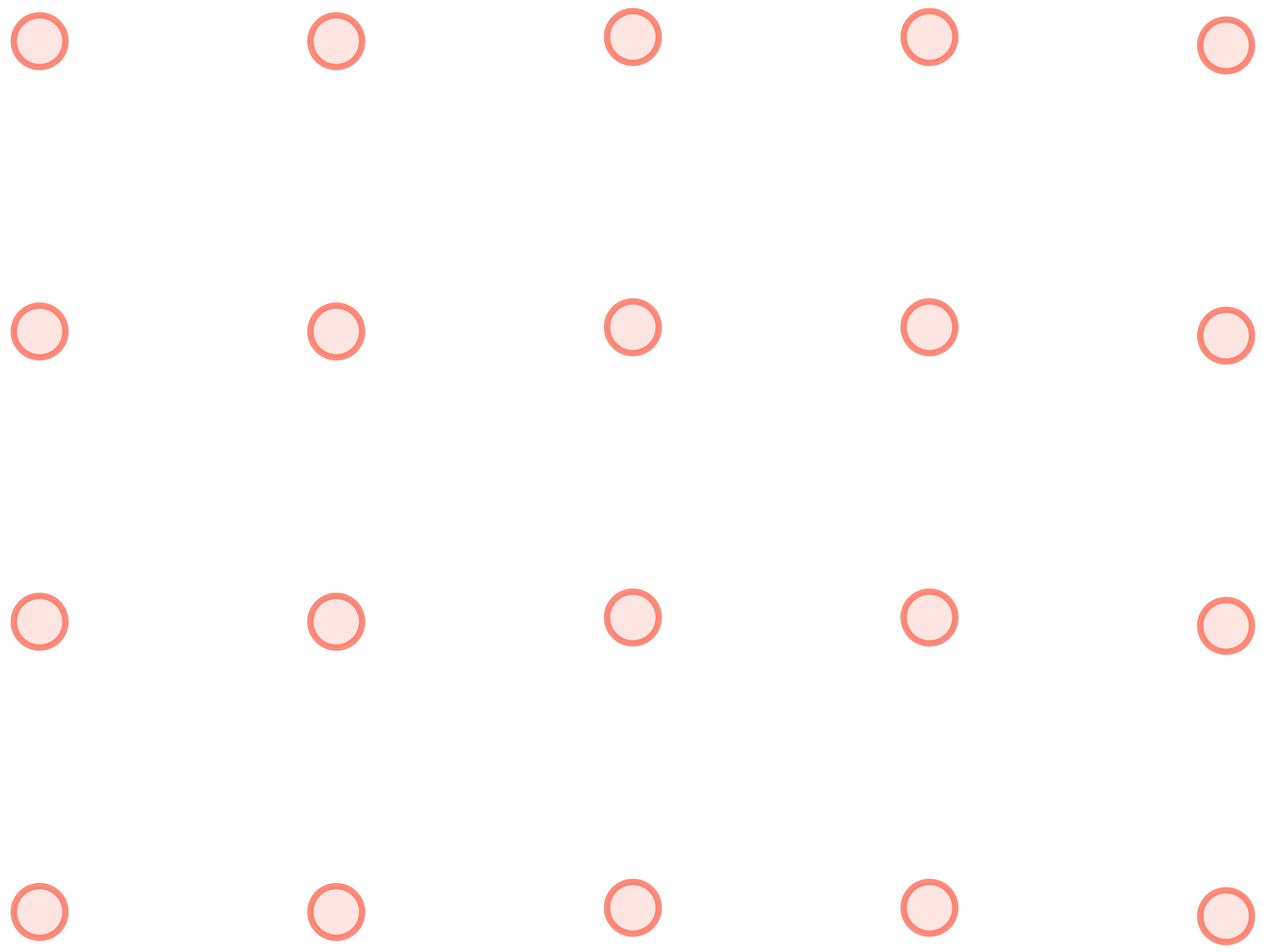
University of California, San Diego

Linear Programming

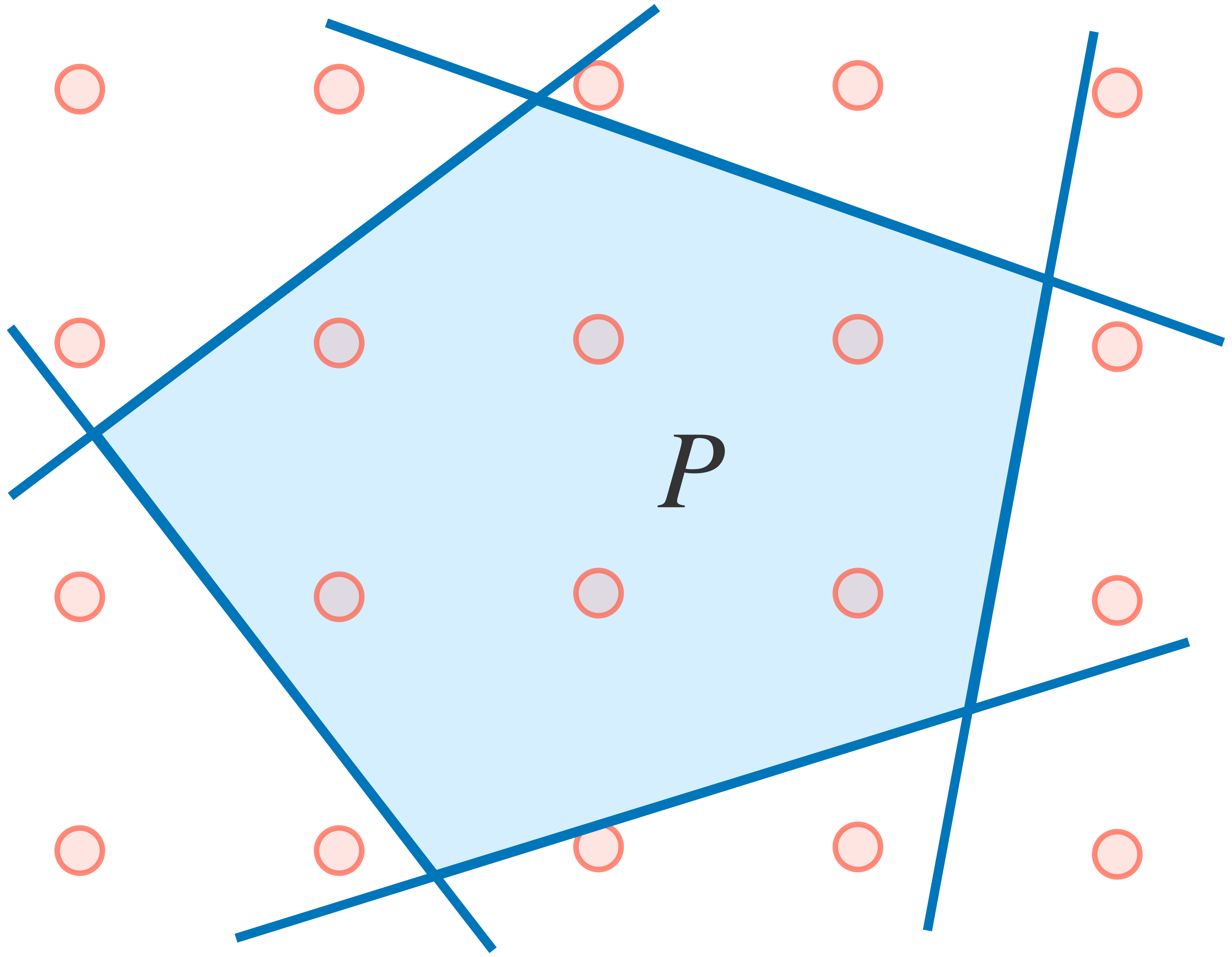
Input: set of linear inequalities $Ax \geq b$



Linear Programming

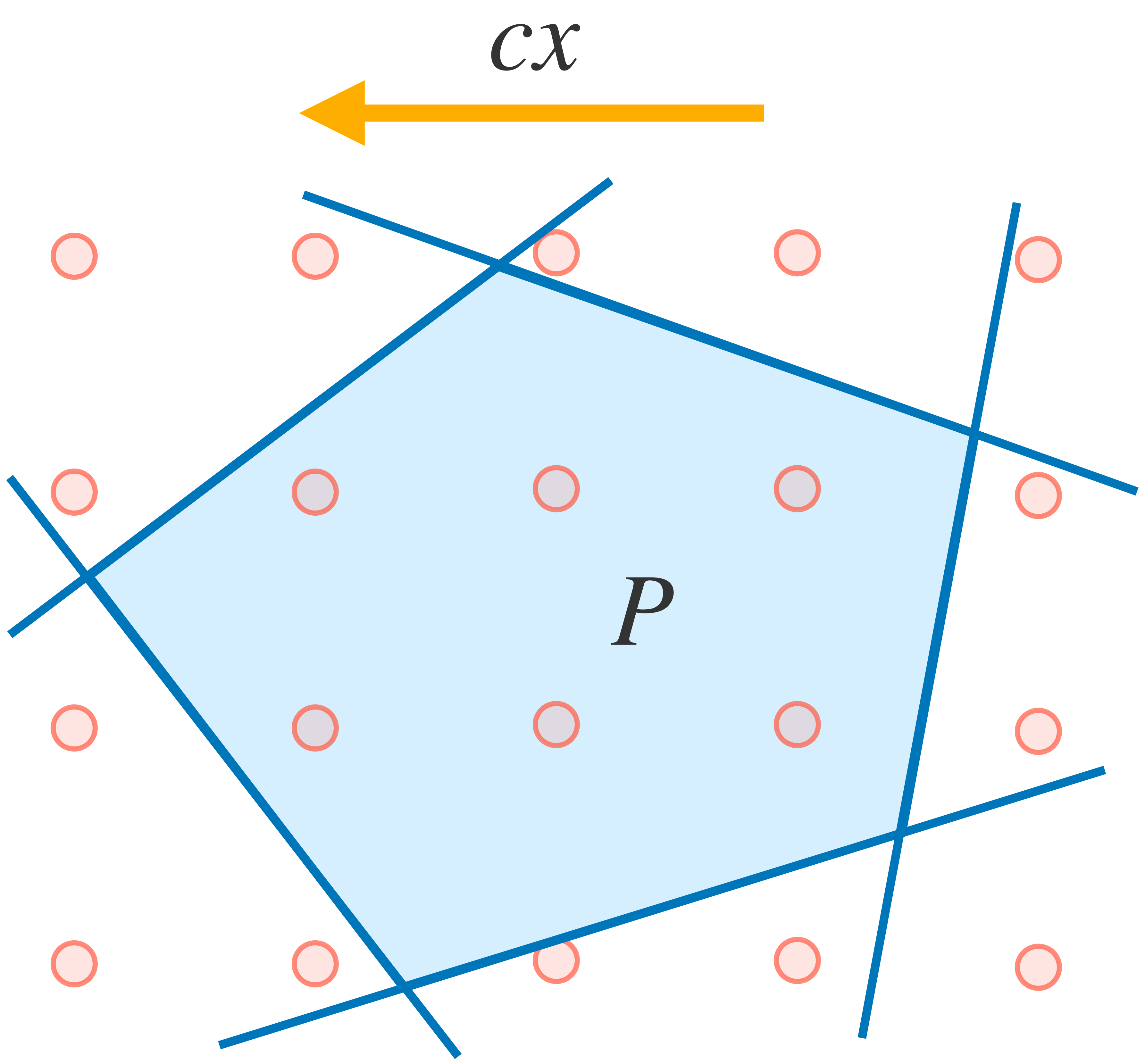
Input: set of linear inequalities $Ax \geq b$

Which defines a polytope $P = \{x : Ax \geq b\}$



Linear Programming

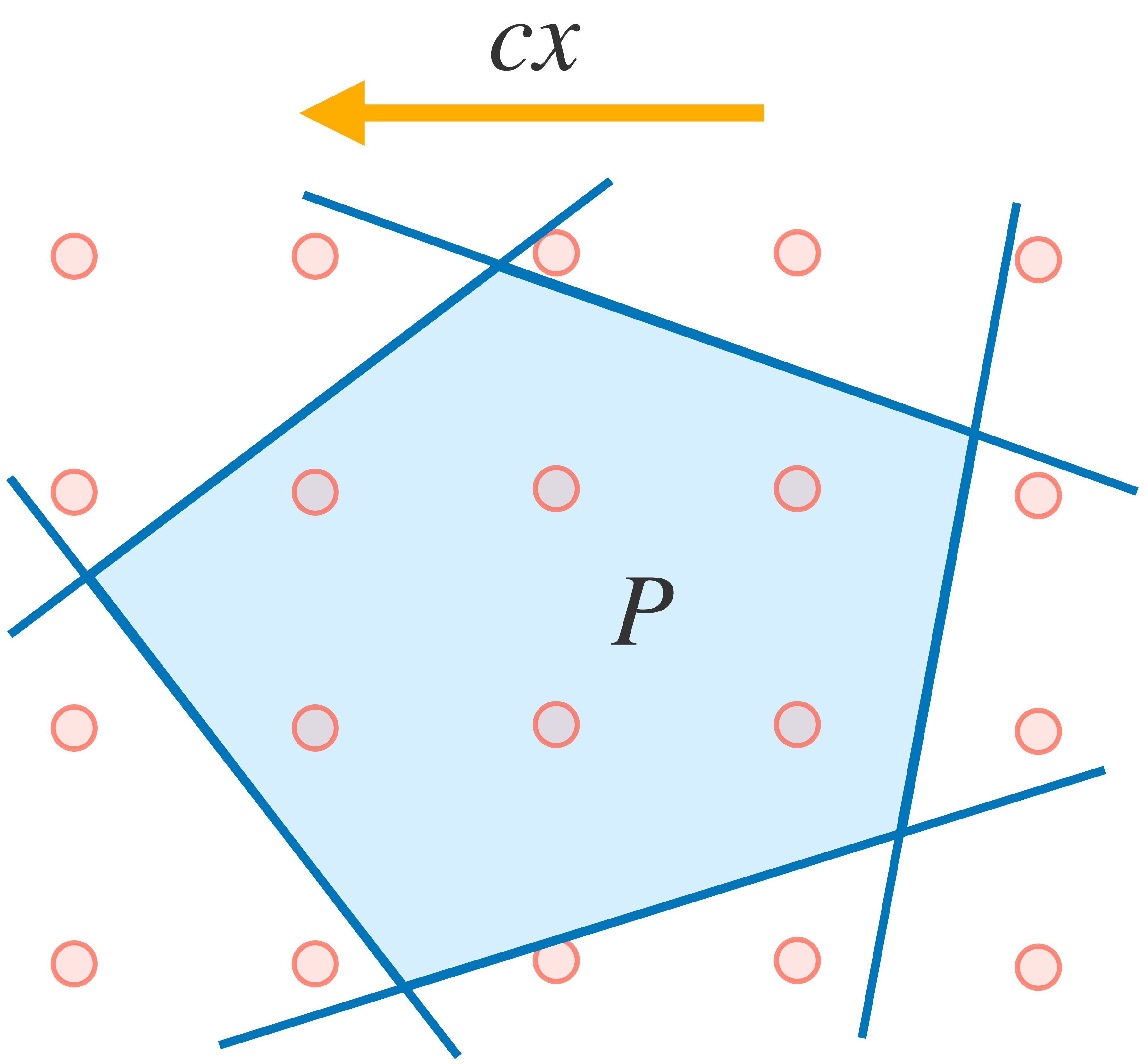
Input: set of linear inequalities $Ax \geq b$
linear objective function cx



Linear Programming

<u>Input:</u> set of linear inequalities $Ax \geq b$
linear objective function cx

<u>Output:</u> Solution $z \in \mathbb{R}^n$
Satisfying $Az \geq b$
Maximizing cz



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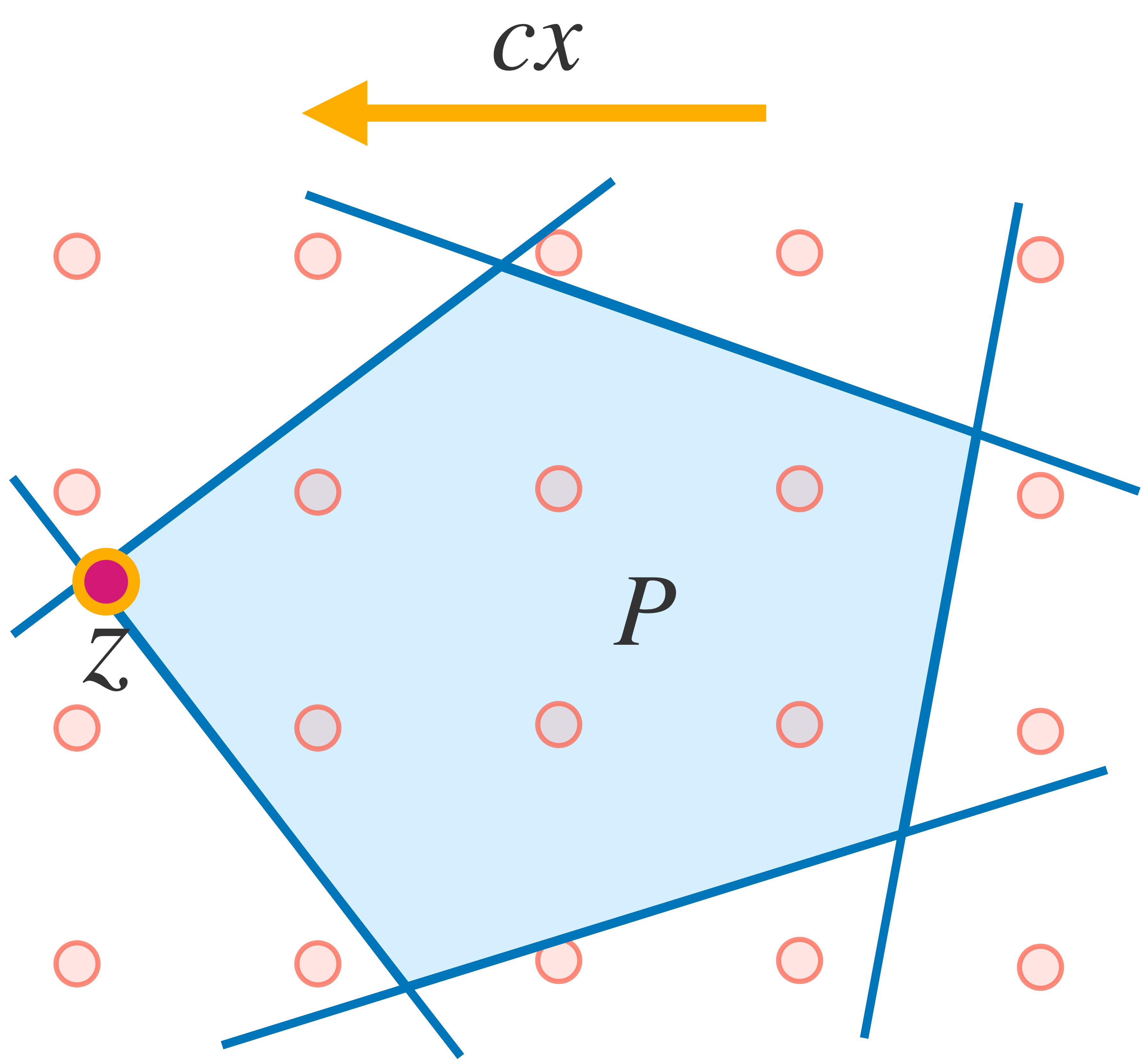

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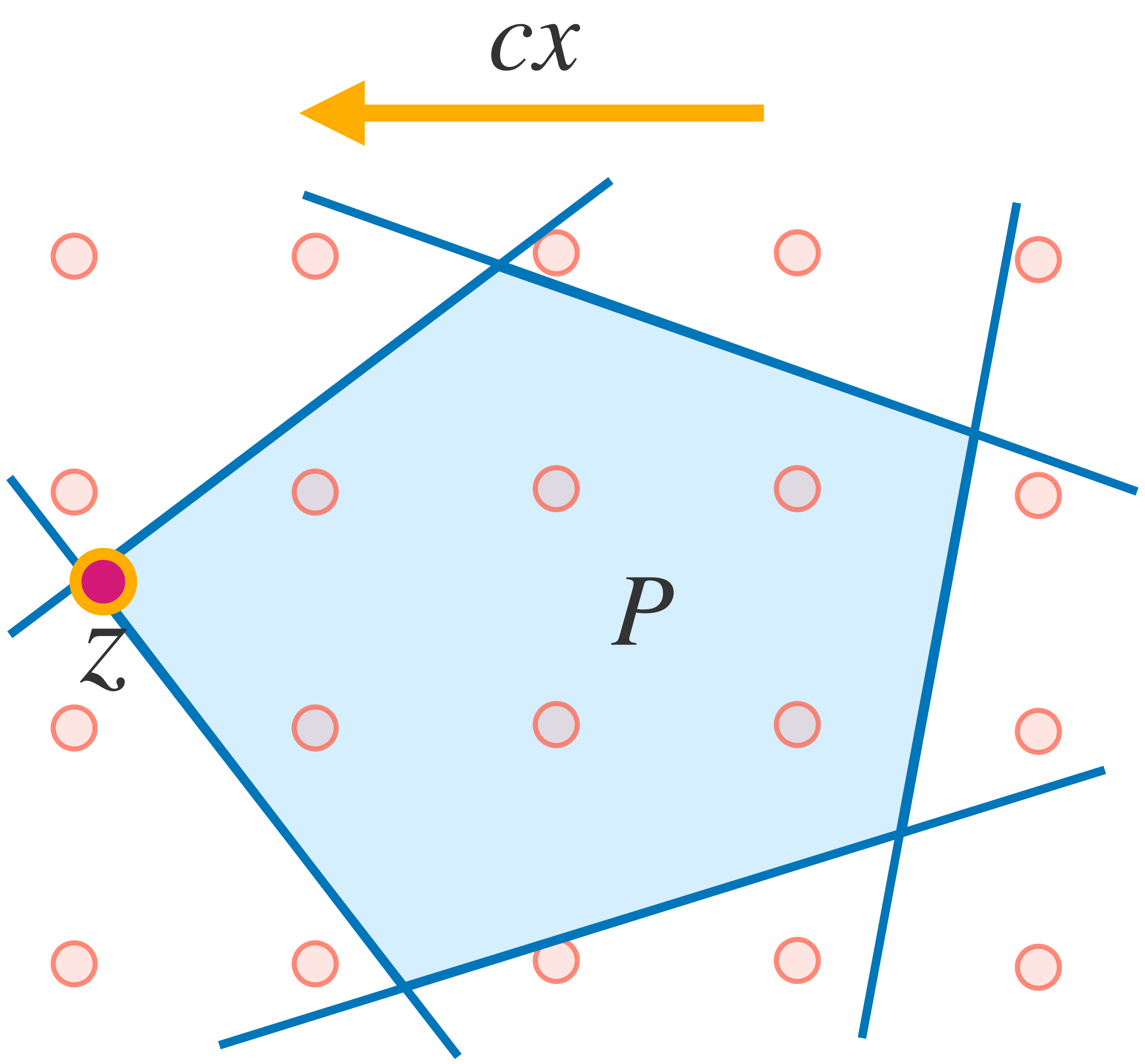
- Broadly applicable framework for optimization.
- Efficiently solvable!



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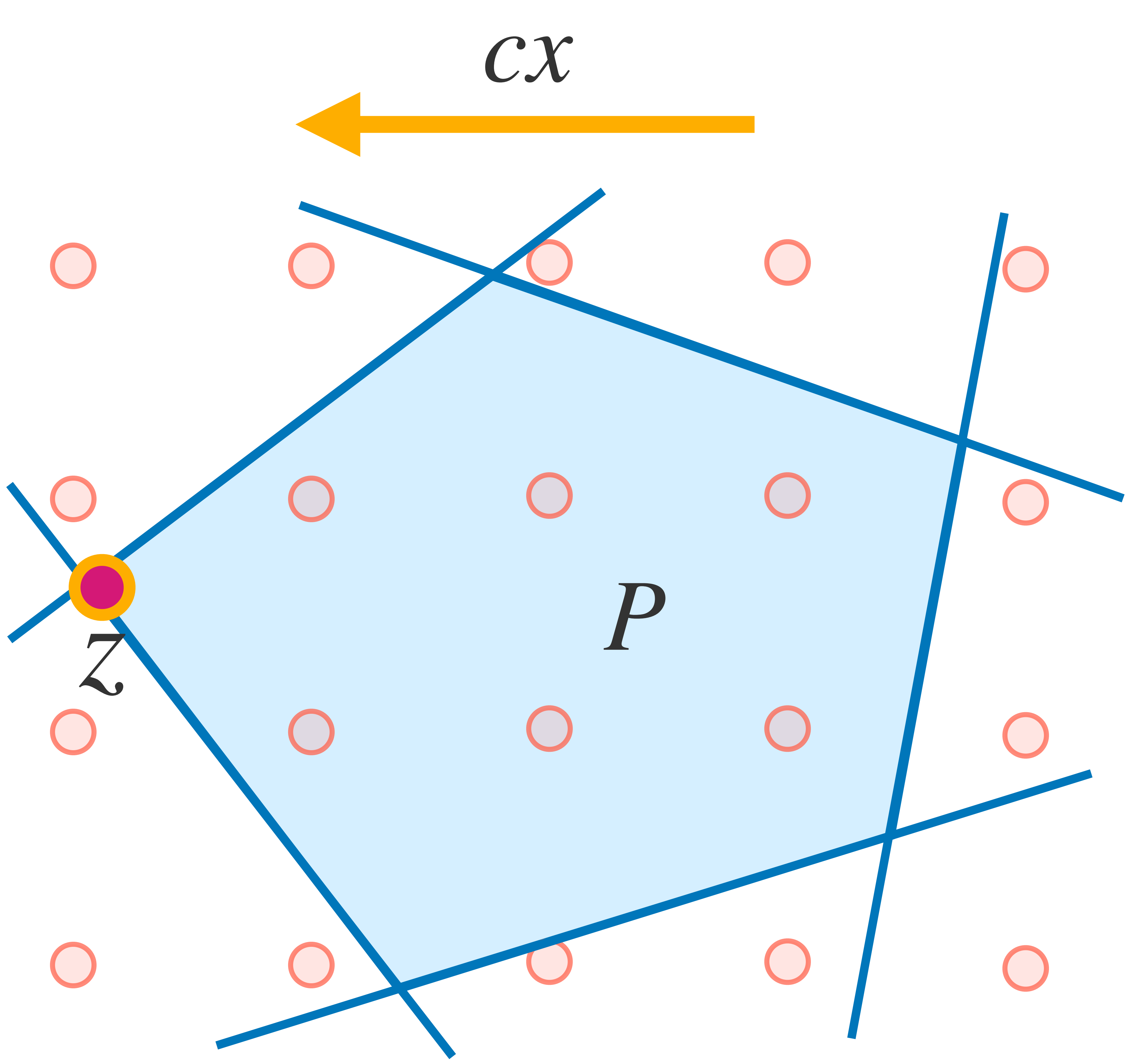

- Broadly applicable framework for optimization.
- Efficiently solvable!

However, many **important problems** phrased most naturally as finding **integer solutions** to a linear program

- e.g. maxCut, maxSAT, maxClique, etc.

Integer Programming (IP)

<u>Input:</u> set of linear inequalities $Ax \geq b$
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<u>Output:</u> Integer solution $z \in \mathbb{Z}^n$
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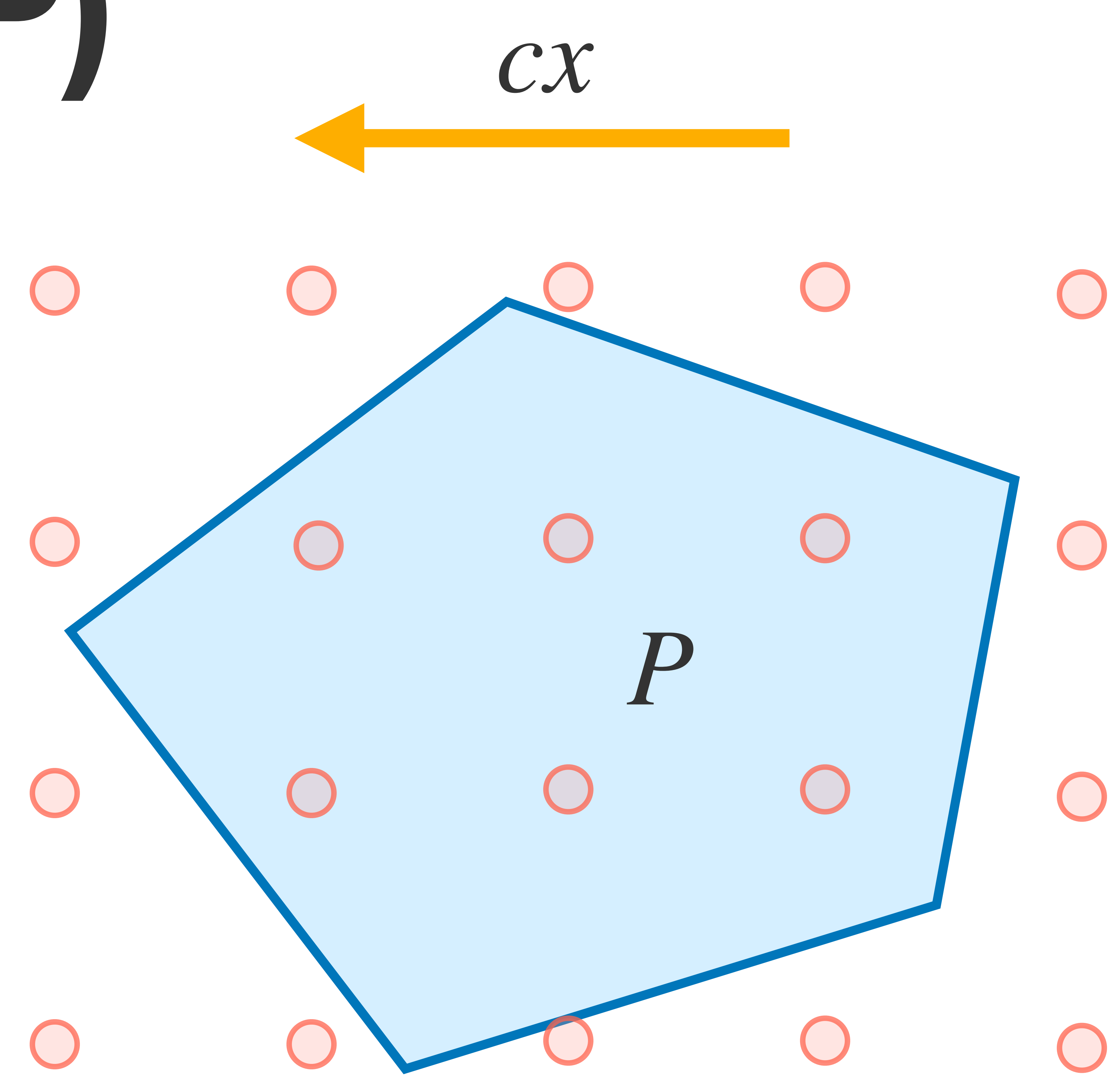

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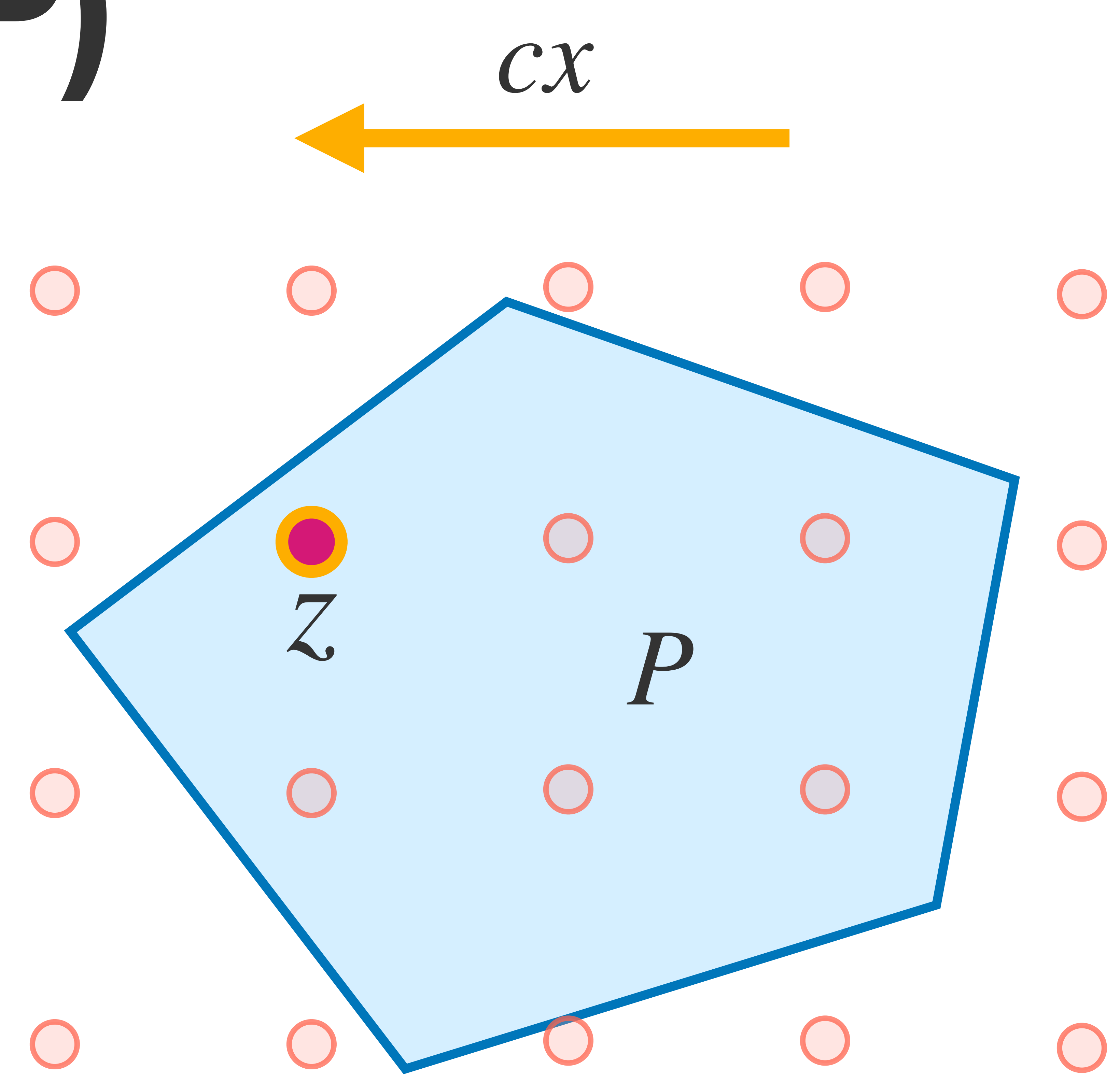

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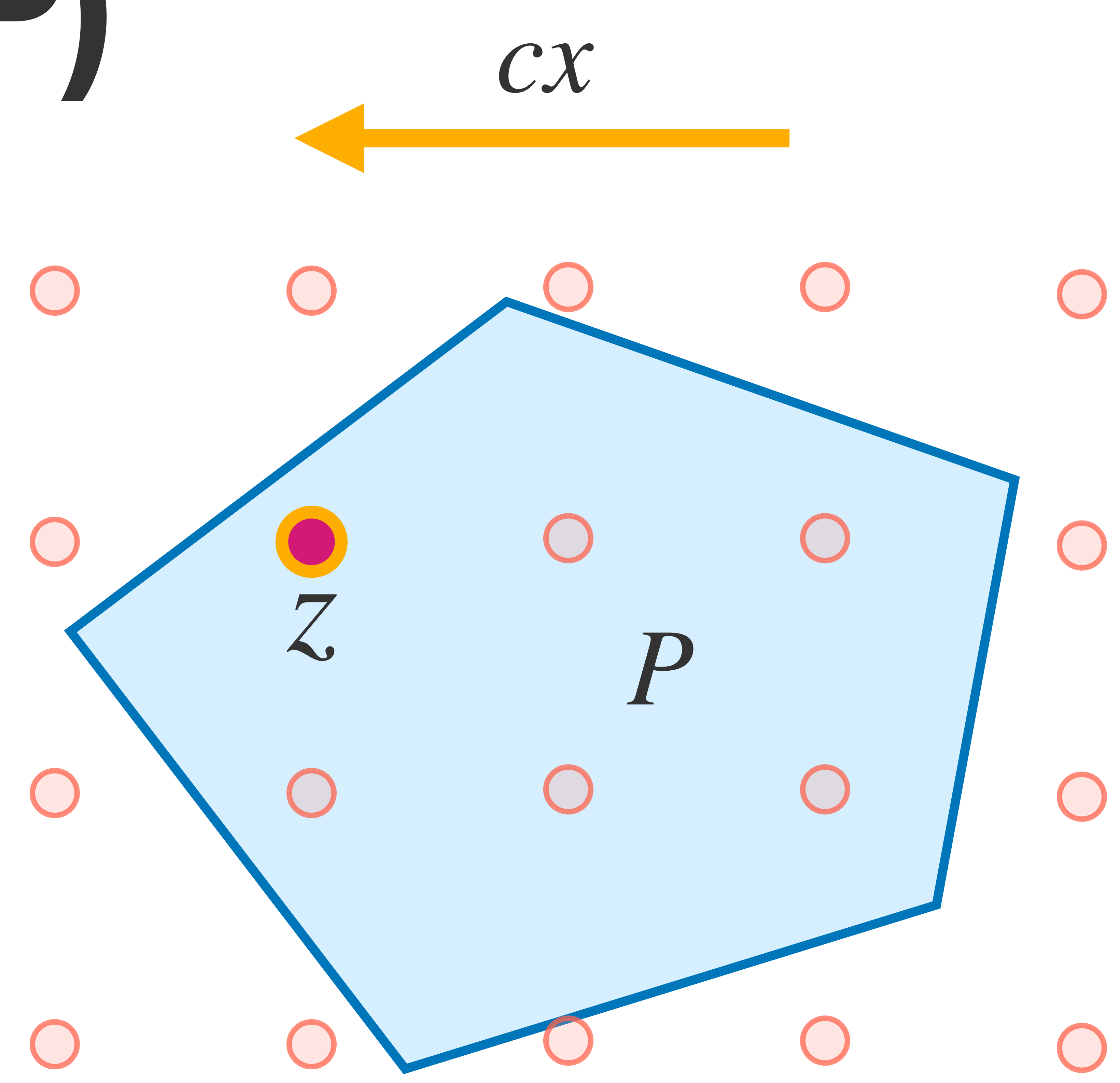
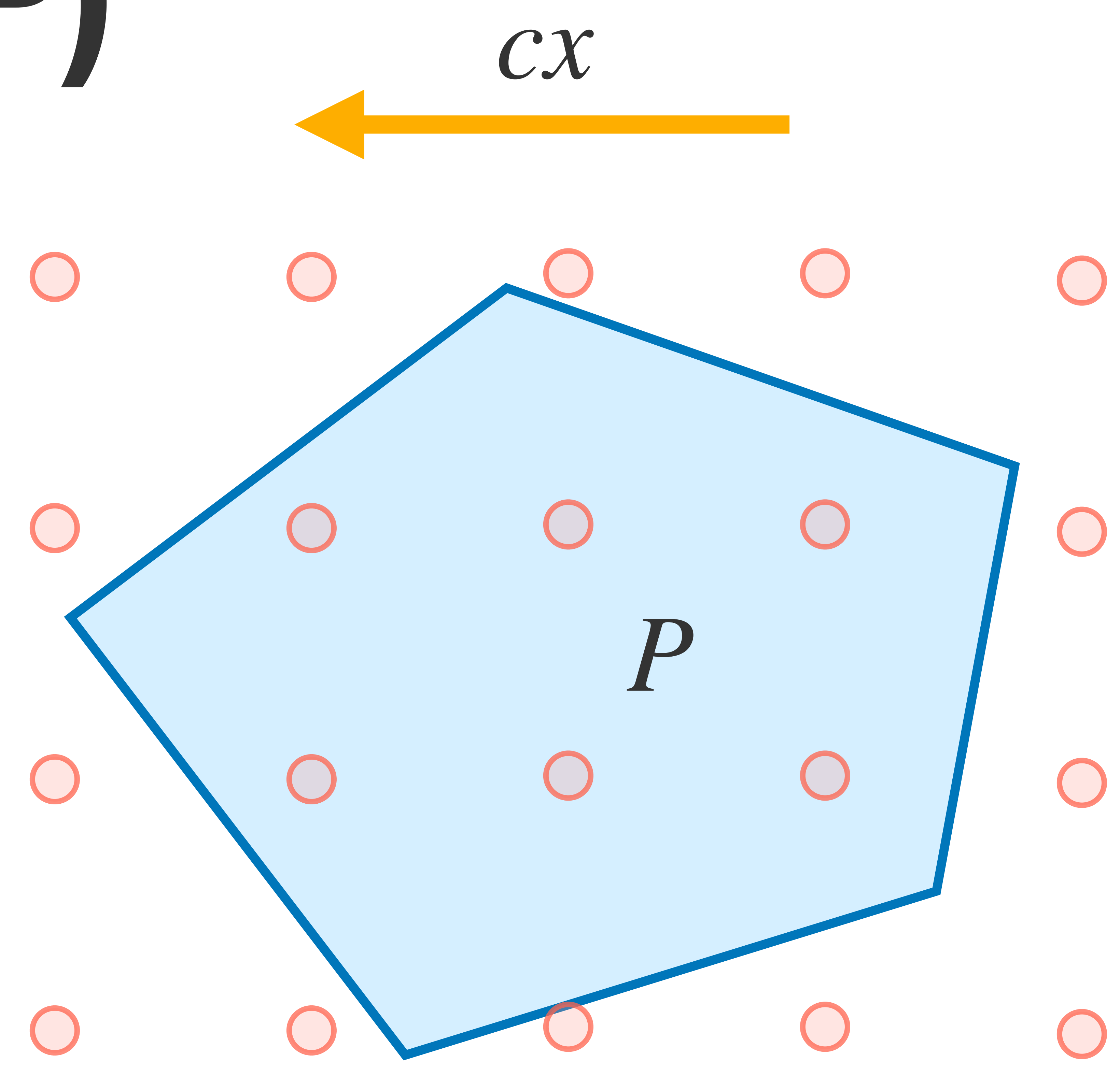


Extremely general framework!
… but **NP**-complete.

Integer Programming (IP)

Nonetheless, Integer programs are solved **extremely** fast in **practice!**



Integer Programming (IP)

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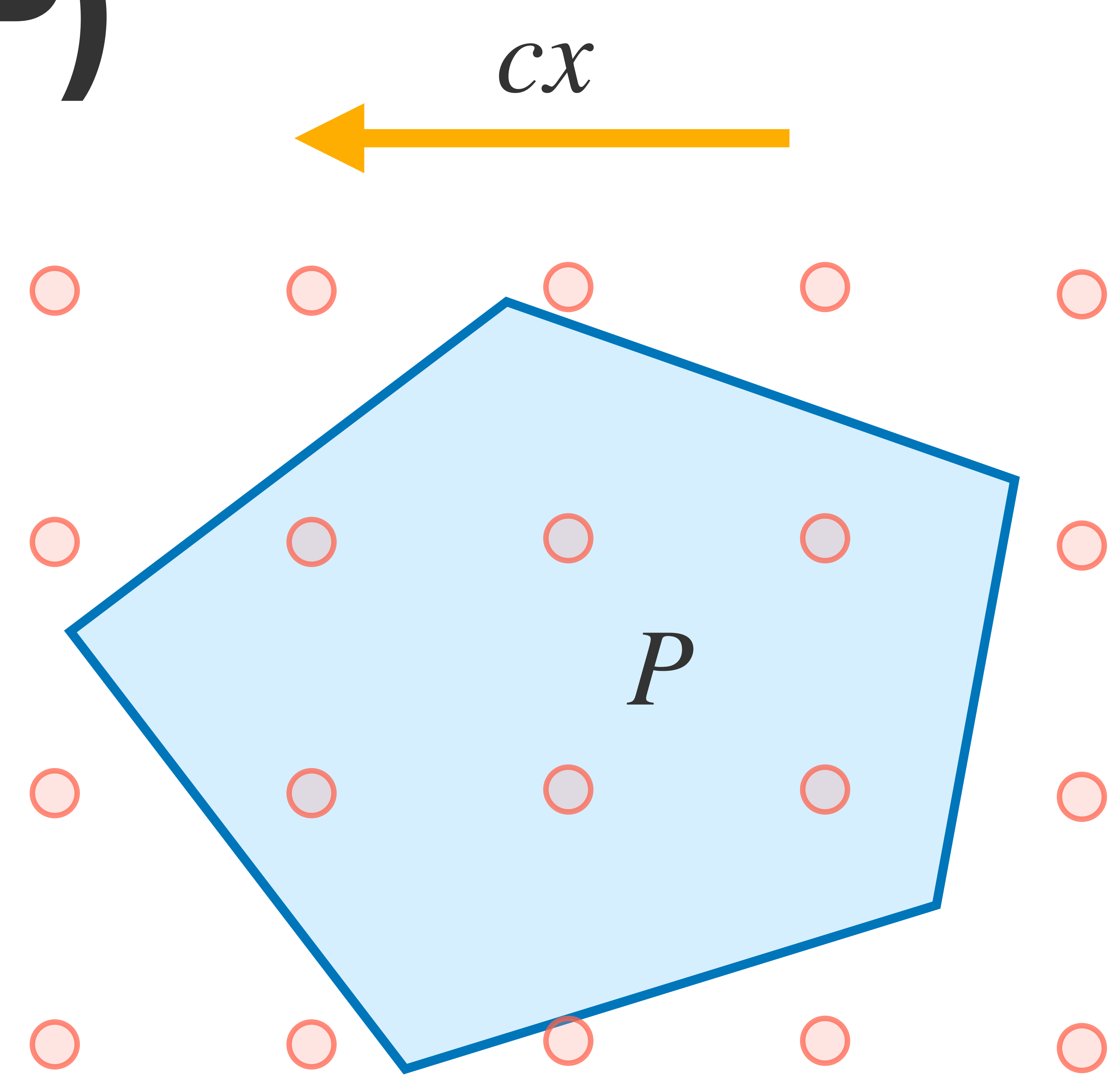
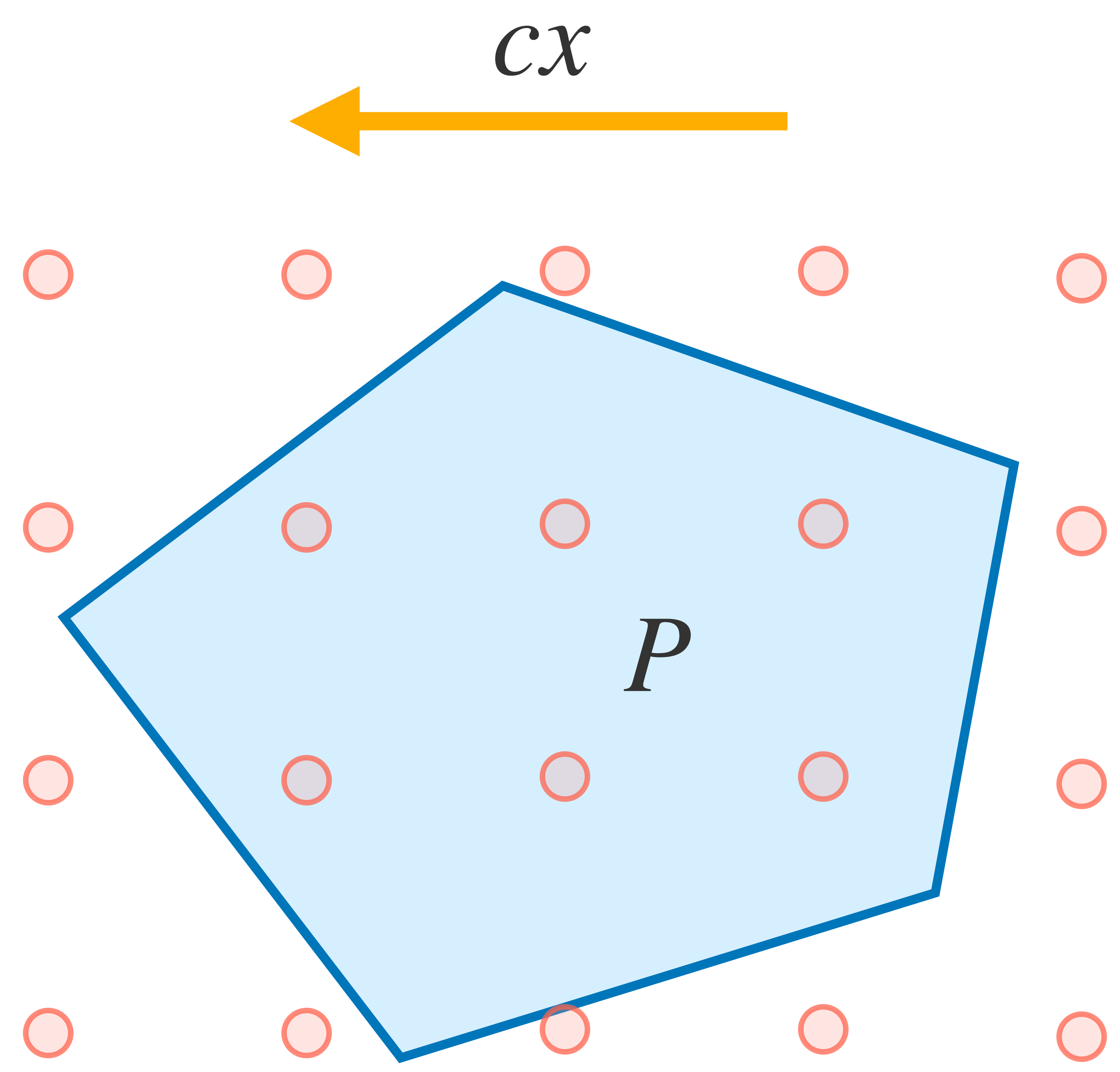
How? Branch-and-Cut!



Branch-and-Cut

Idea: Try to use **linear** programming to solve **integer** programming!



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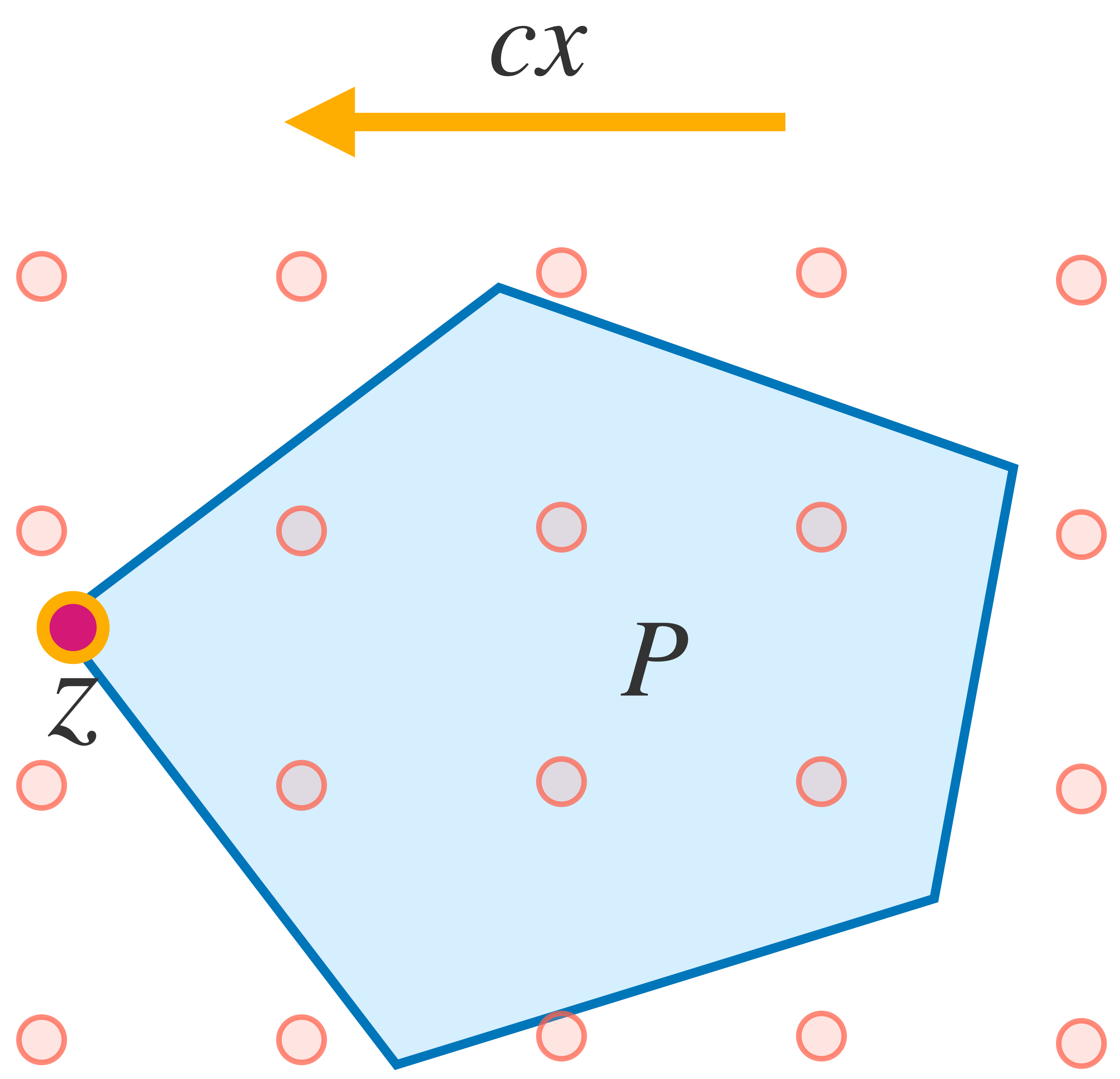
Run a linear program



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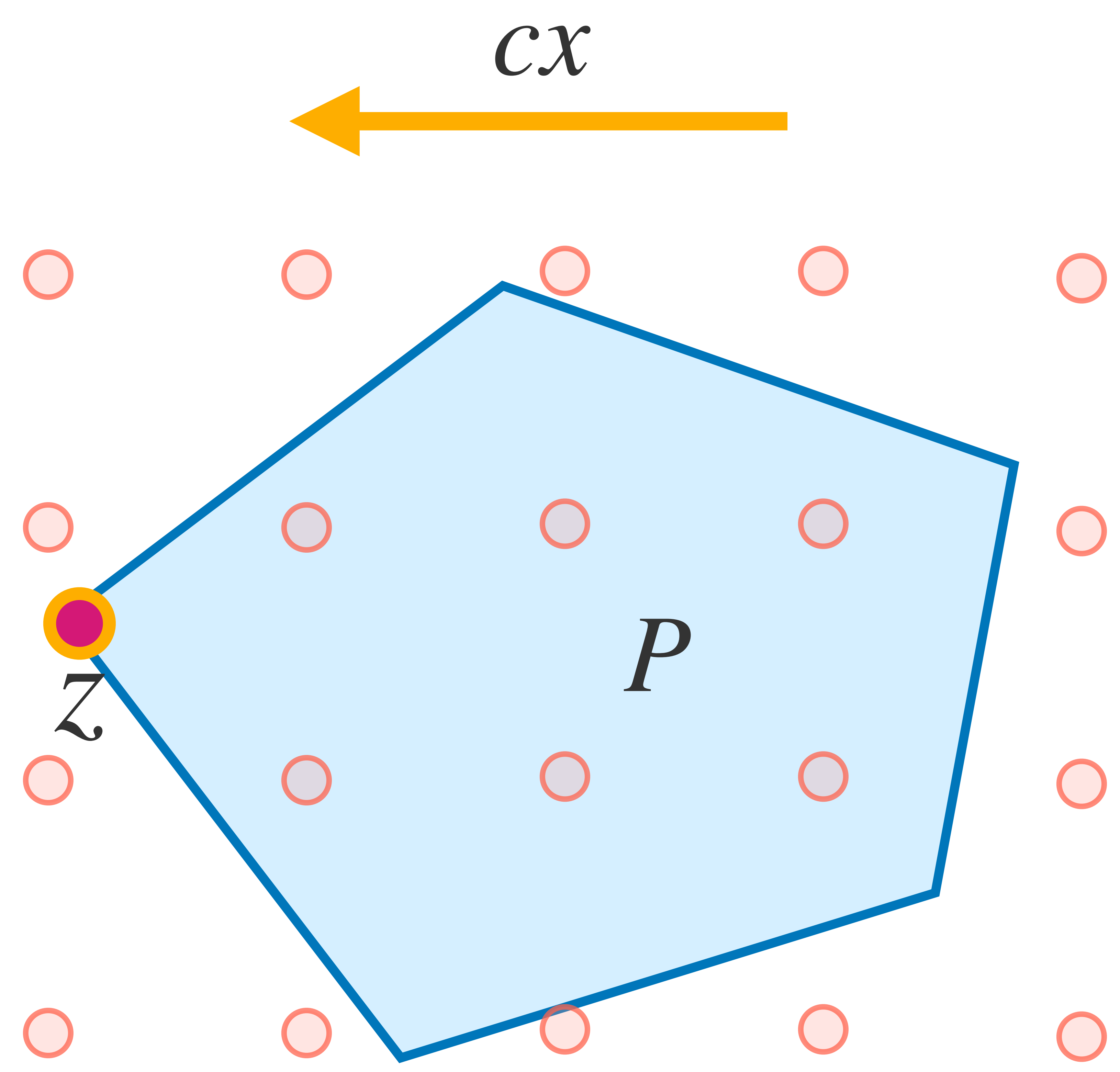

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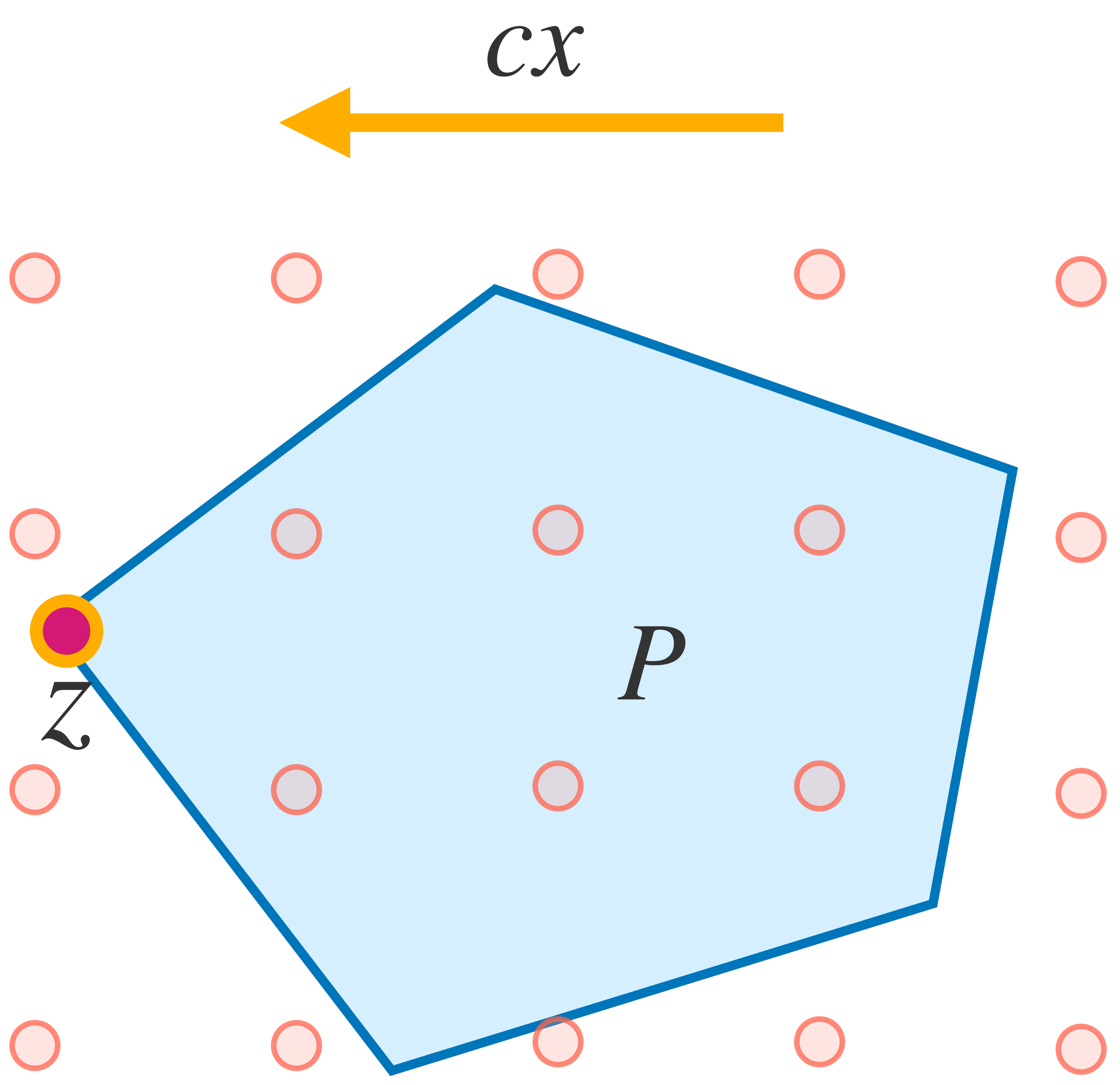

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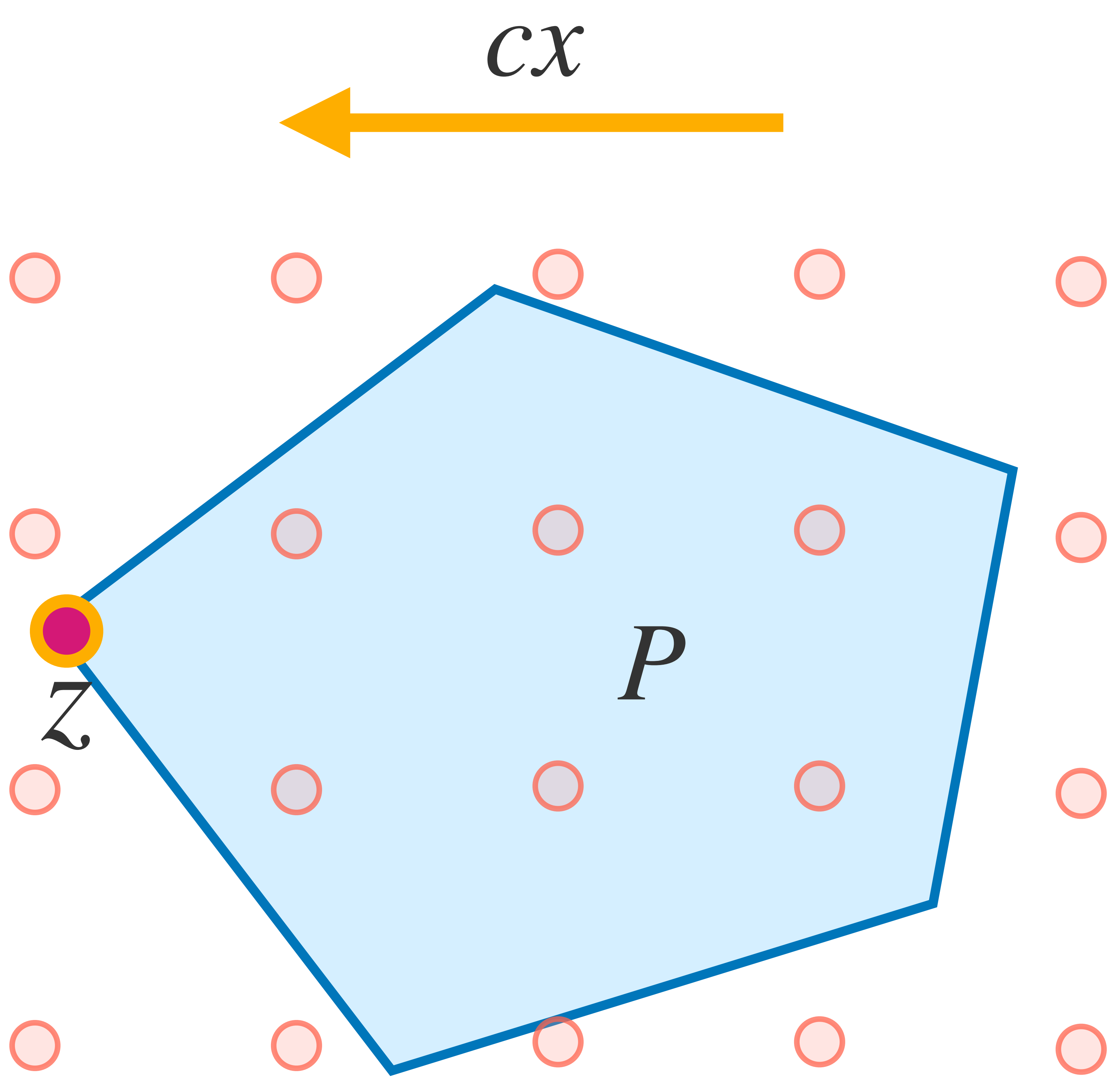

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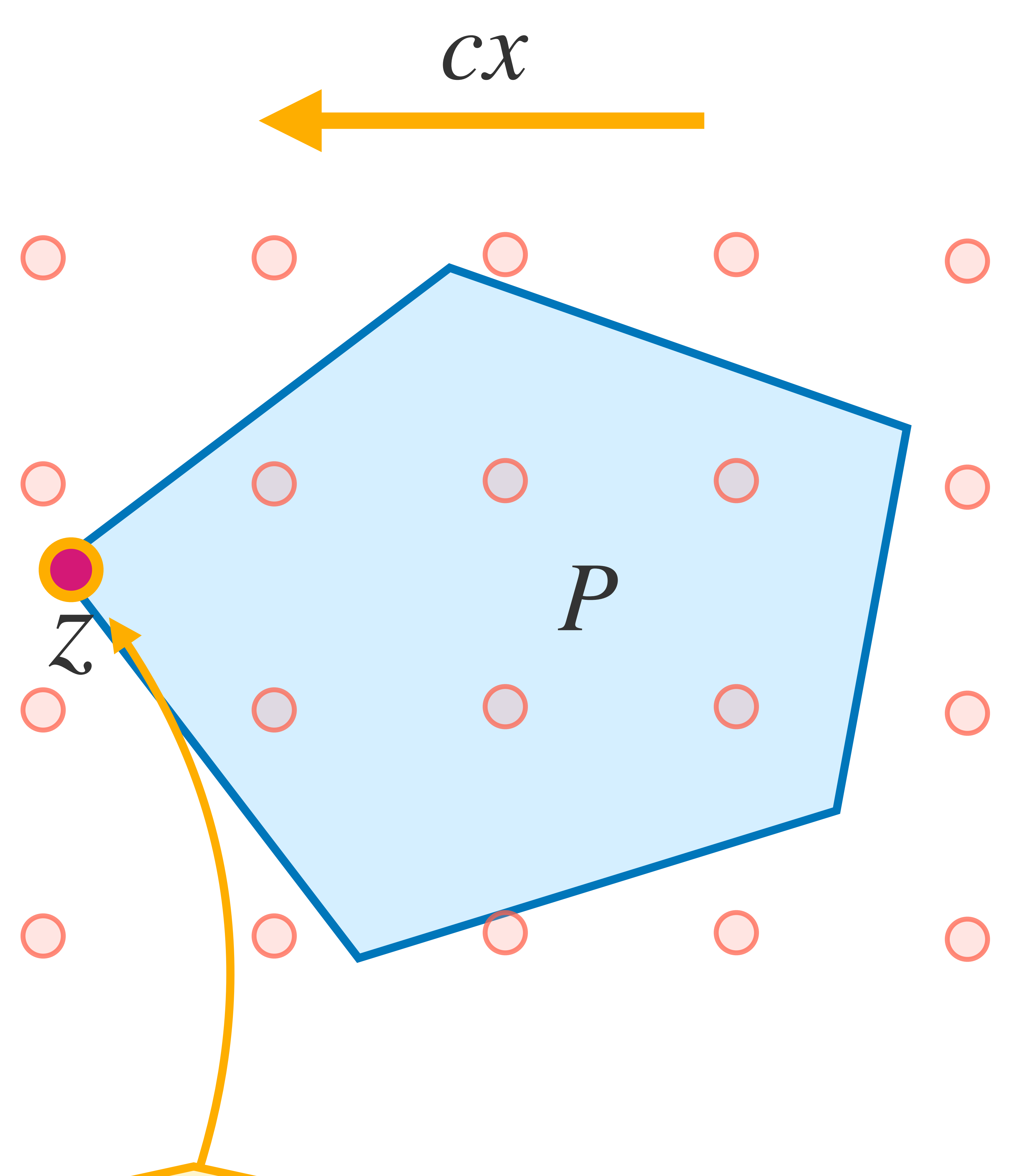

Remove this point so **better** solutions can be found!

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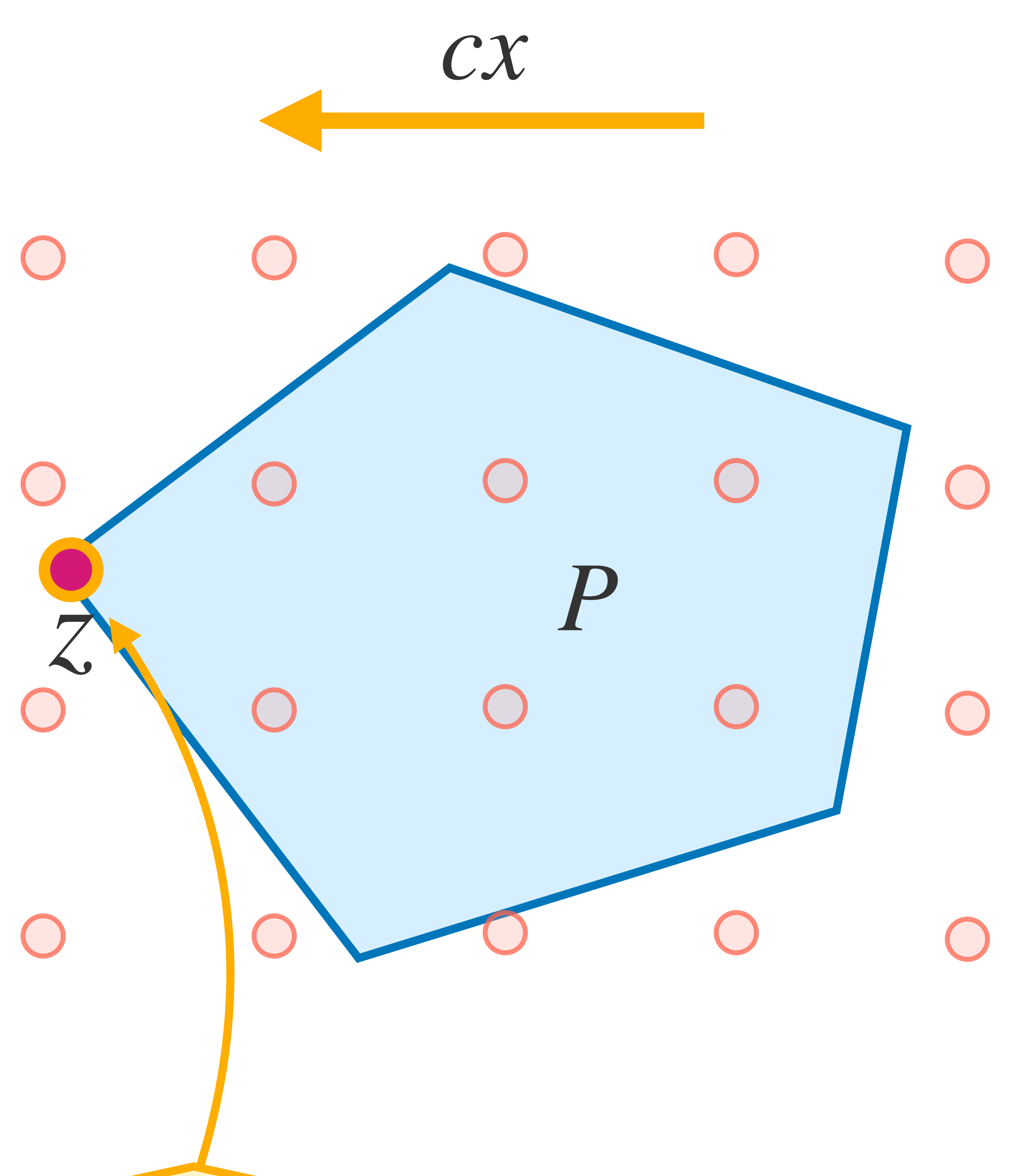
Run a linear program

- If solution is integral, **done**!
- Otherwise, **refine** the polytope until an integer solution can be found by linear programming
  $\rightarrow$ **remove non-integer** solutions by adding **additional constraints** to P
- Recuse

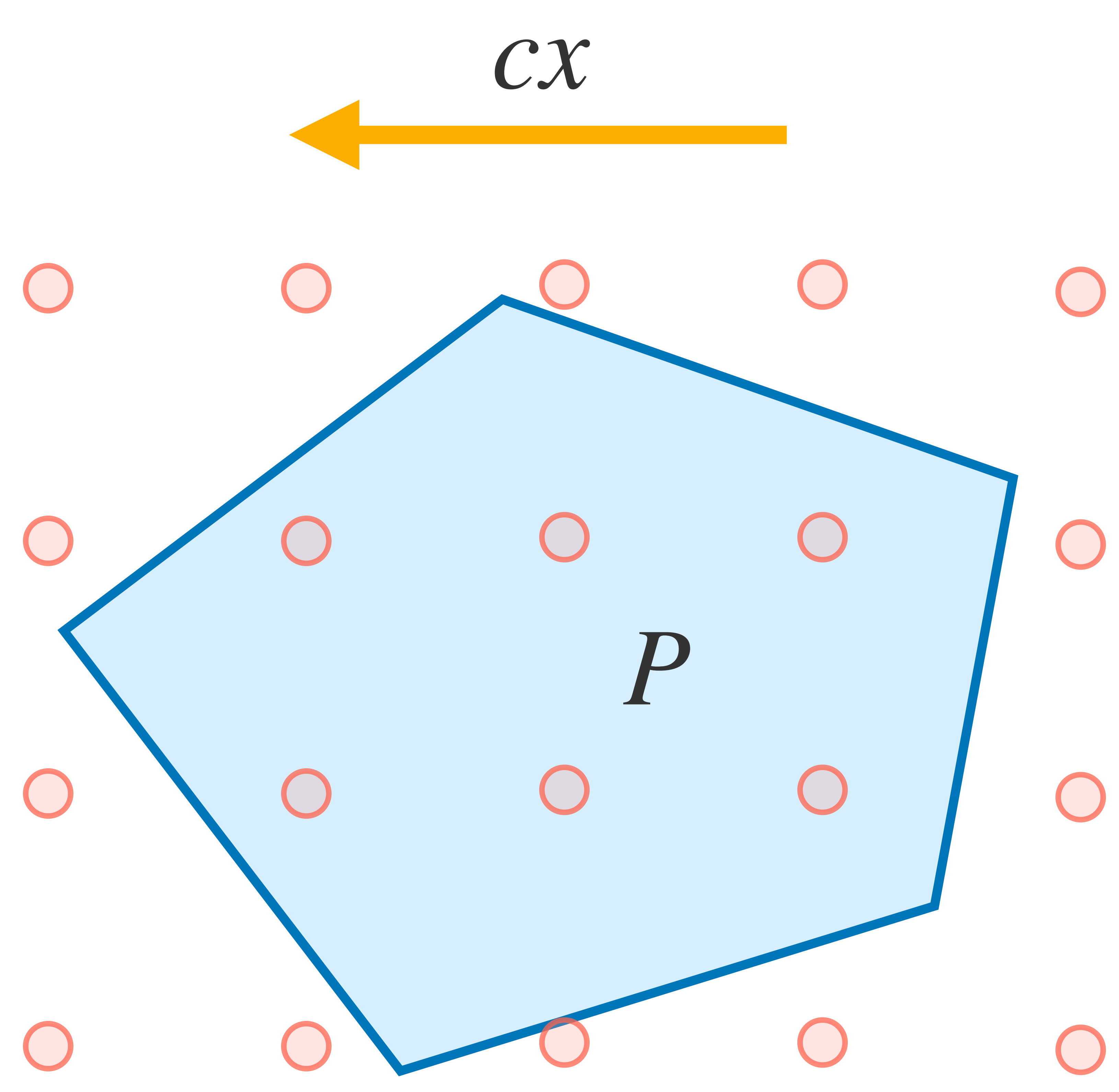


Remove this point so **better** solutions can be found!

Branch-and-Cut

Branch-and-Cut has two ways of **removing non-integer points** from P:

1. DPLL-style **branch**ing on linear inequalities
2. **Cut**ting planes



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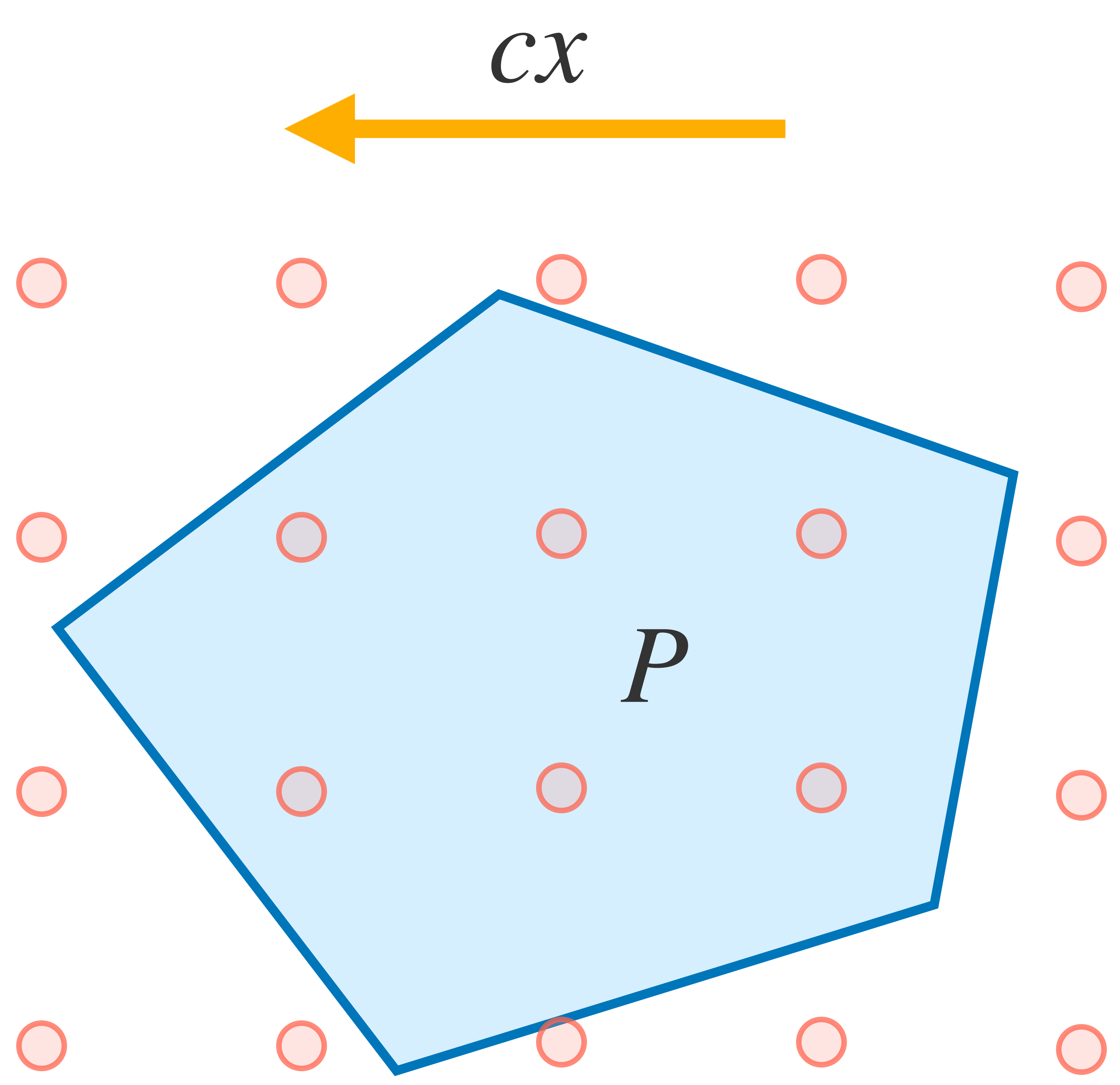

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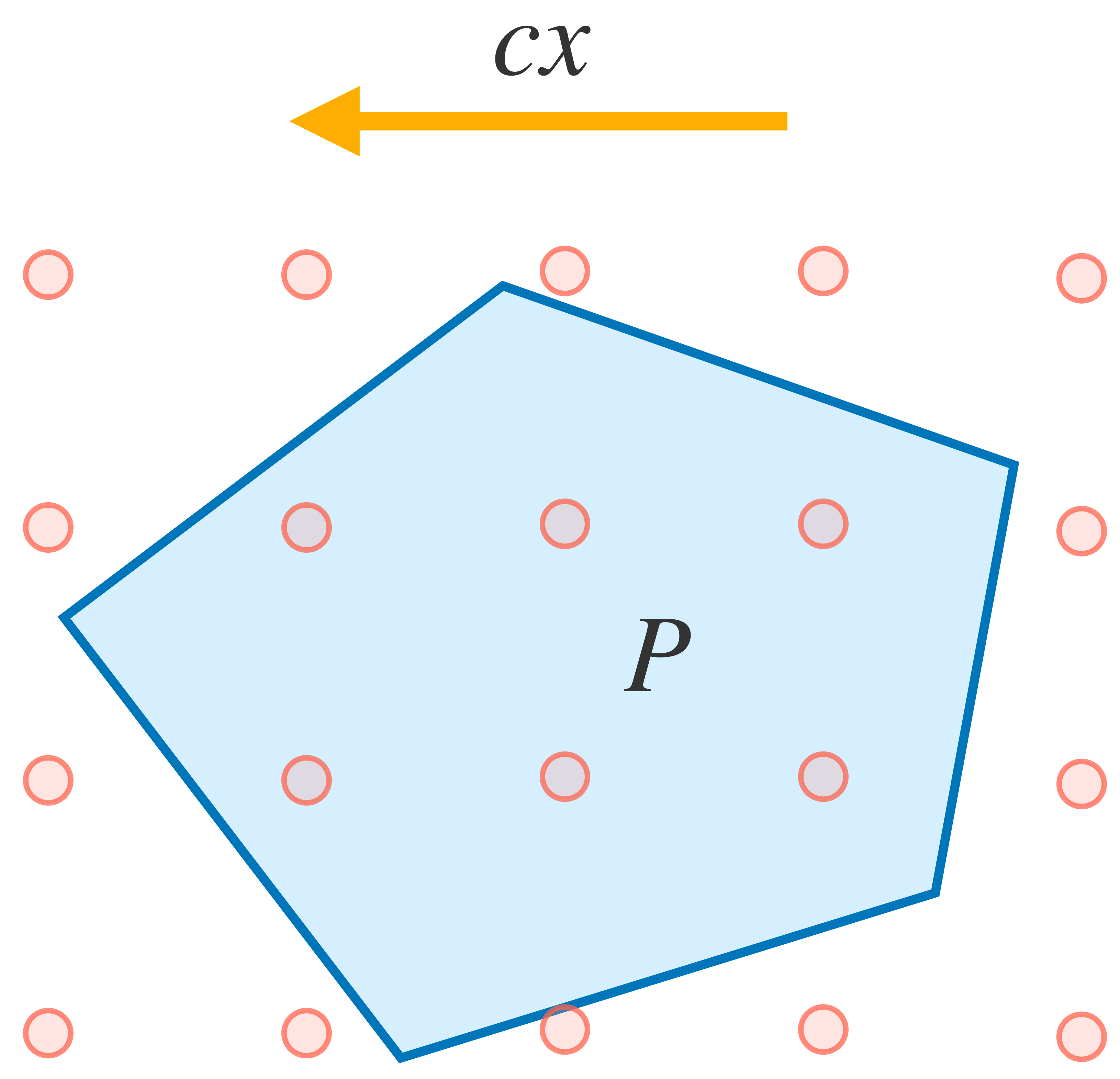

Branching —"Break P up into subpolytopes"

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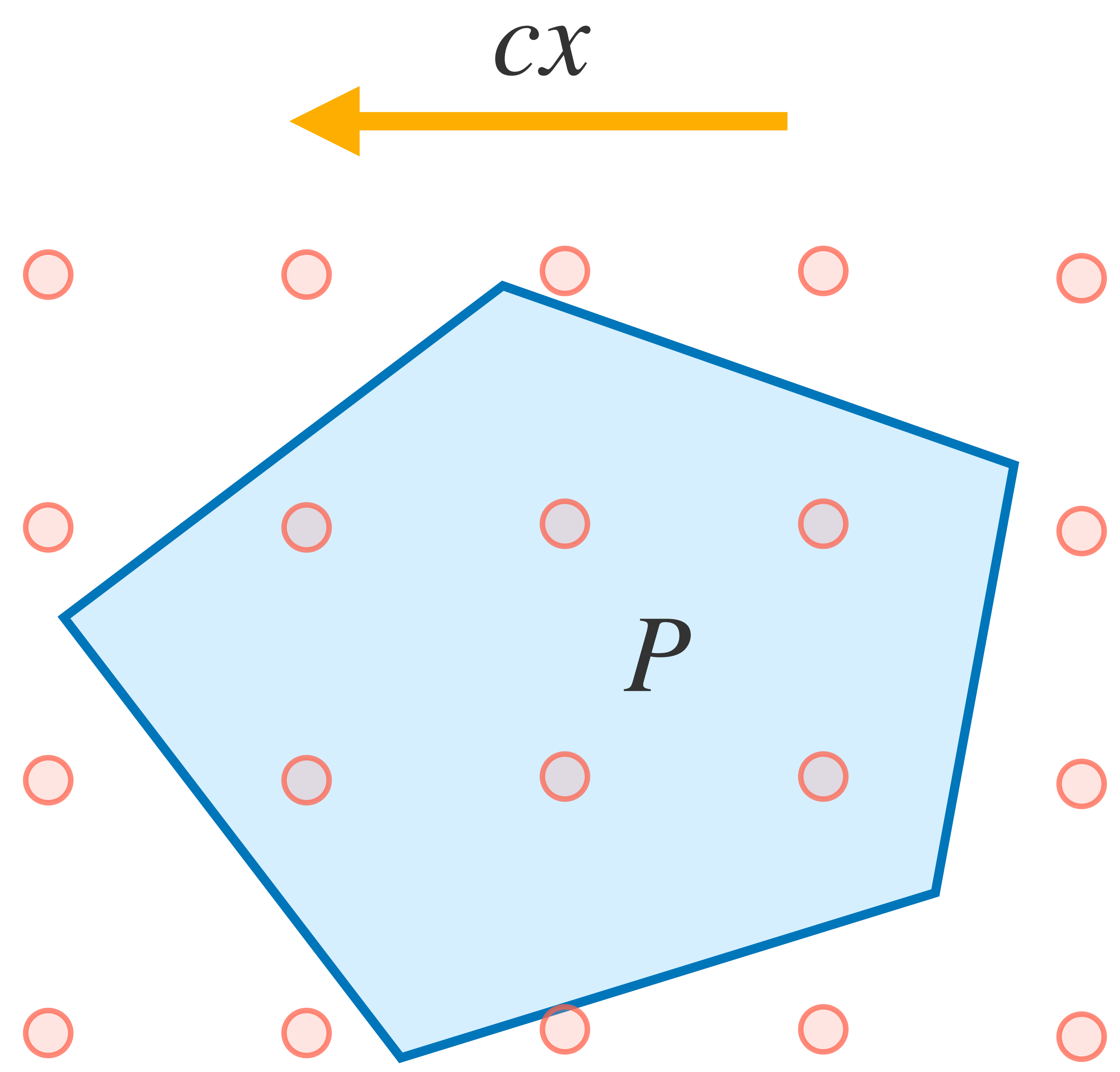

<u>**Branching**</u> —"Break P up into subpolytopes"

1. Heuristically choose an **integer**-linear inequality $ax \geq b$
2. Break P into $P \cap \{ax \geq b\}$ and $P \cap \{ax \leq b - 1\}$

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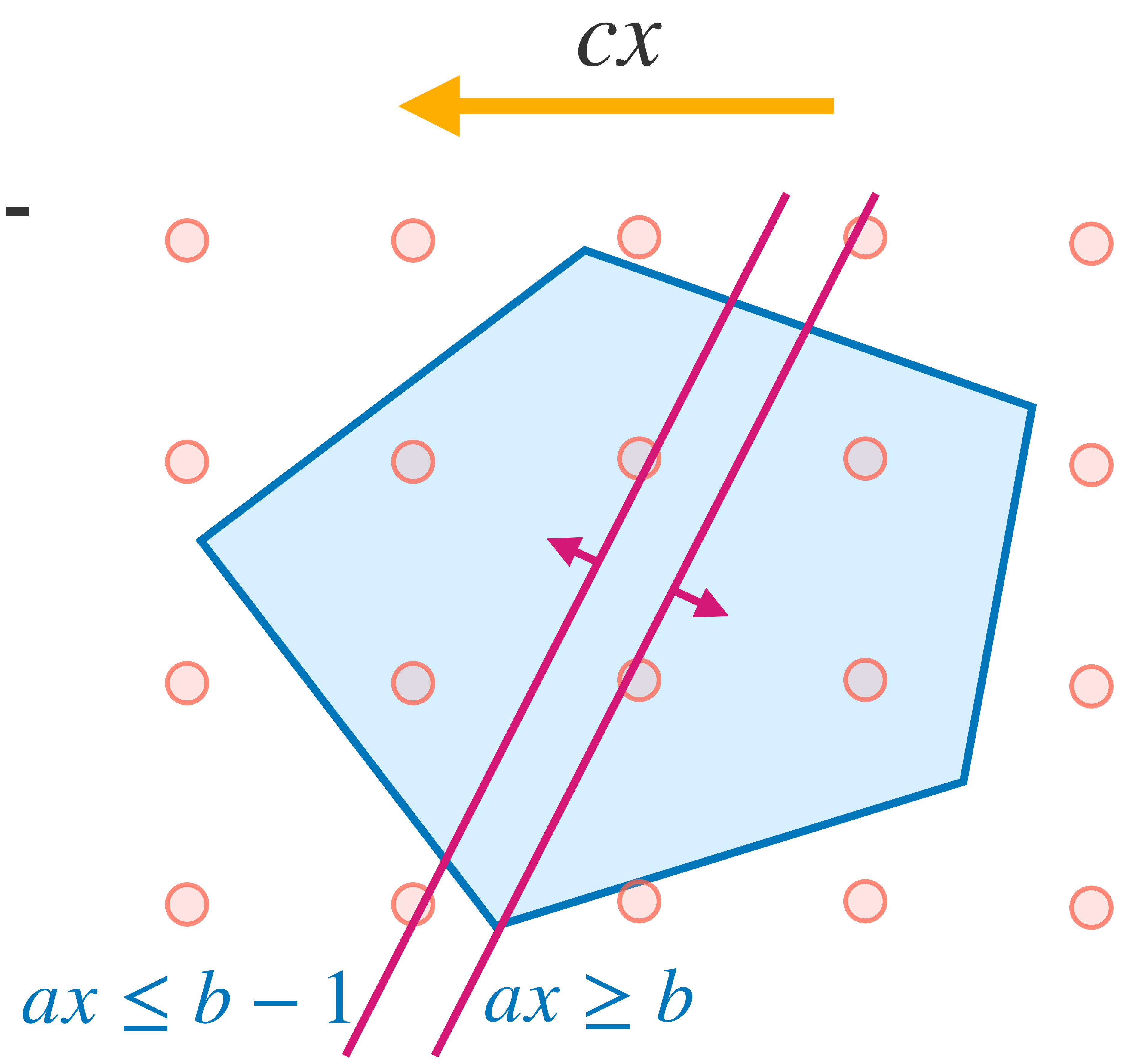

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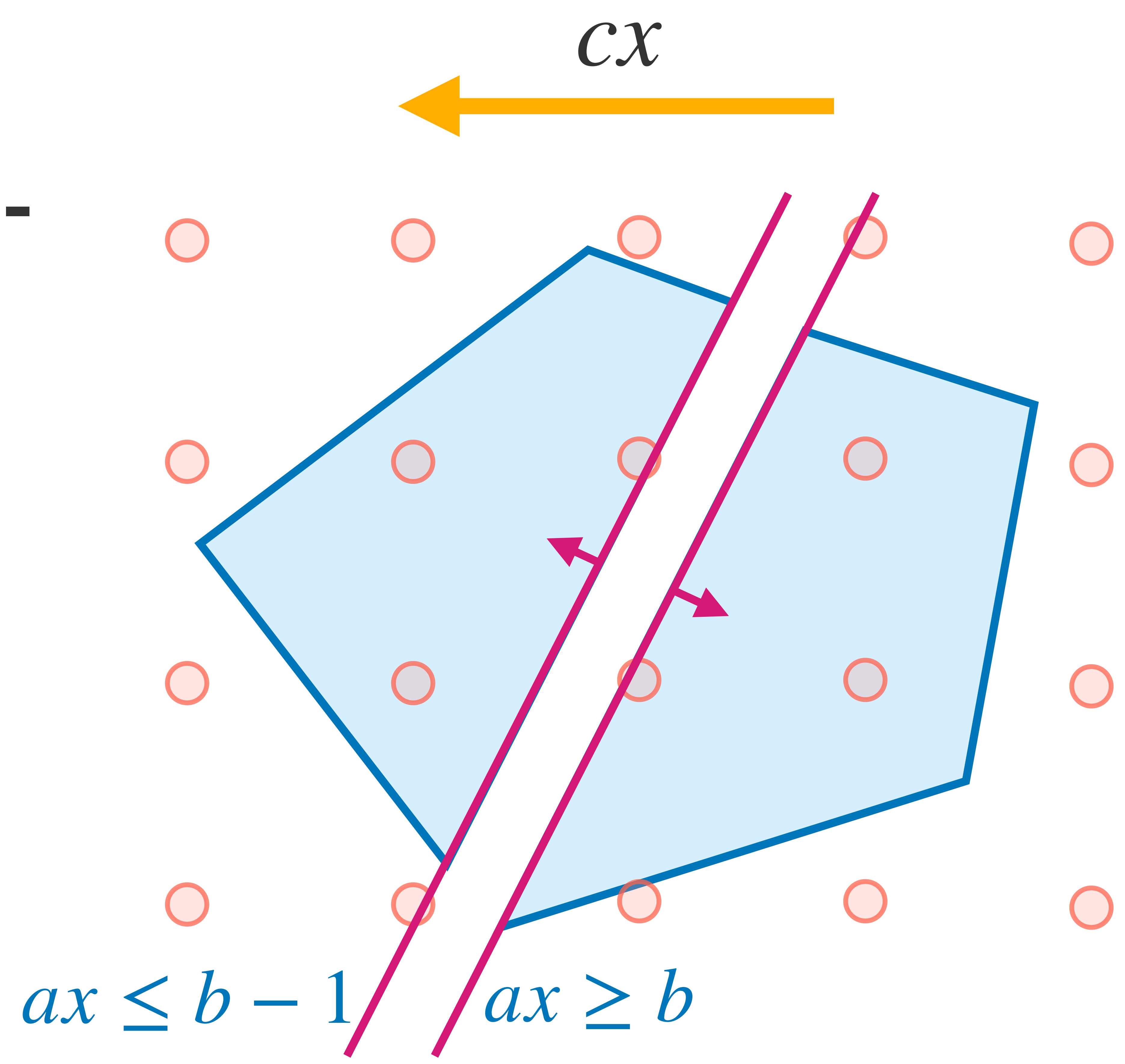

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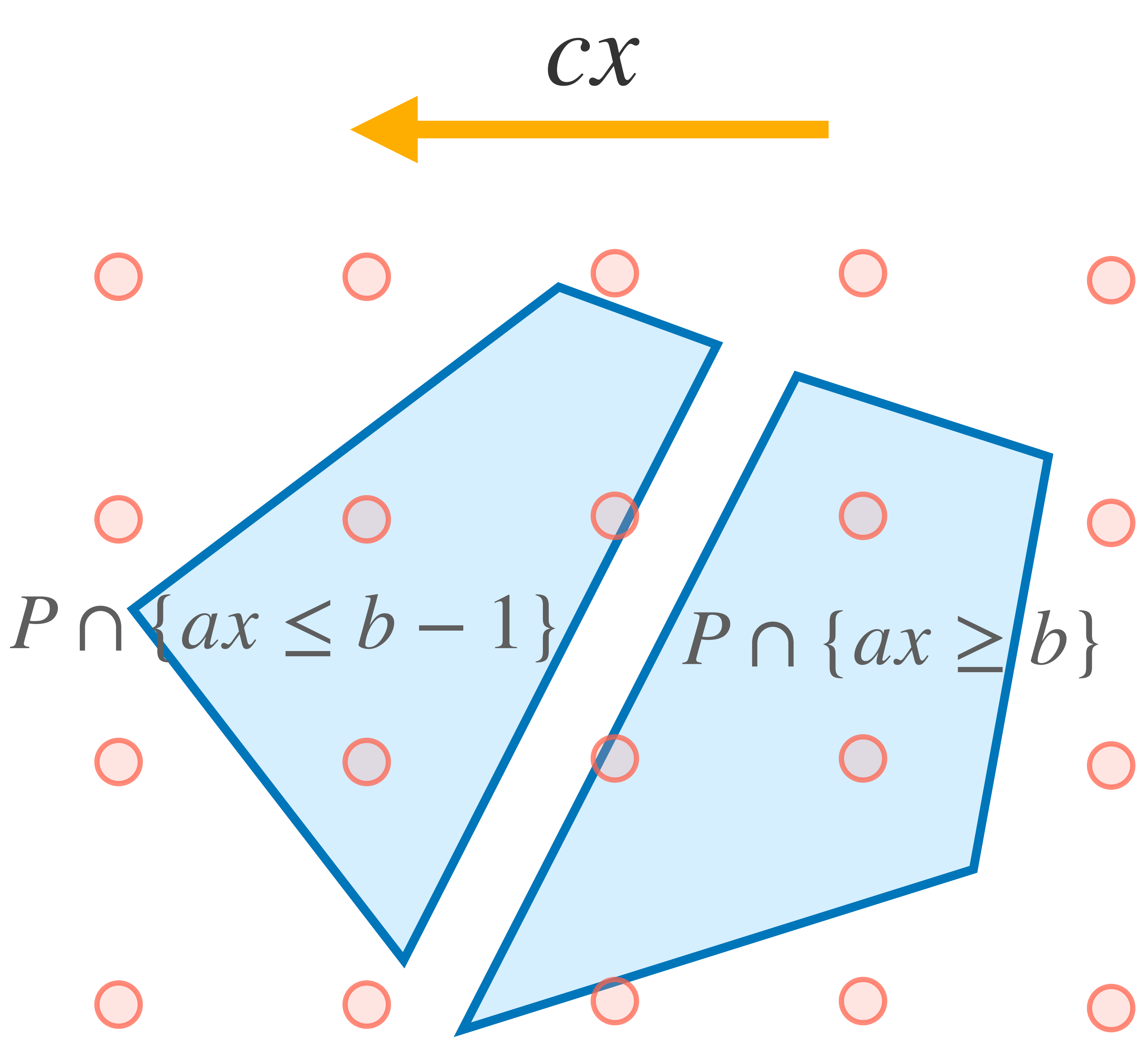

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2. Break P into $P \cap \{ax \geq b\}$ and $P \cap \{ax \leq b-1\}$

Preserves integer solutions! $x \in \mathbb{Z}^n$ satisfies $ax \leq b-1$ or $ax \geq b$

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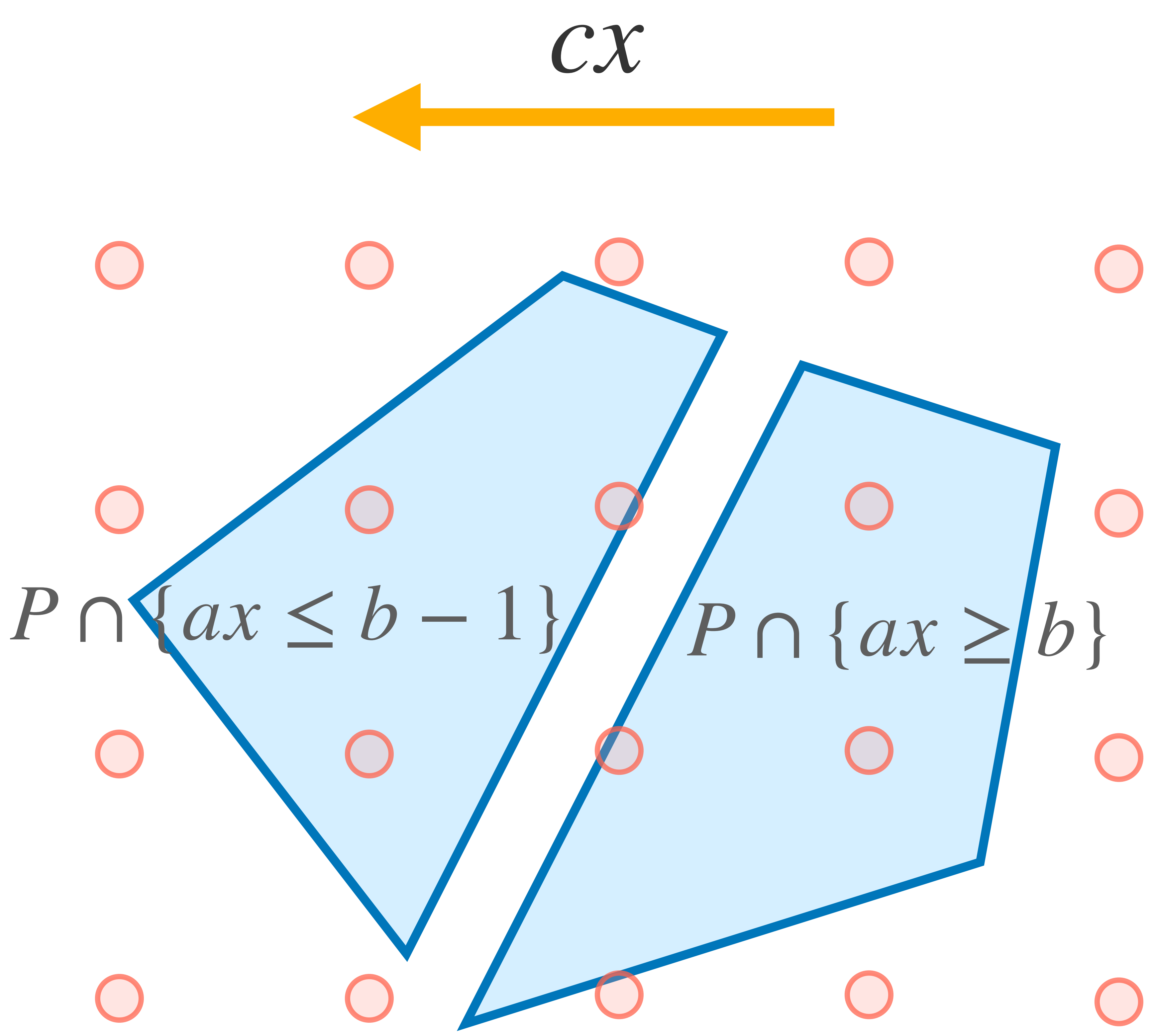

<u>**Branching**</u> —"Break P up into subpolytopes"

1. Heuristically choose an **integer**-linear inequality $ax \ge b$
2. **Recurse** on $P \cap \{ax \ge b\}$ and $P \cap \{ax \le b - 1\}$

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Branch-and-Cut has two ways of **removing non-integer points** from P:

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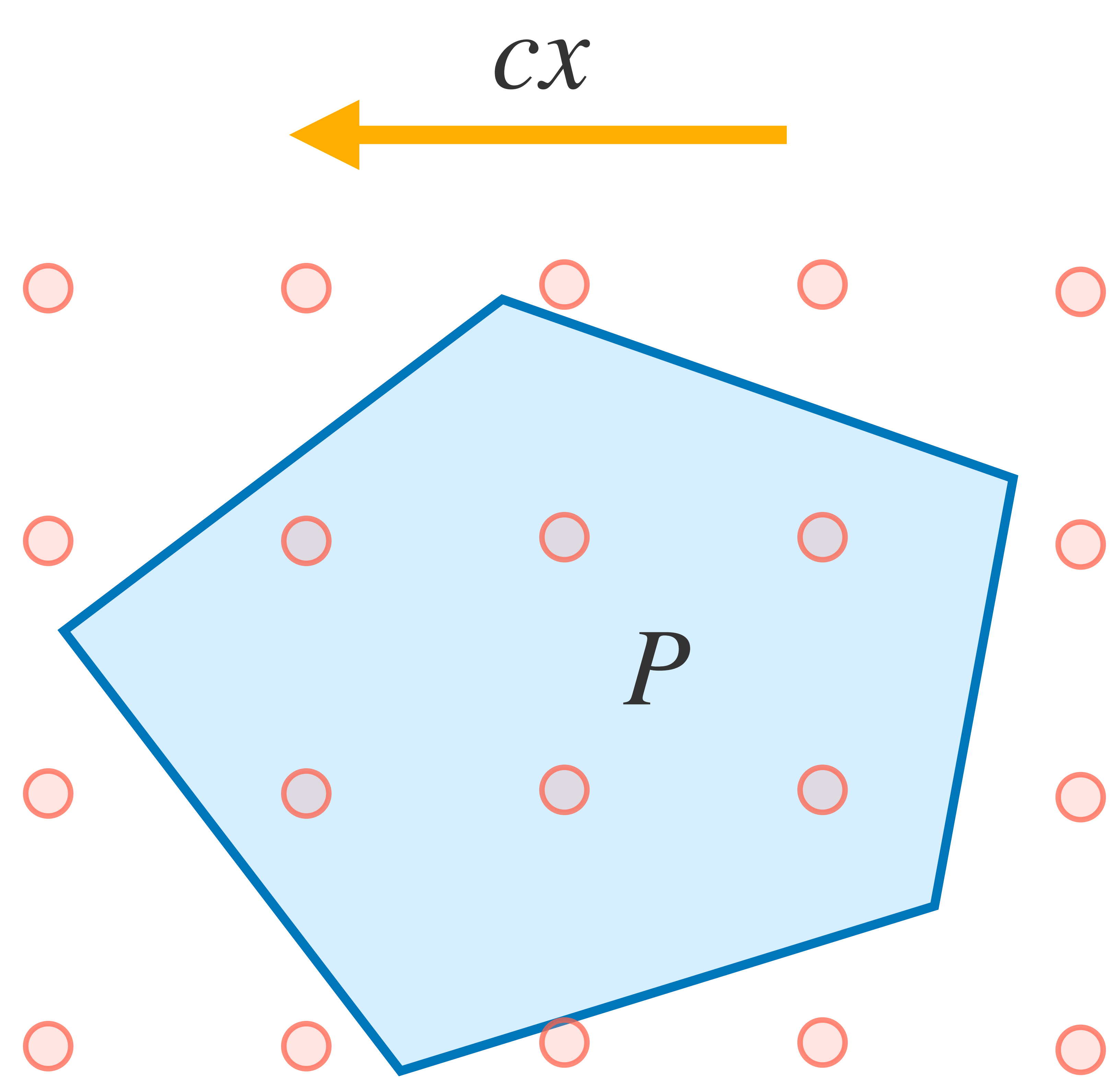
<u>**Cutting Planes**</u> — "Remove corners of P"



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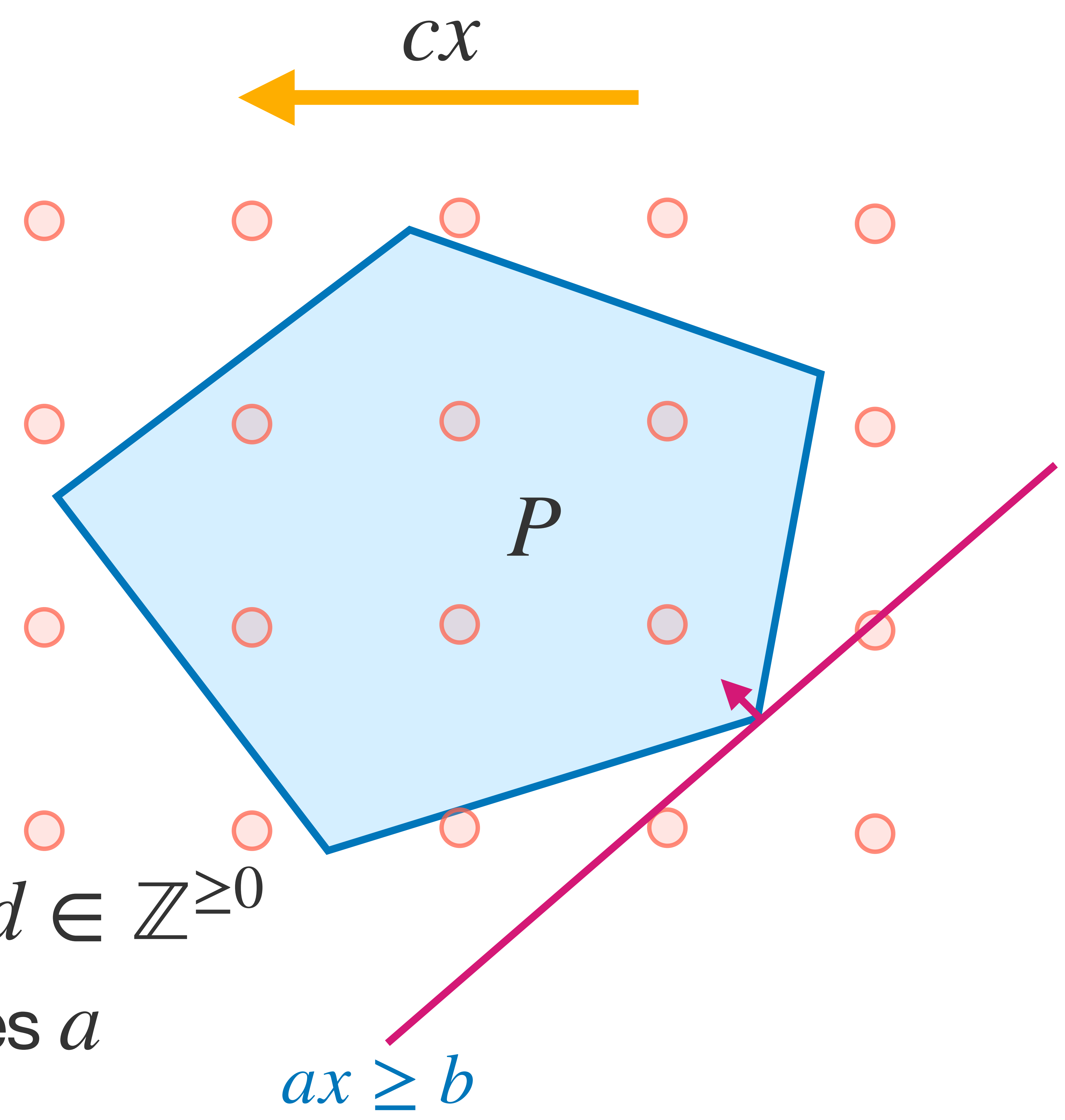
1. Choose an integer-linear inequality $ax \geq b$, and $d \in \mathbb{Z}^{\geq 0}$ s.t. every point in P satisfies $ax \geq b$ and d divides a



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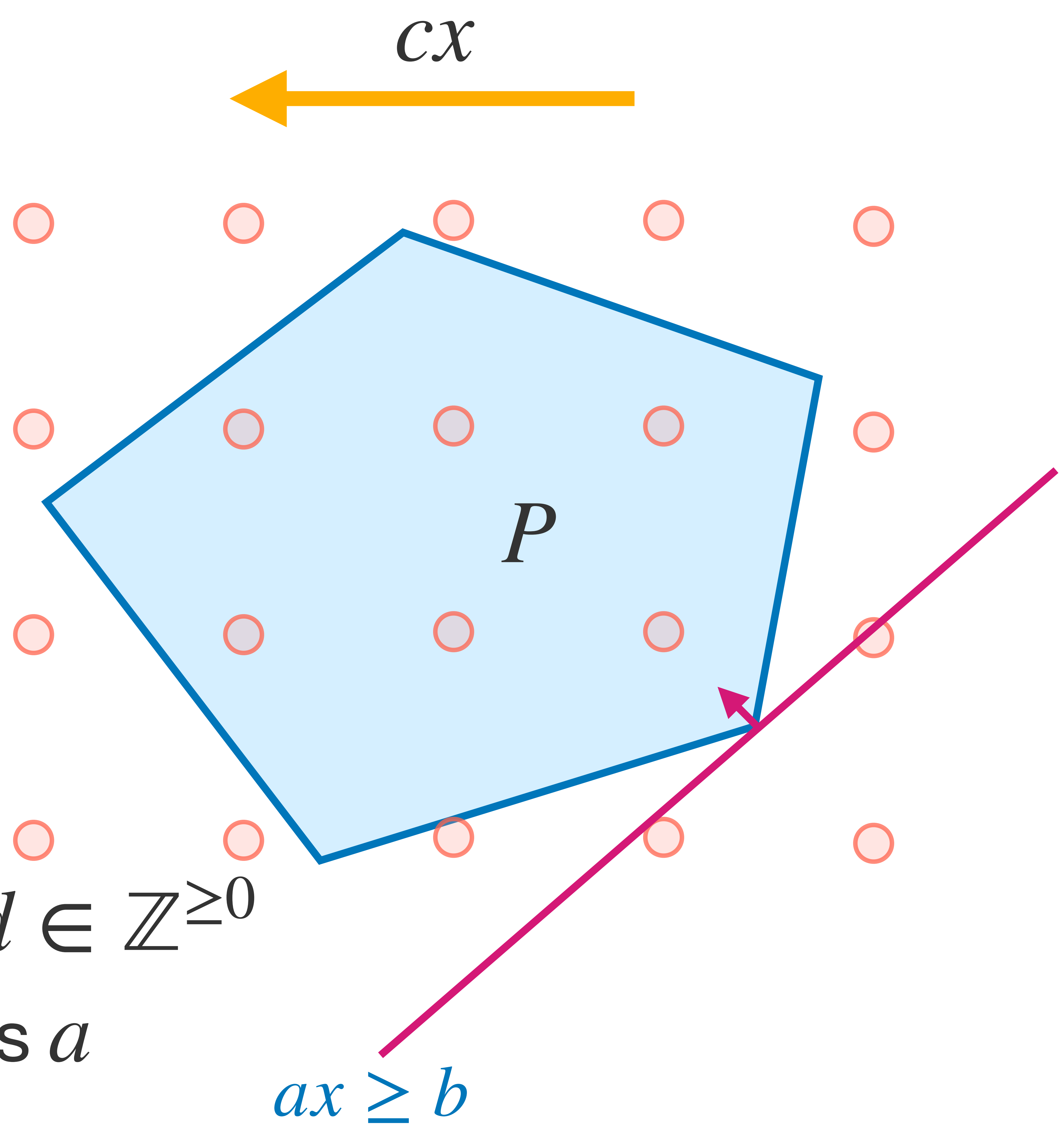

Cutting Planes — "Remove corners of P"

1. Choose an integer-linear inequality $ax \geq b$, and $d \in \mathbb{Z}^{\geq 0}$ s.t. every point in P satisfies $ax \geq b$ and d divides a
2. Add $(a/d)x \geq \lceil b/d \rceil$

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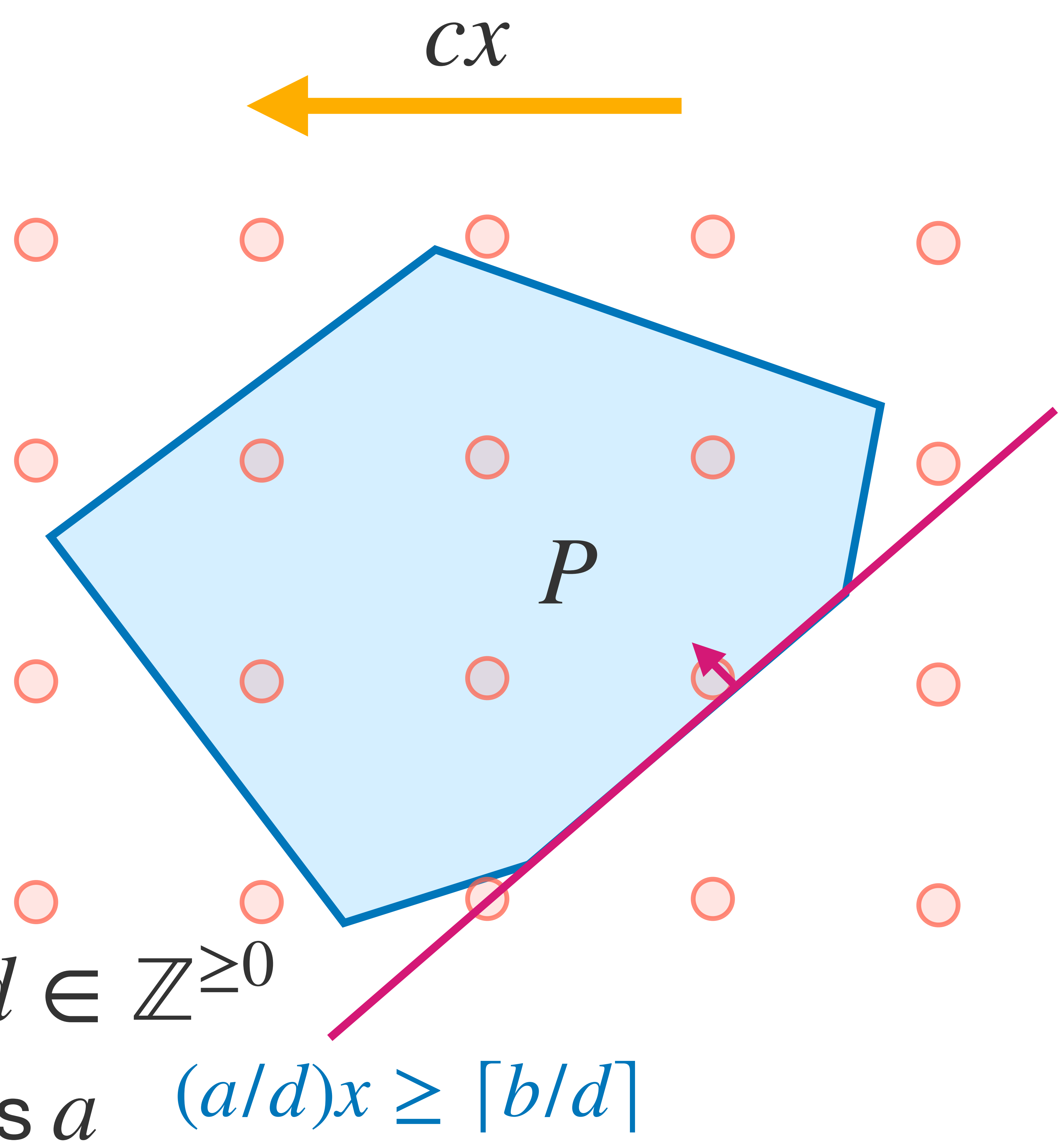

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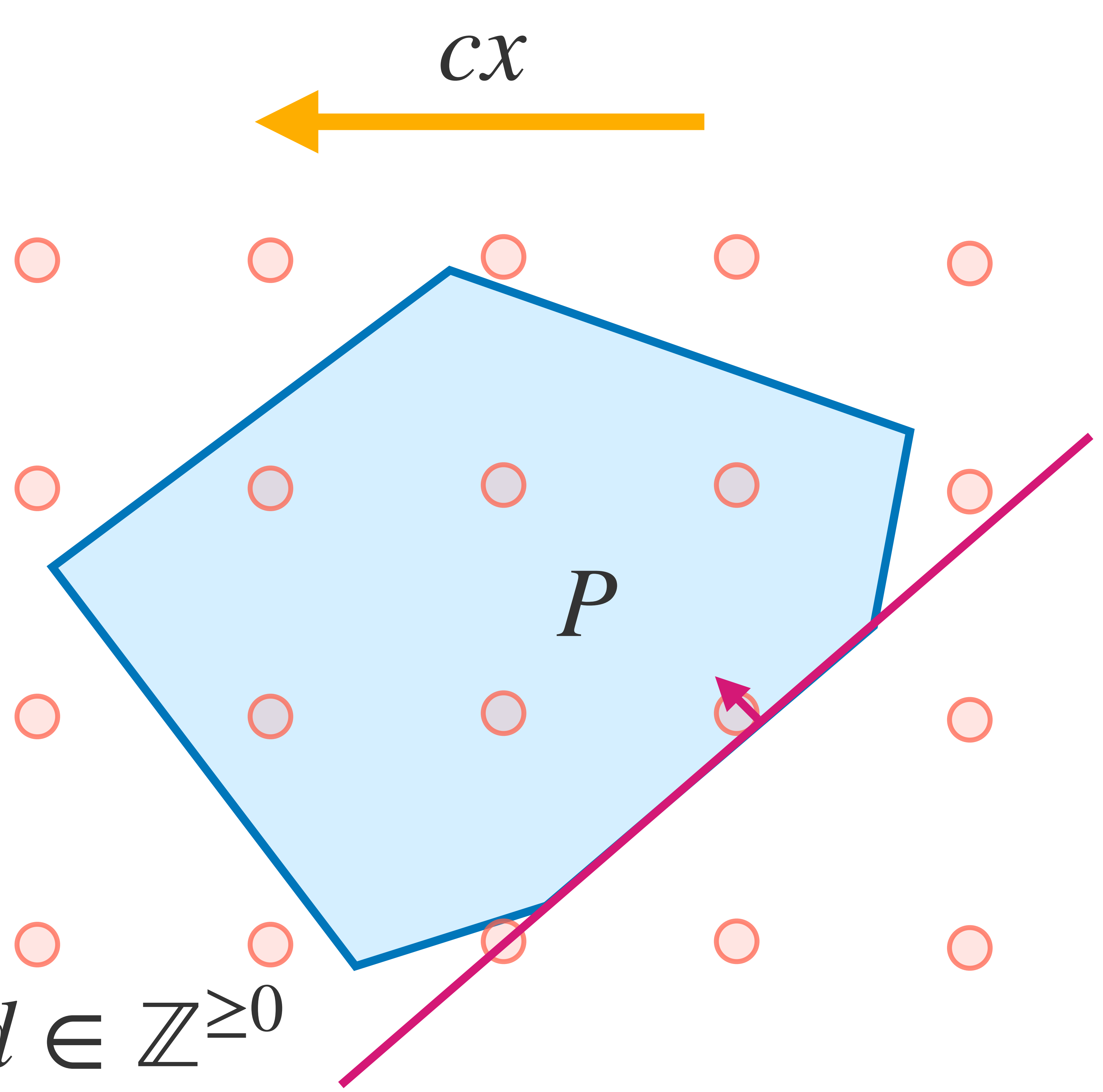

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Preserves **integer** points in P

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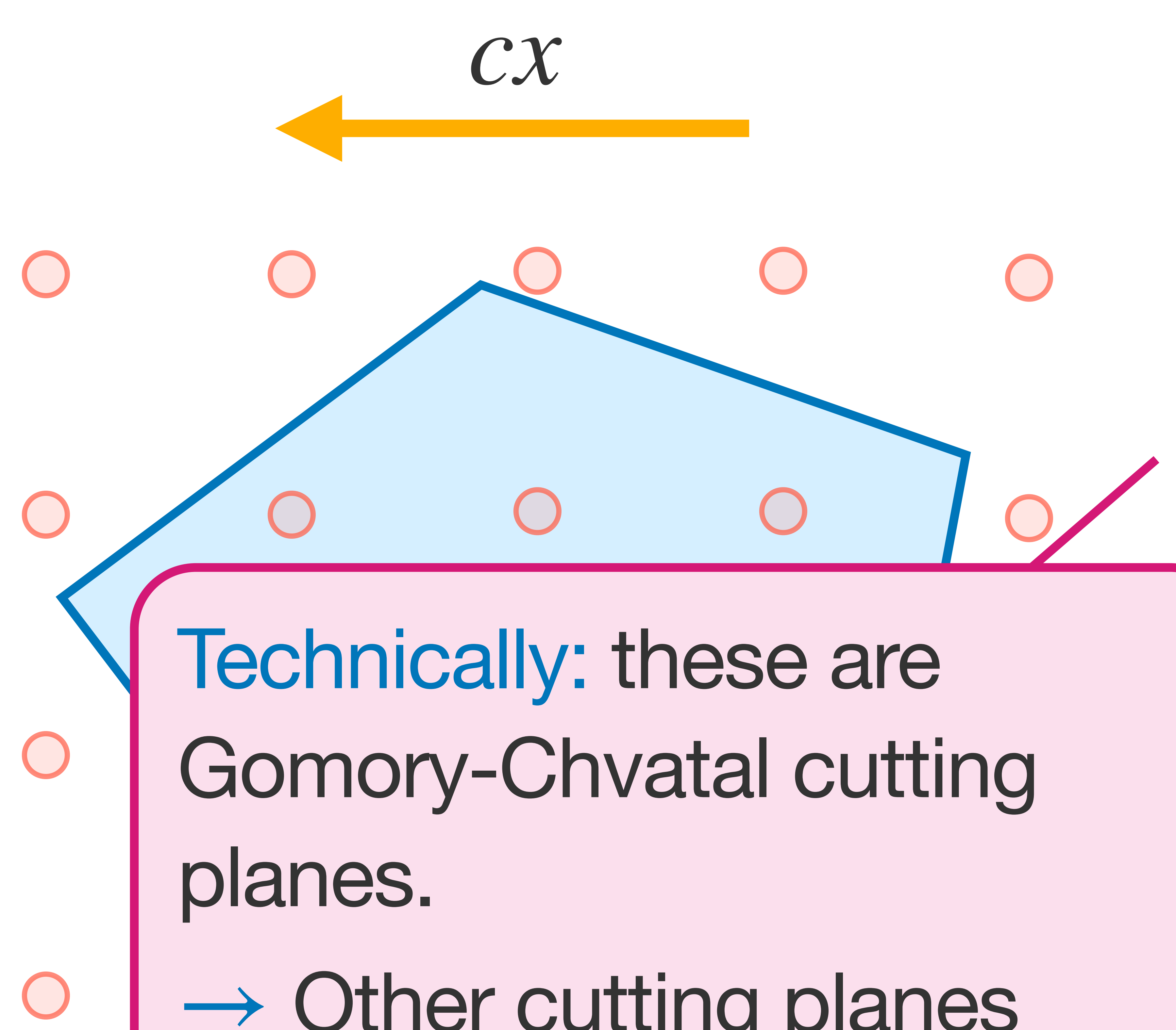

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Preserves **integer** points

Technically: these are Gomory-Chvatal cutting planes.
→ Other cutting planes have been considered as well.
→ What we talk about today applies to them as well

Branch-and-Cut

Idea: Try to use **linear** programming to solve **integer** programming!

Branch and Cut Template

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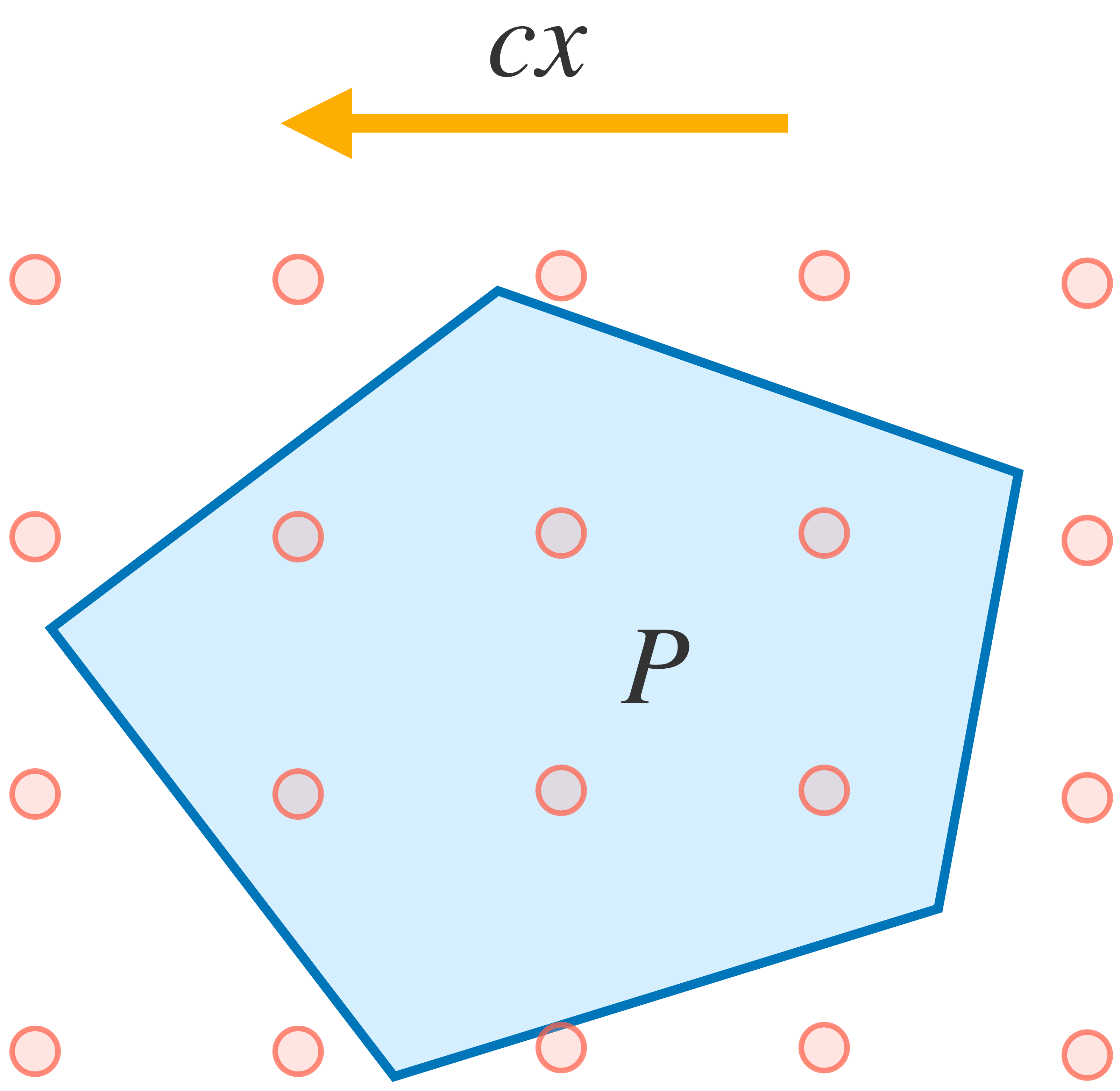
3. Repeat.



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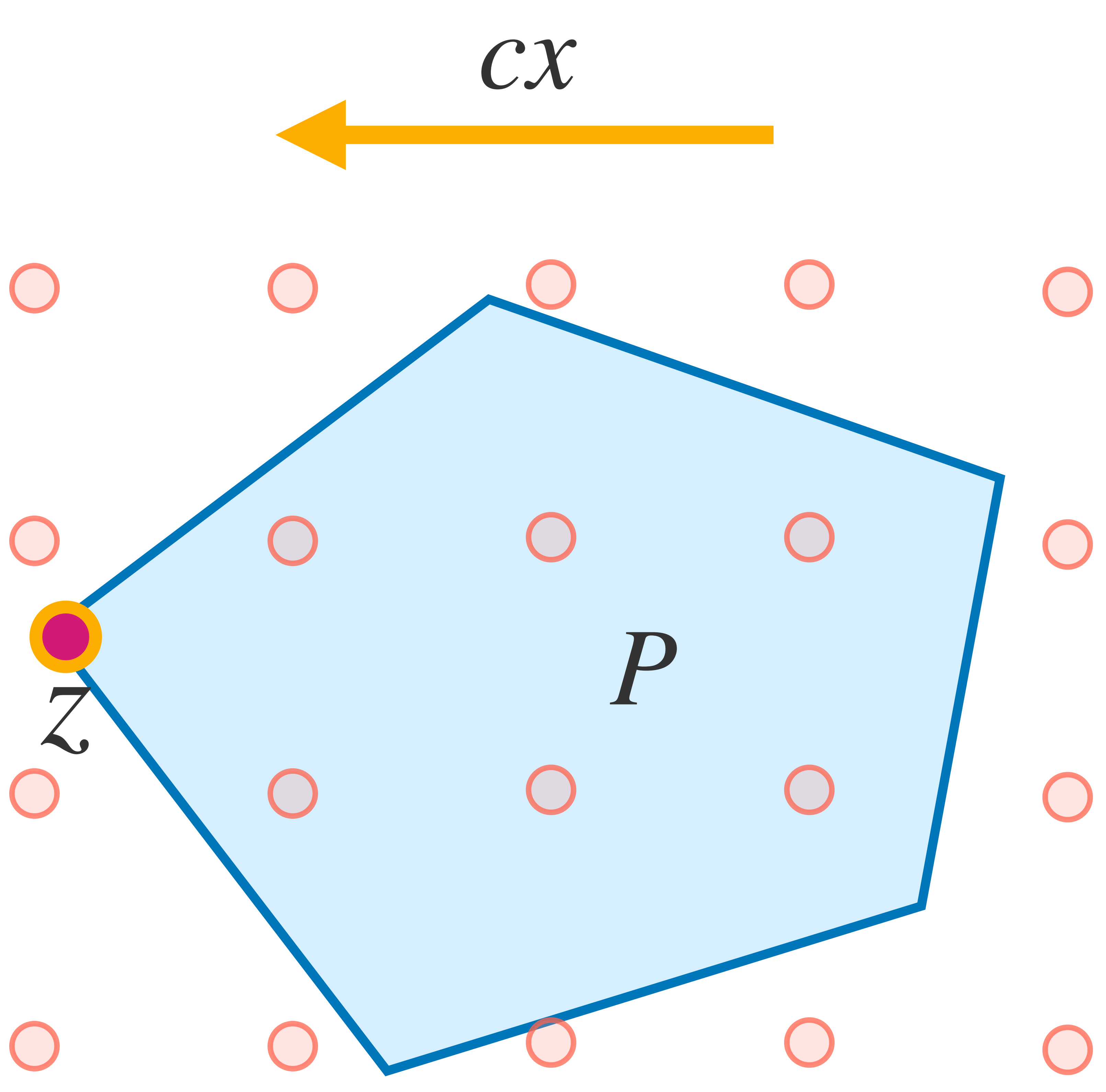

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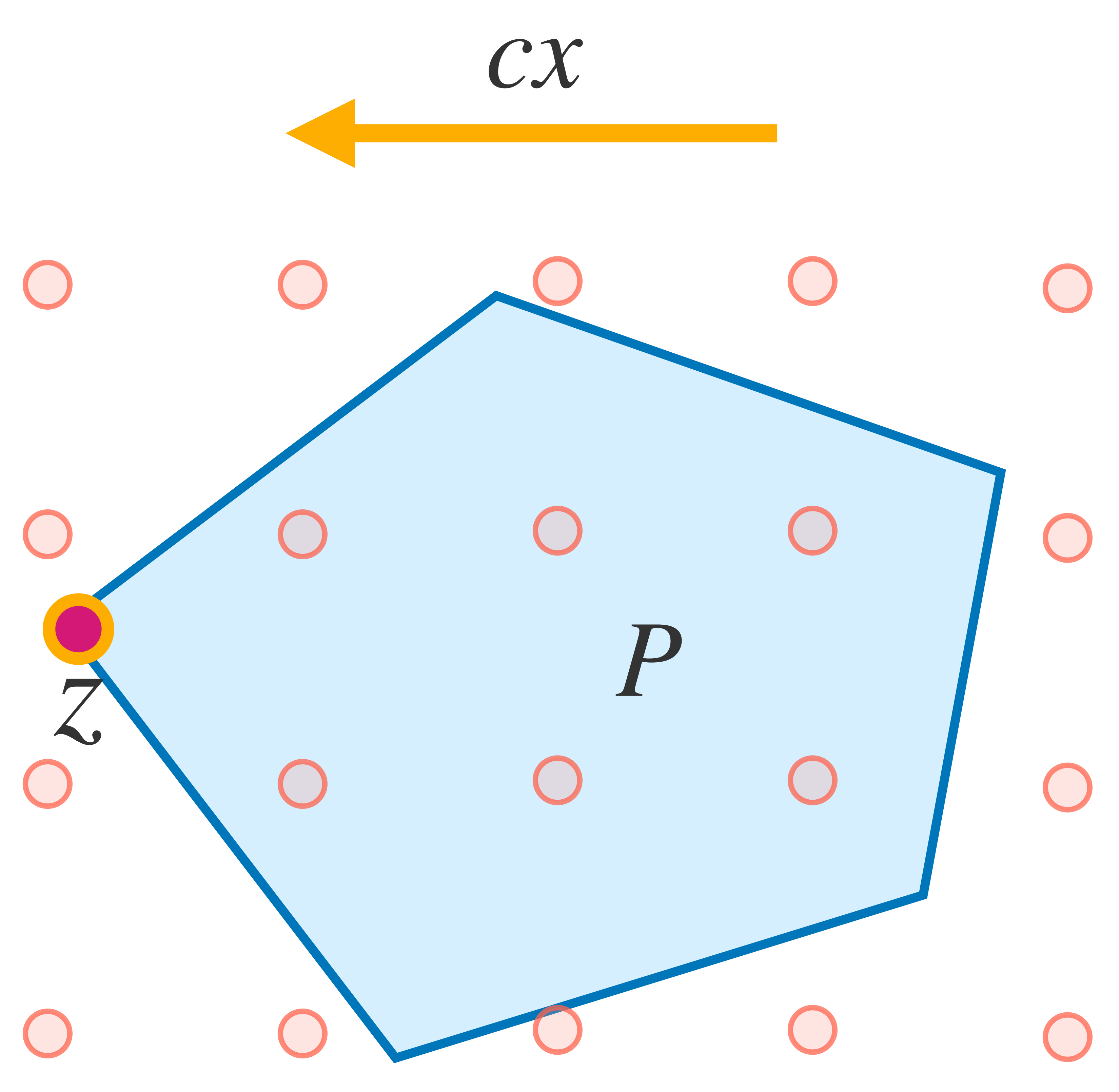

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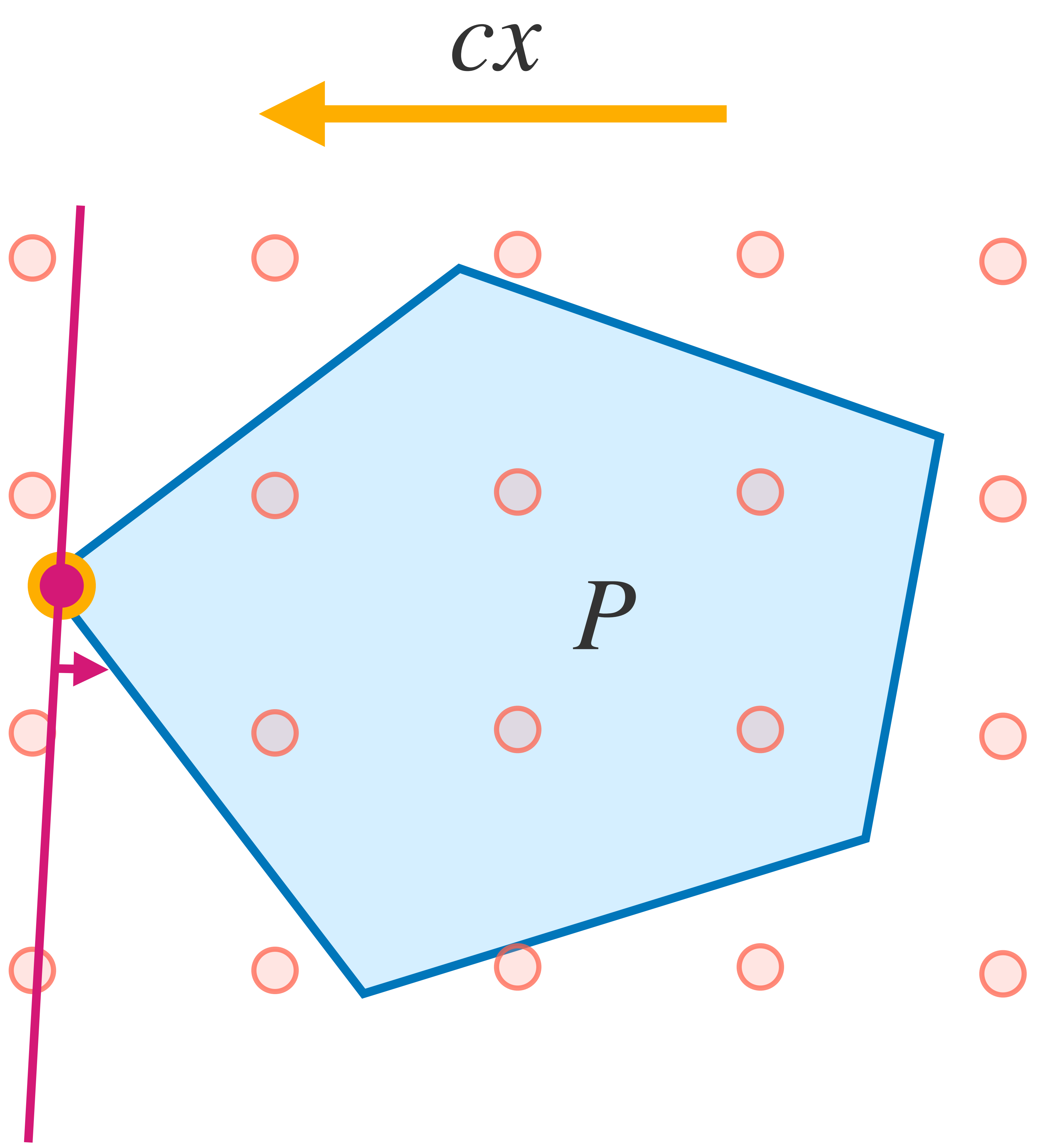

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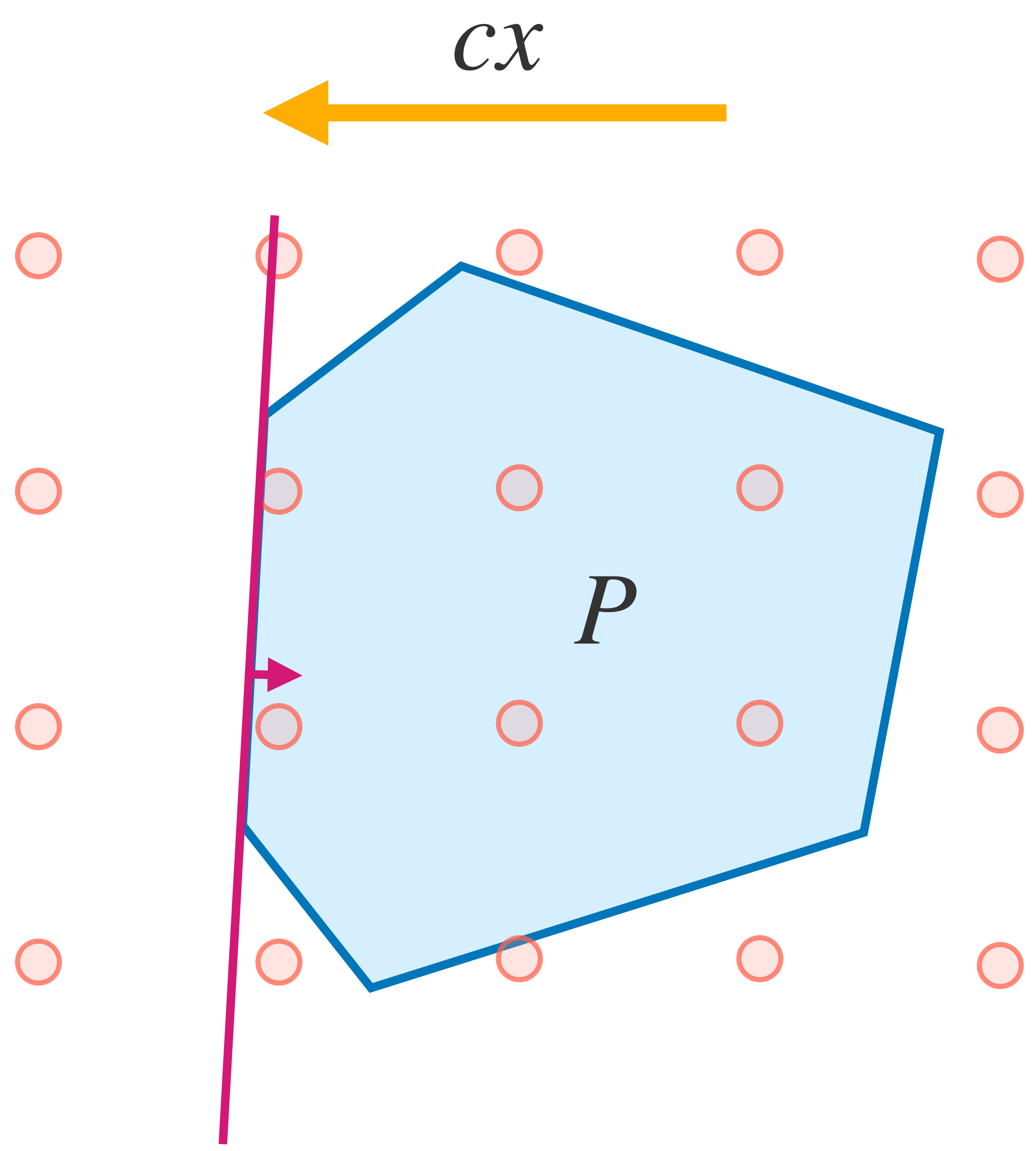

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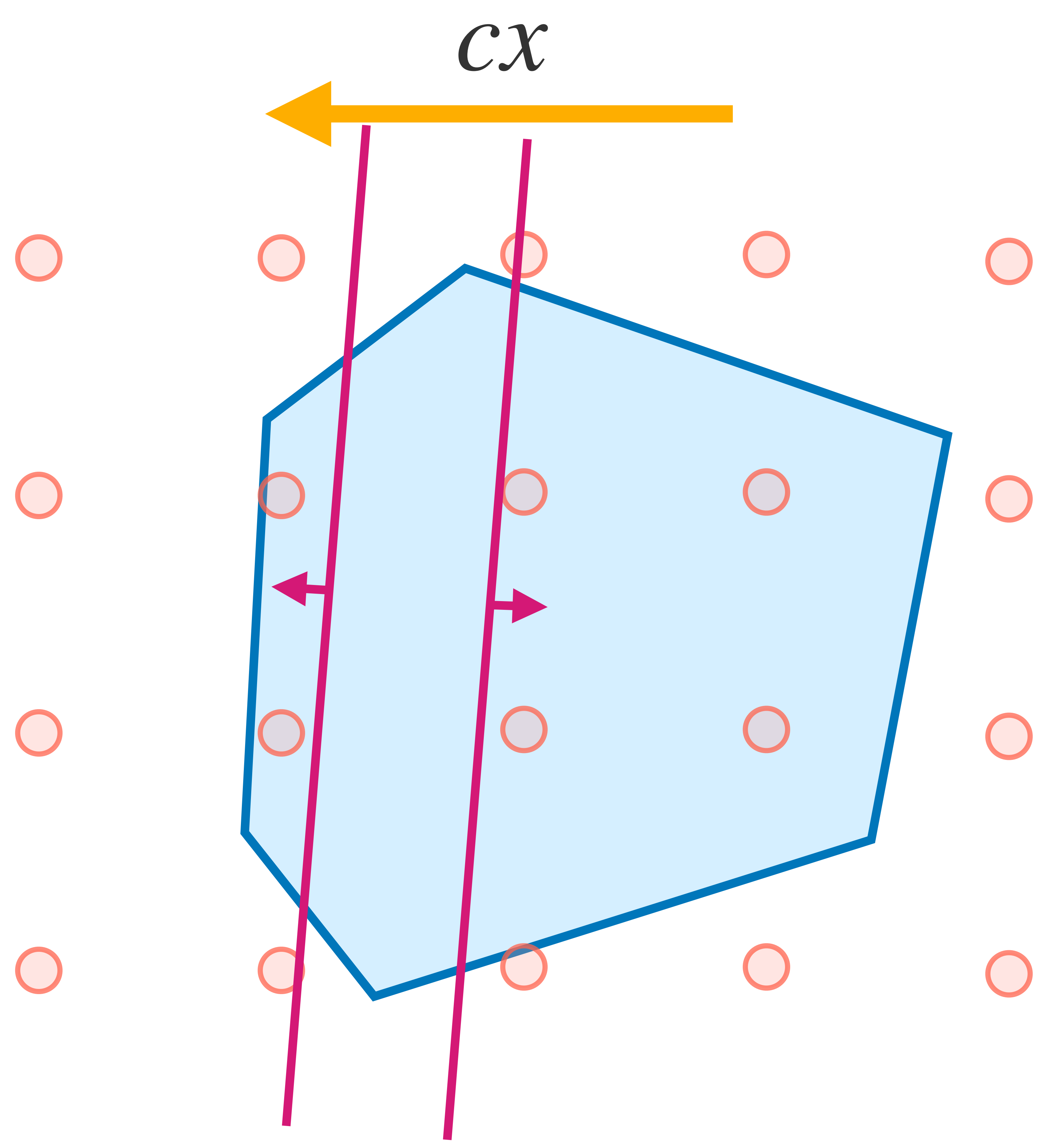

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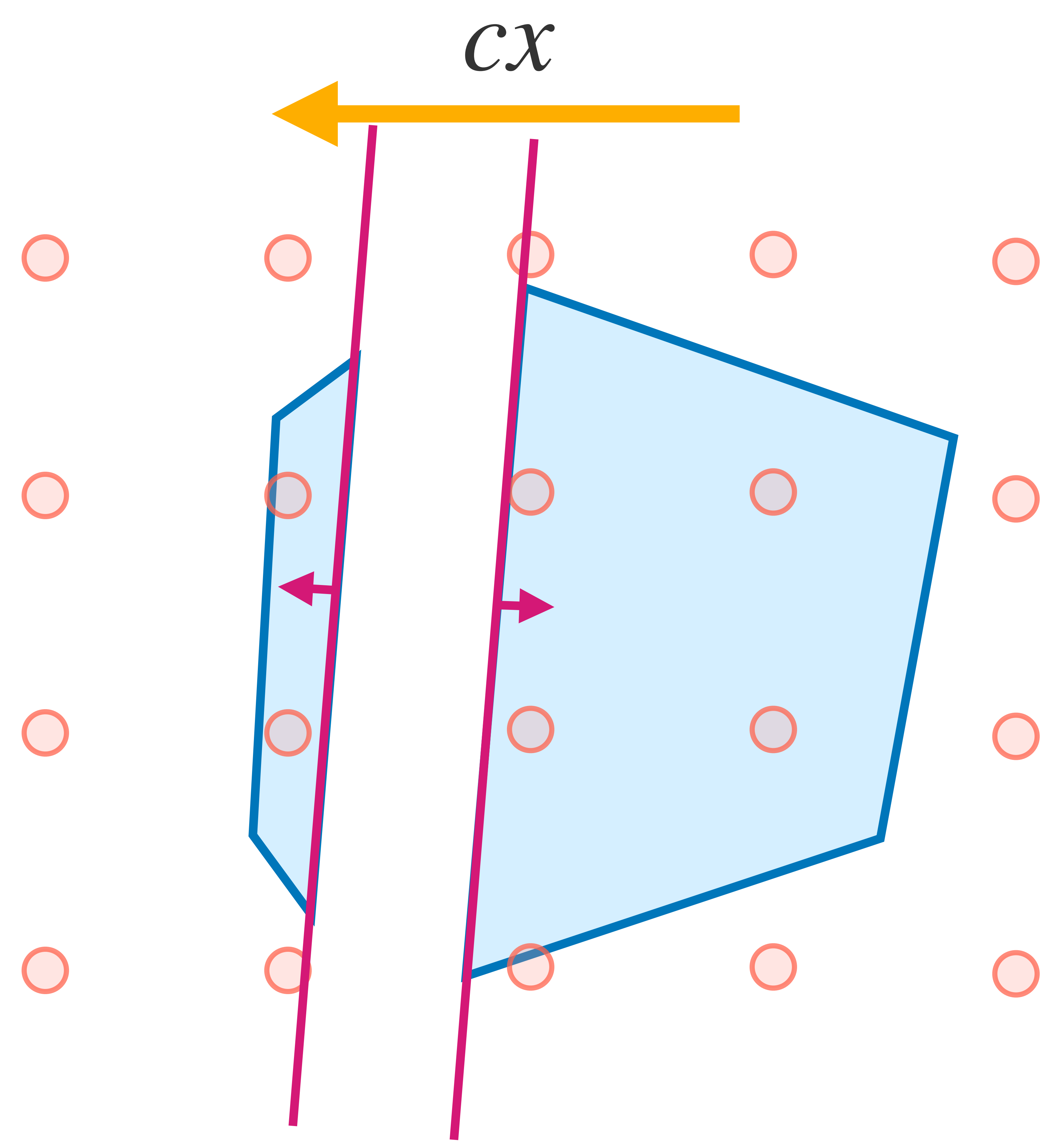

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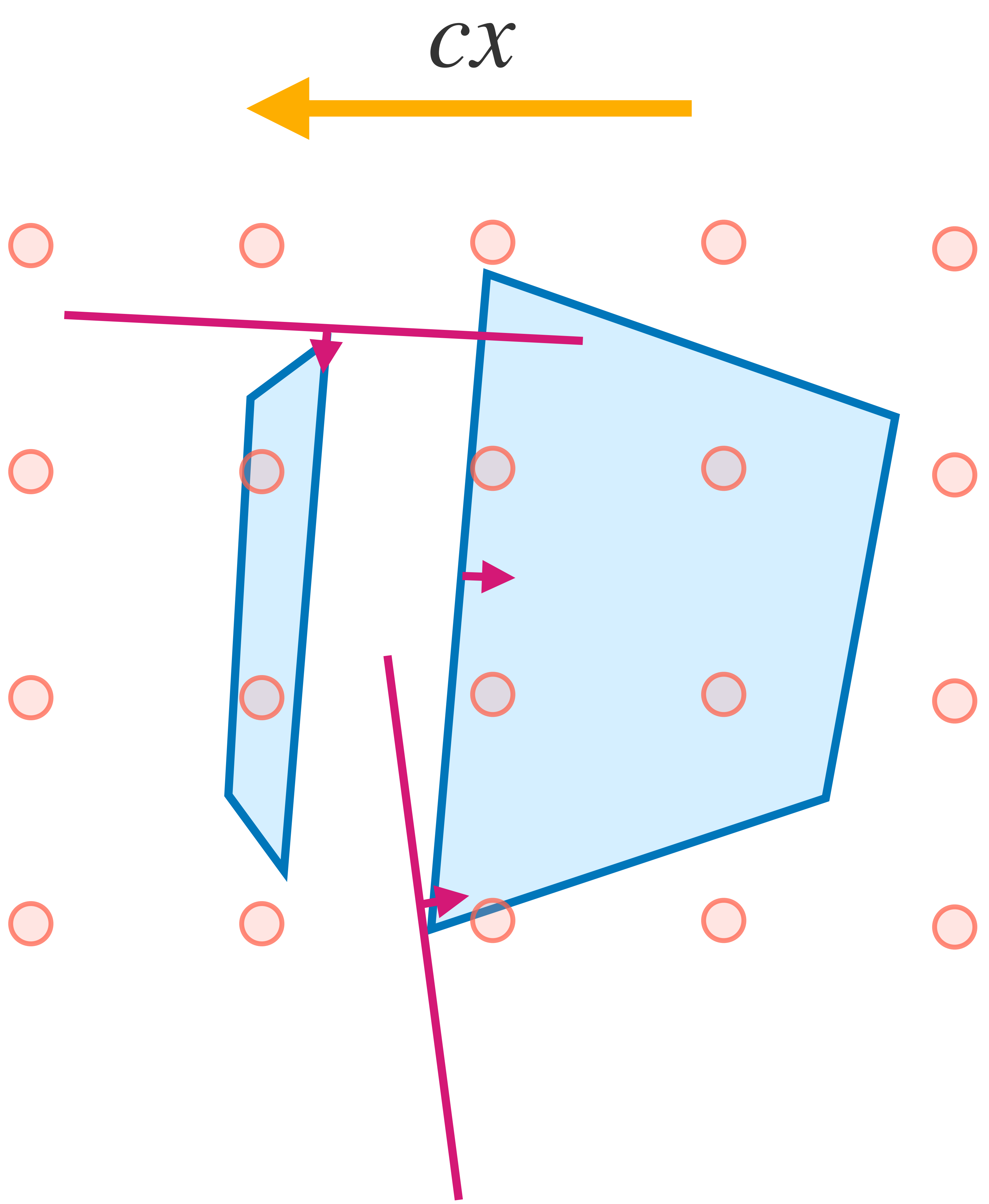

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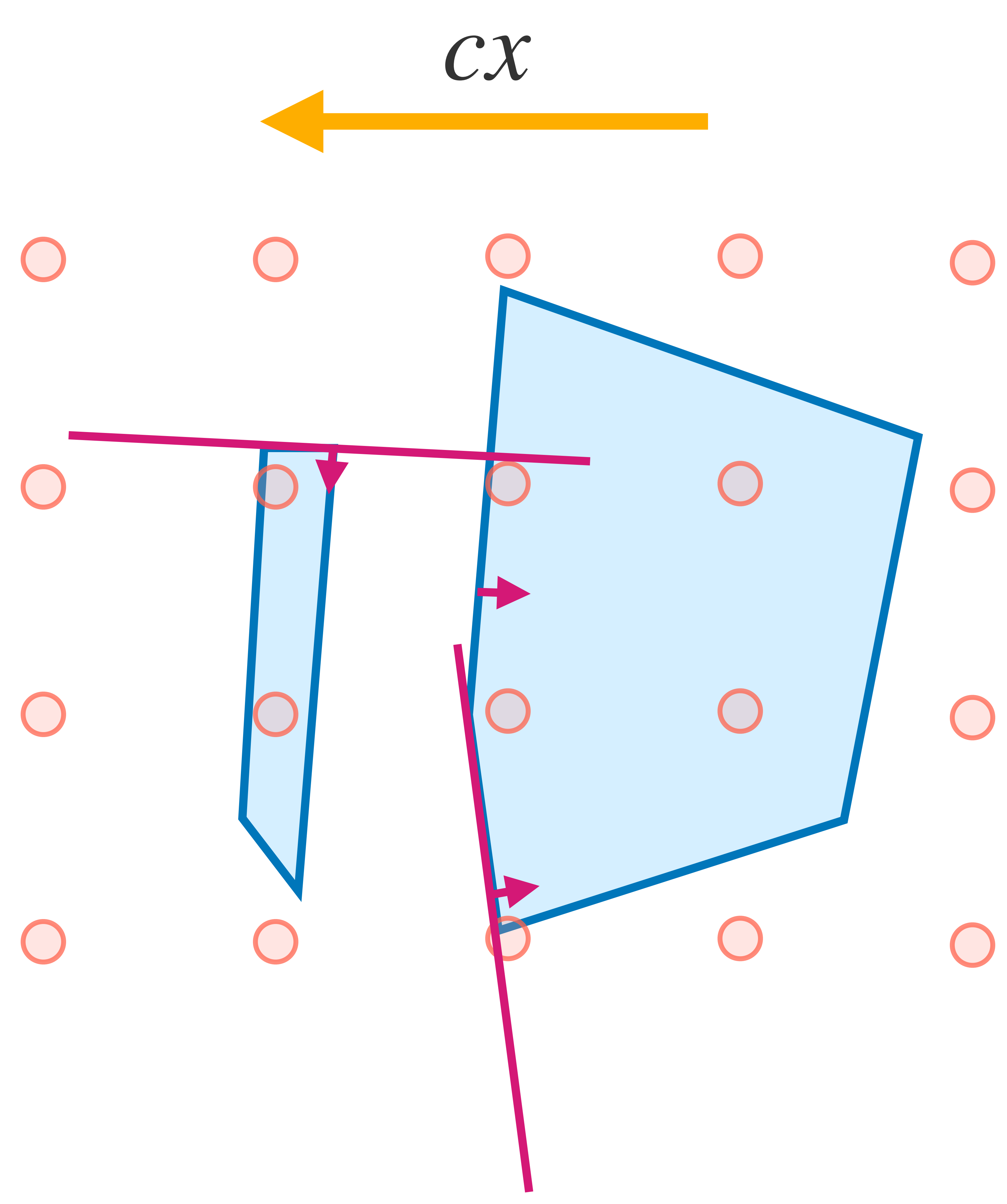

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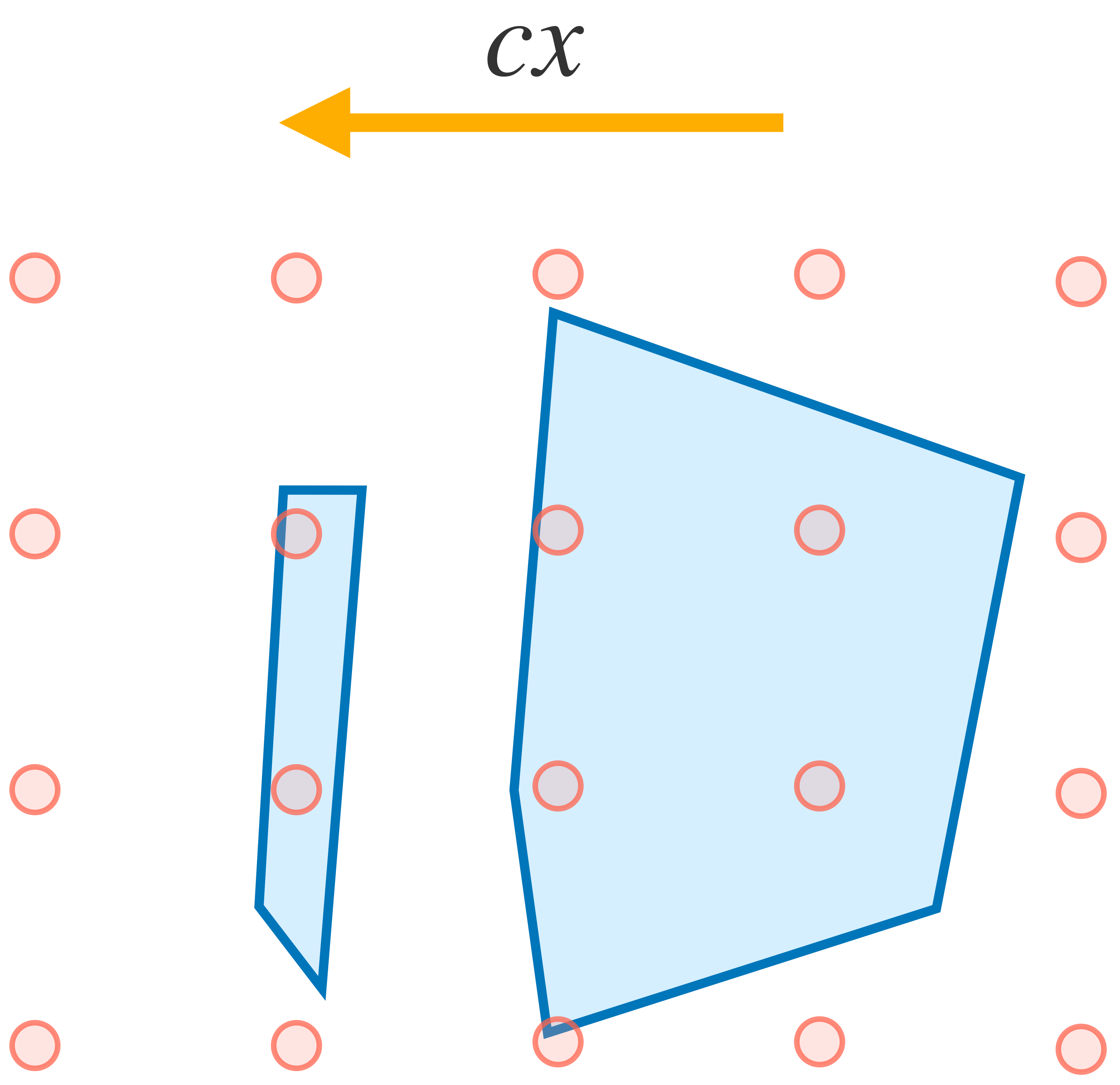
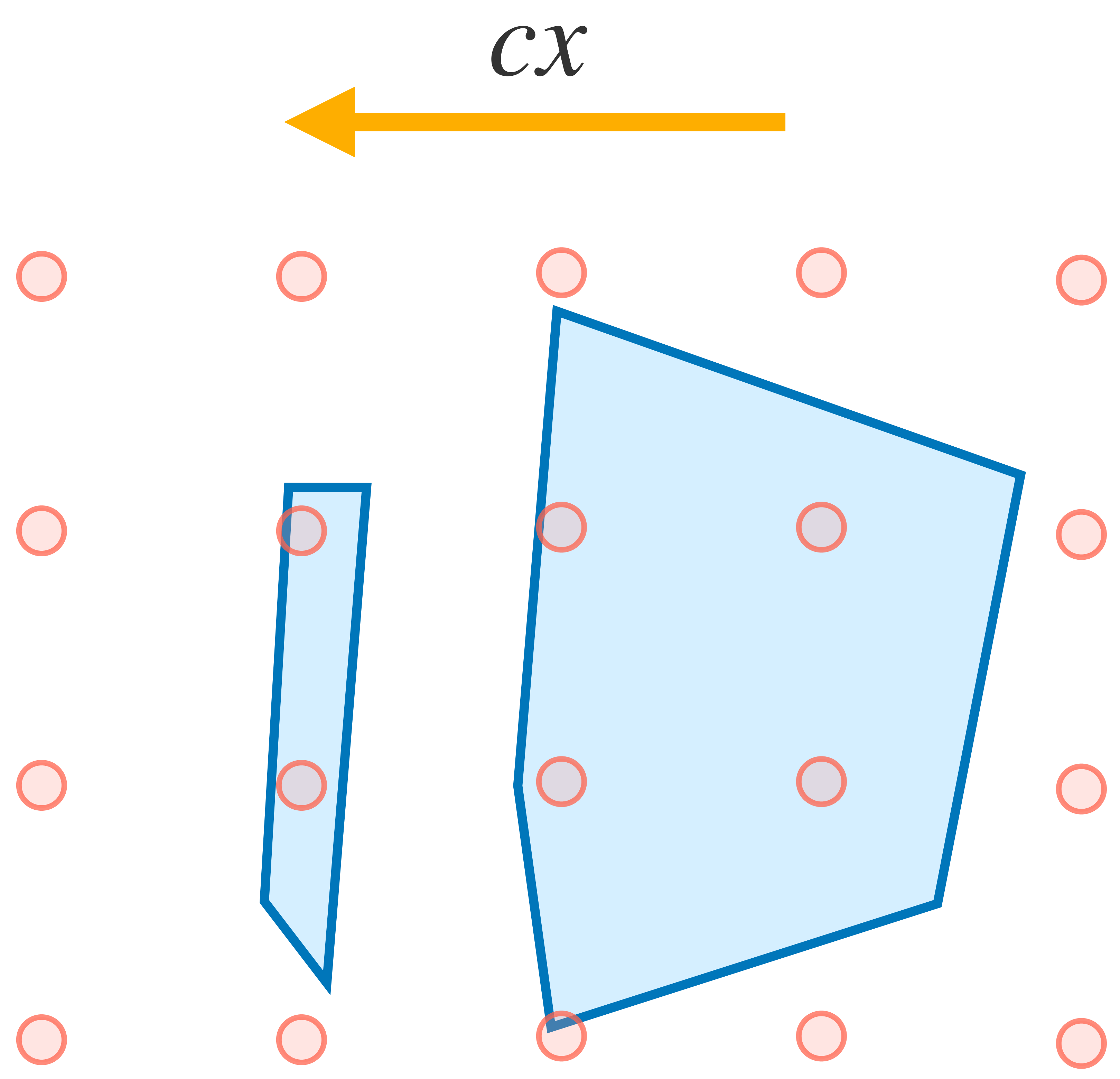


Branch-and-Cut

Idea: Try to use **linear** programming to solve **integer** programming!

Branch and Cut Template

1. Solve the linear program.
2. If solution z is non-integral, refine polytope by:
   i) Branching.
   ii) Cutting.
3. Repeat.



If the polytope is refined only by cutting, then this is known as a **cutting planes algorithm**

Formalizing Modern IP Solvers

[Chvatal73] Introduced the **Cutting Planes** proof system to formalize cutting planes algorithms.

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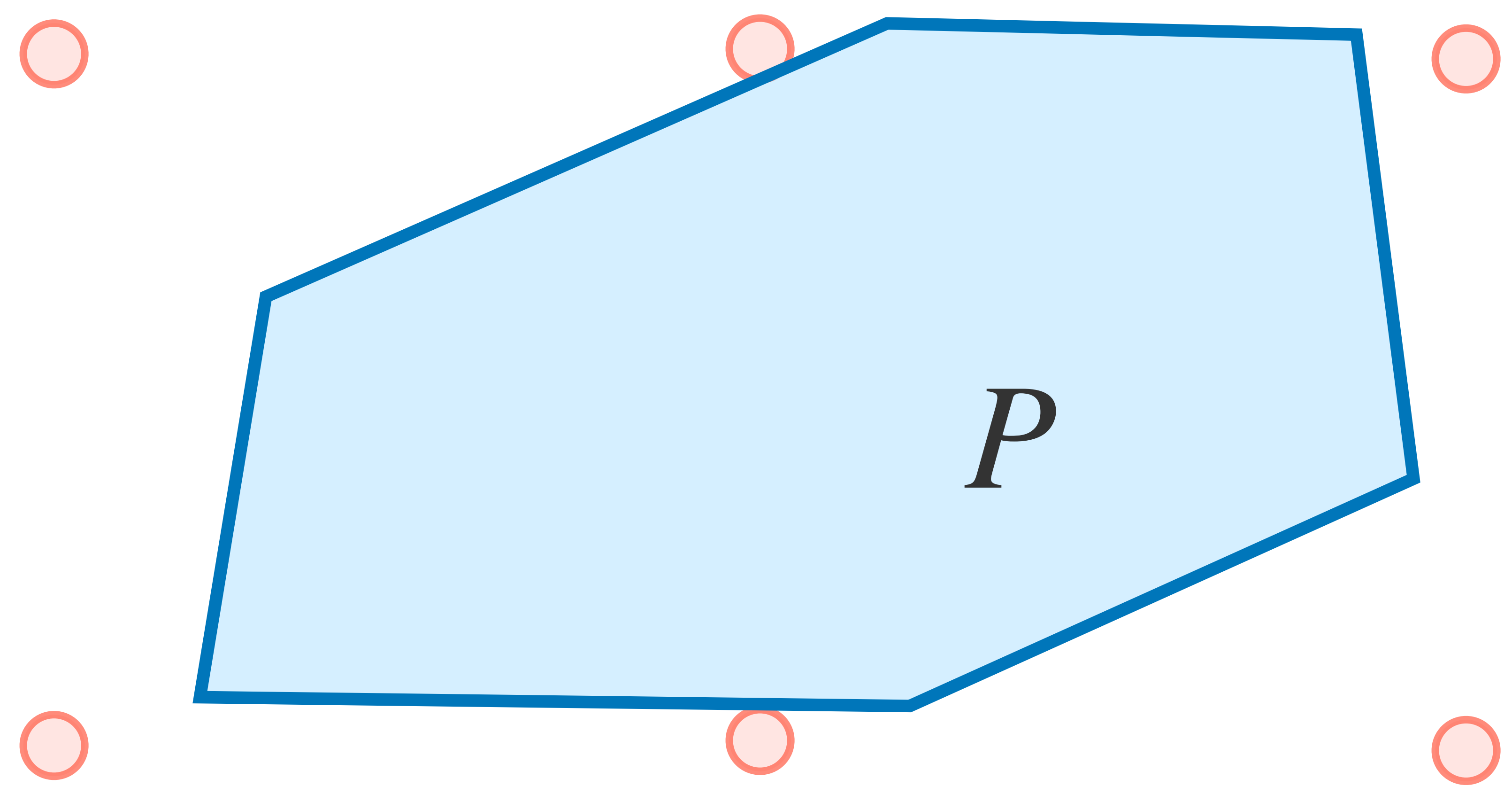
- Even so, it is an important and heavily studied proof system!

Cutting Planes Proofs

Suppose $Ax \geq b$ has **no integer solutions**



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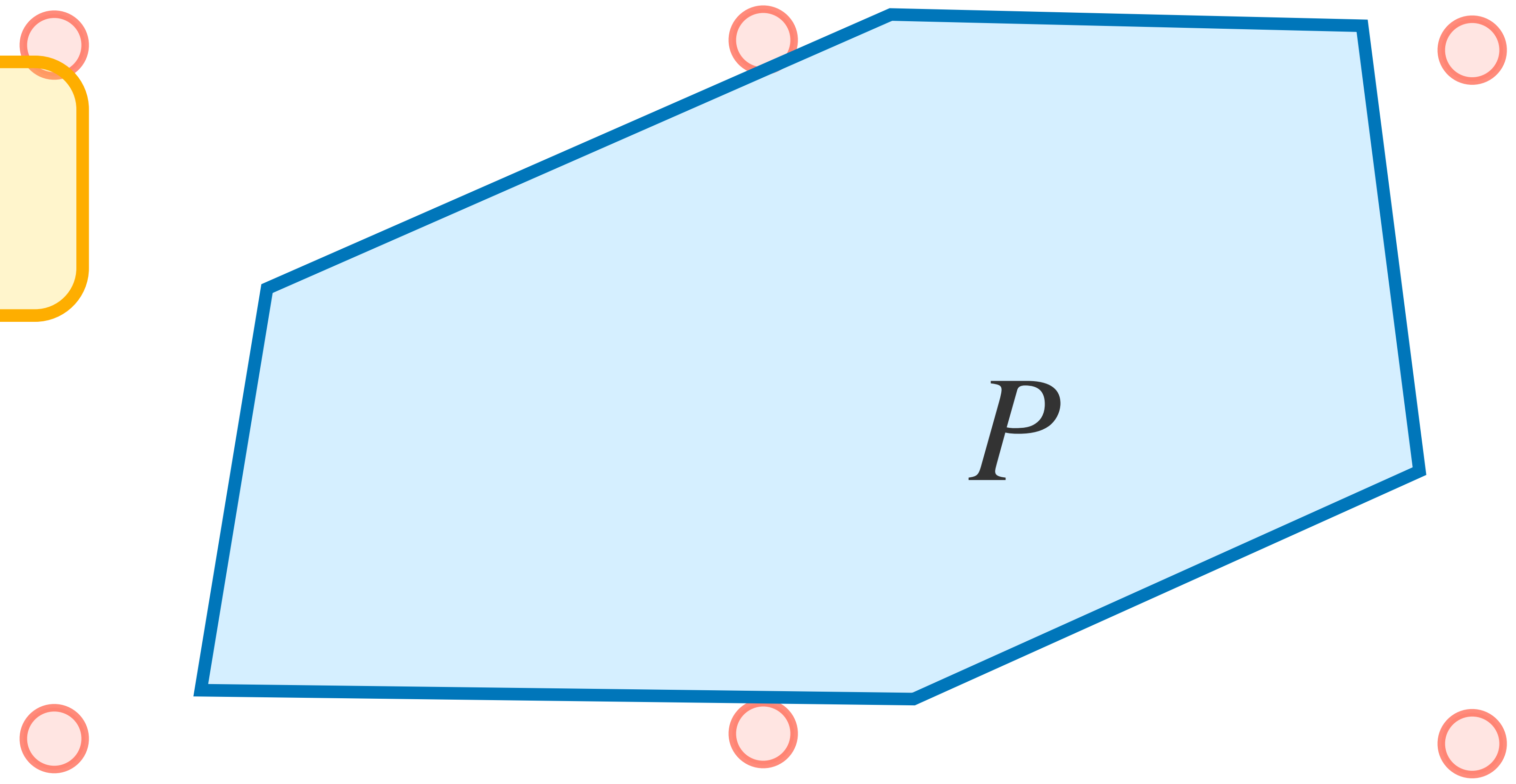
Analogous to running DPLL on unsatisfiable CNF



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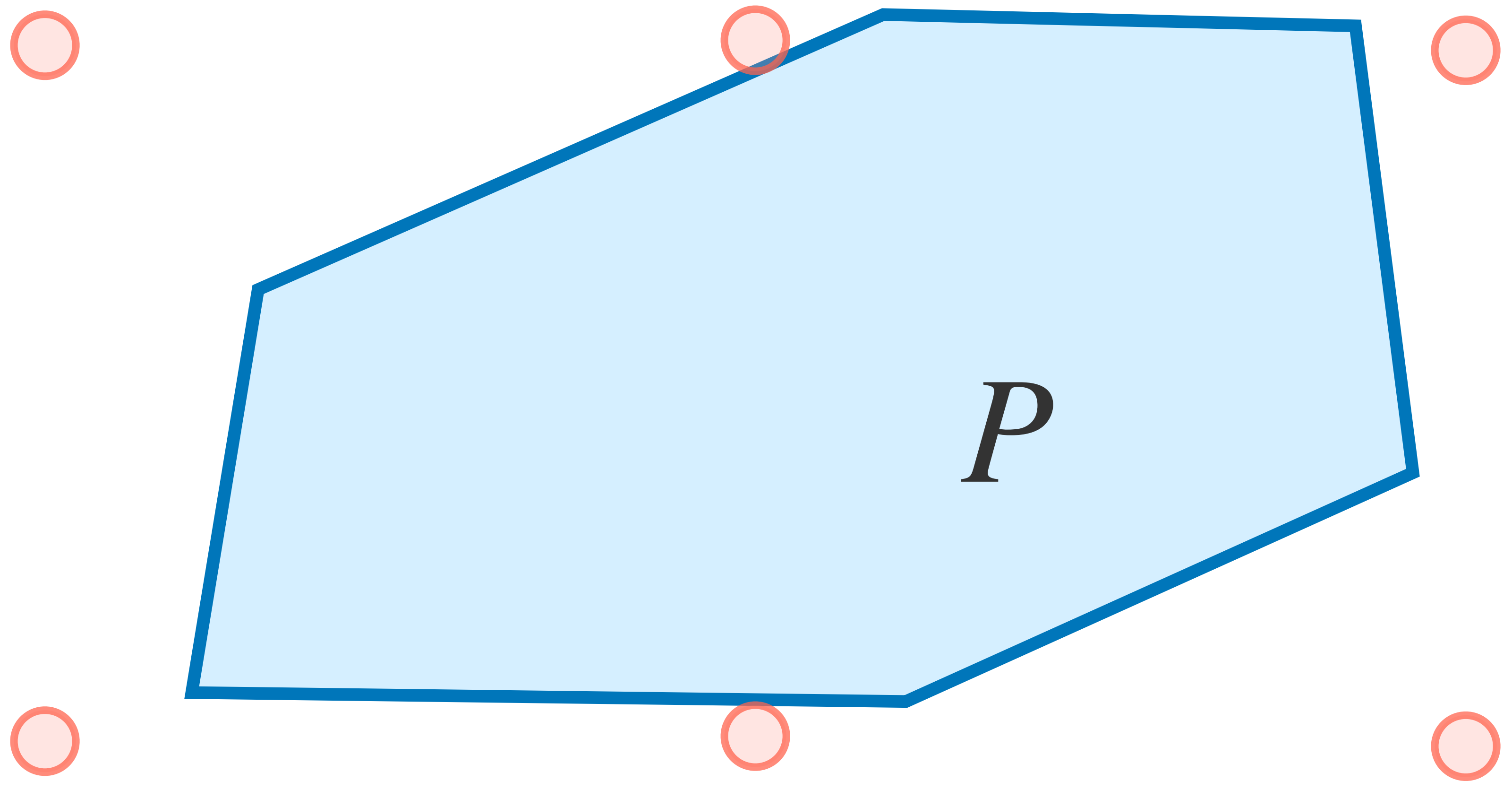

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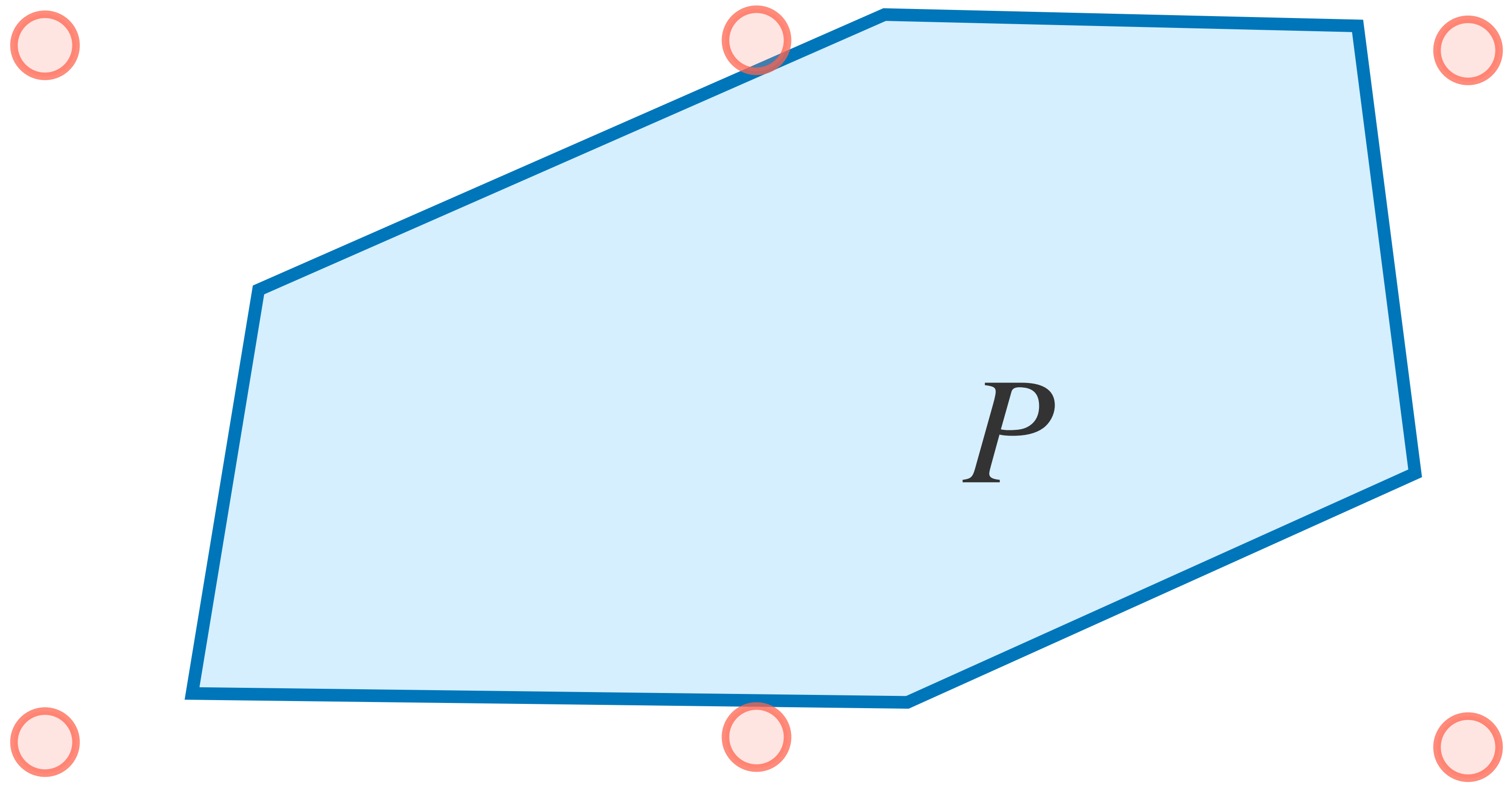

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Derive new integer-inequalities from old ones by:

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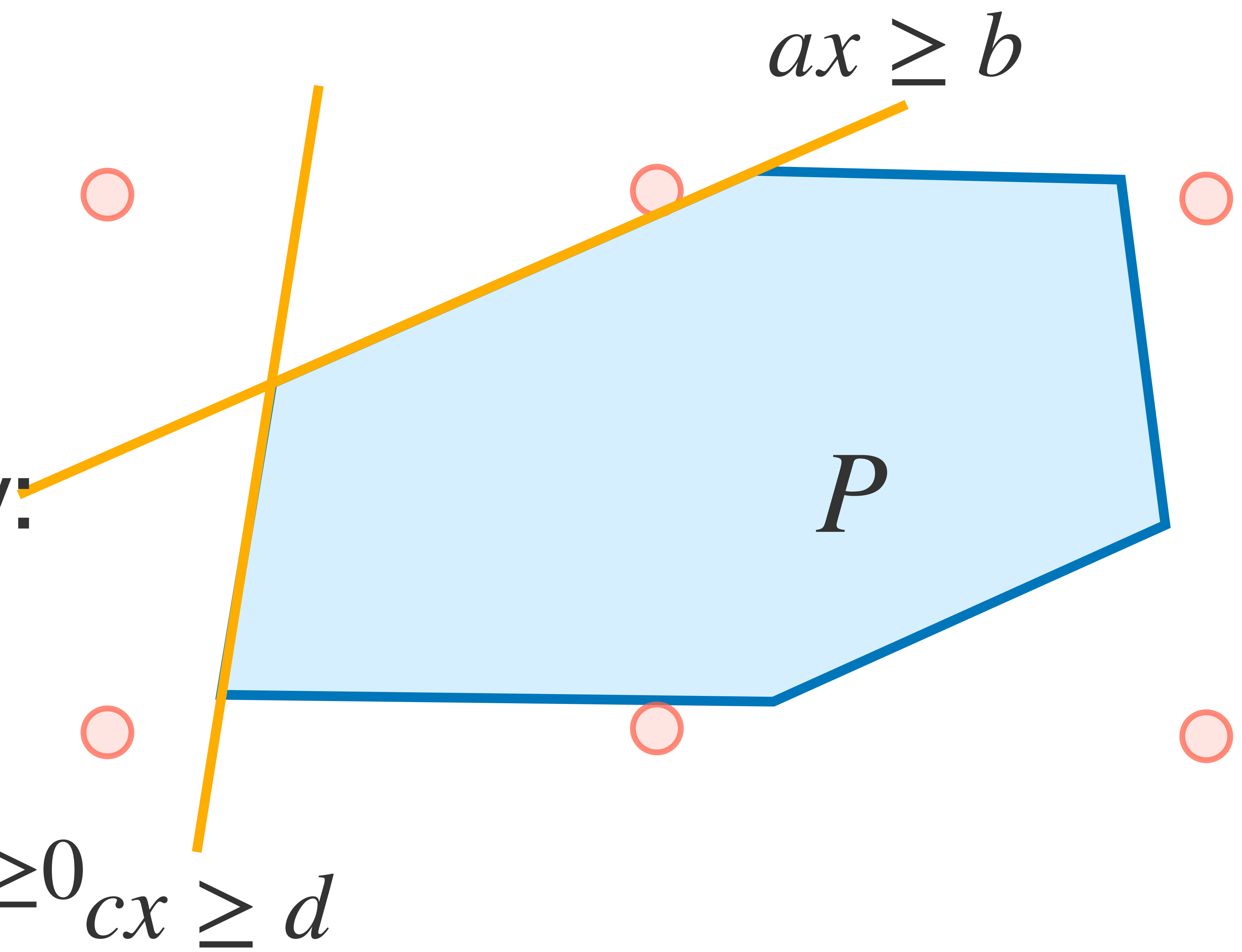

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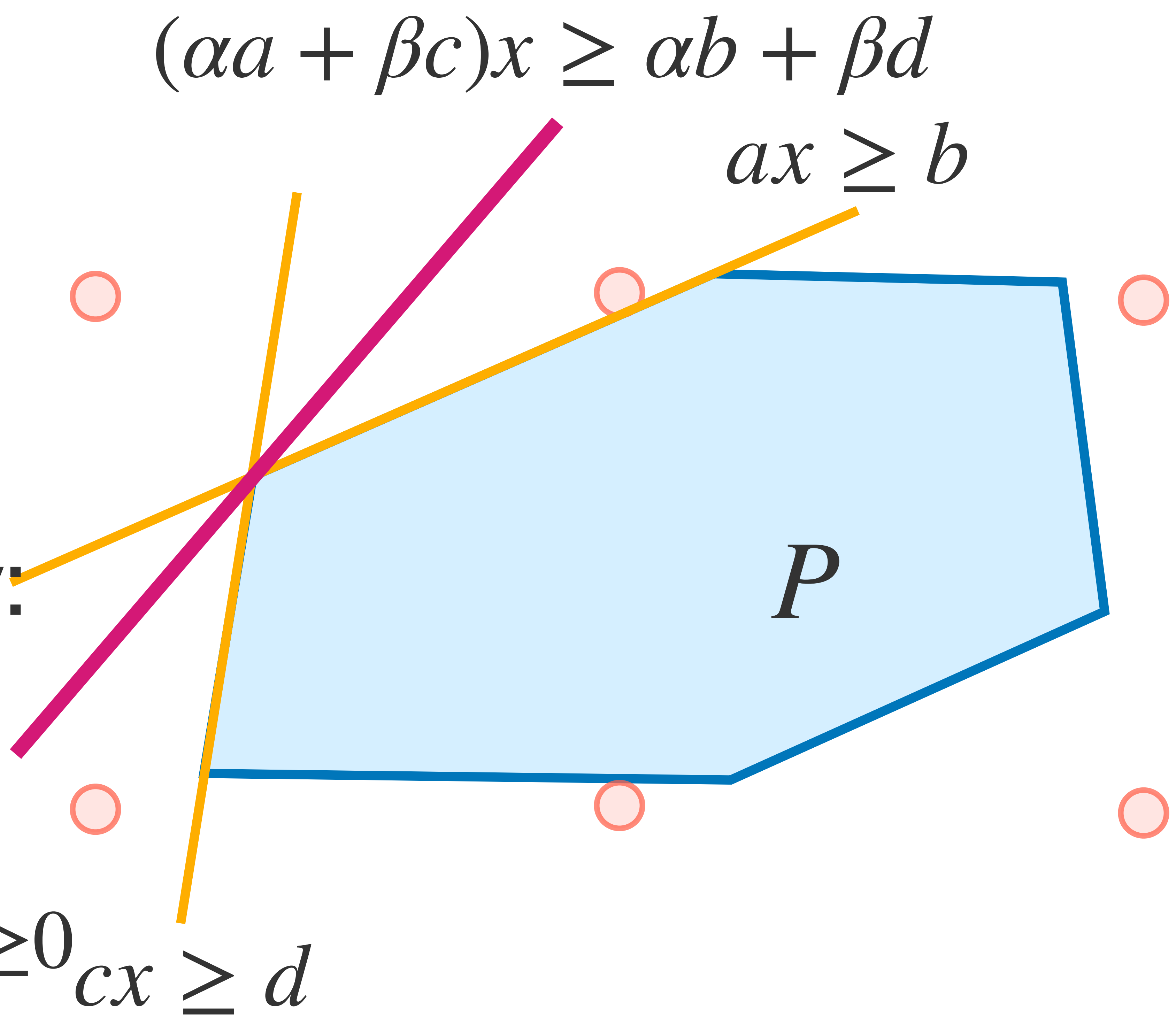

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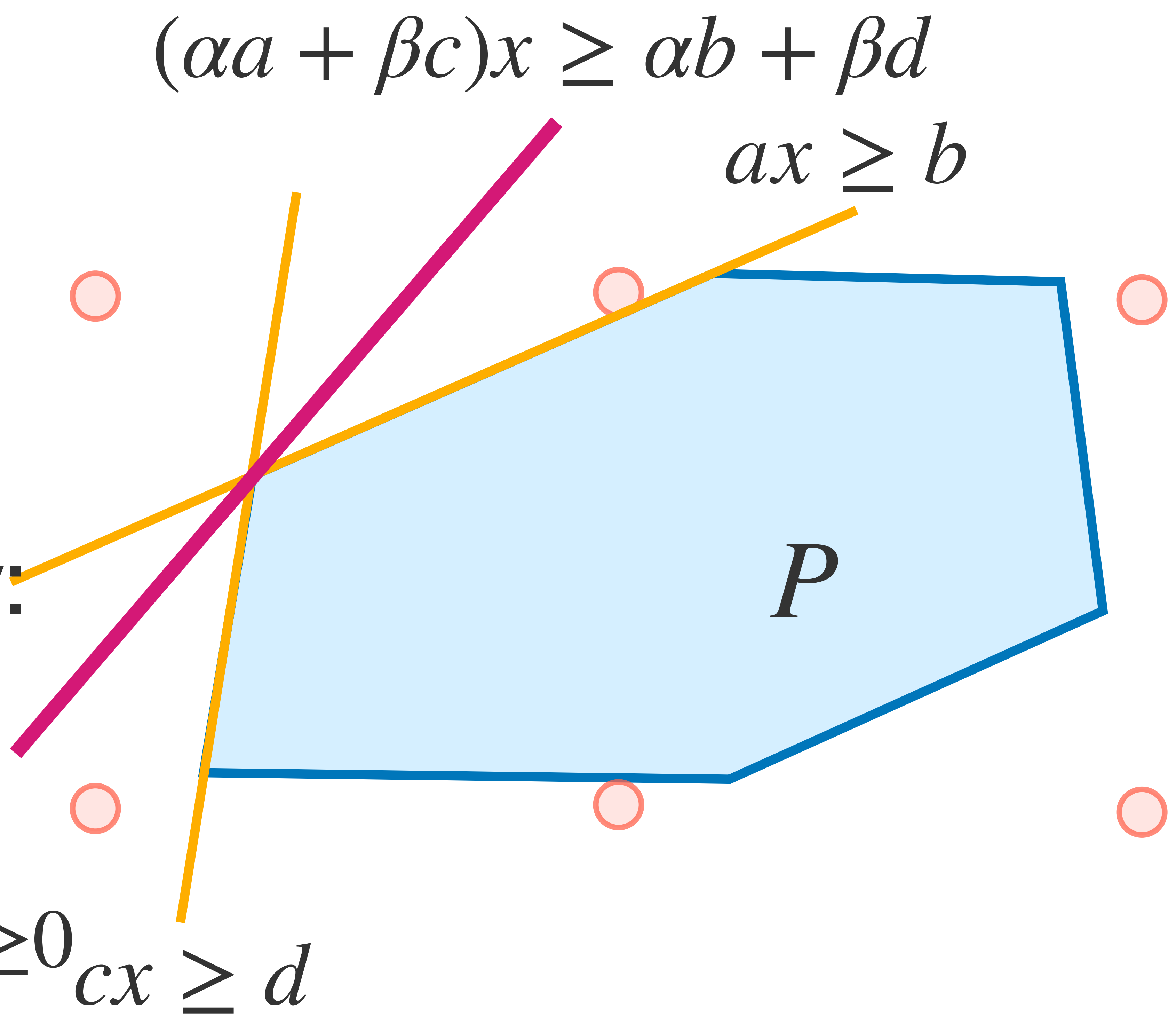

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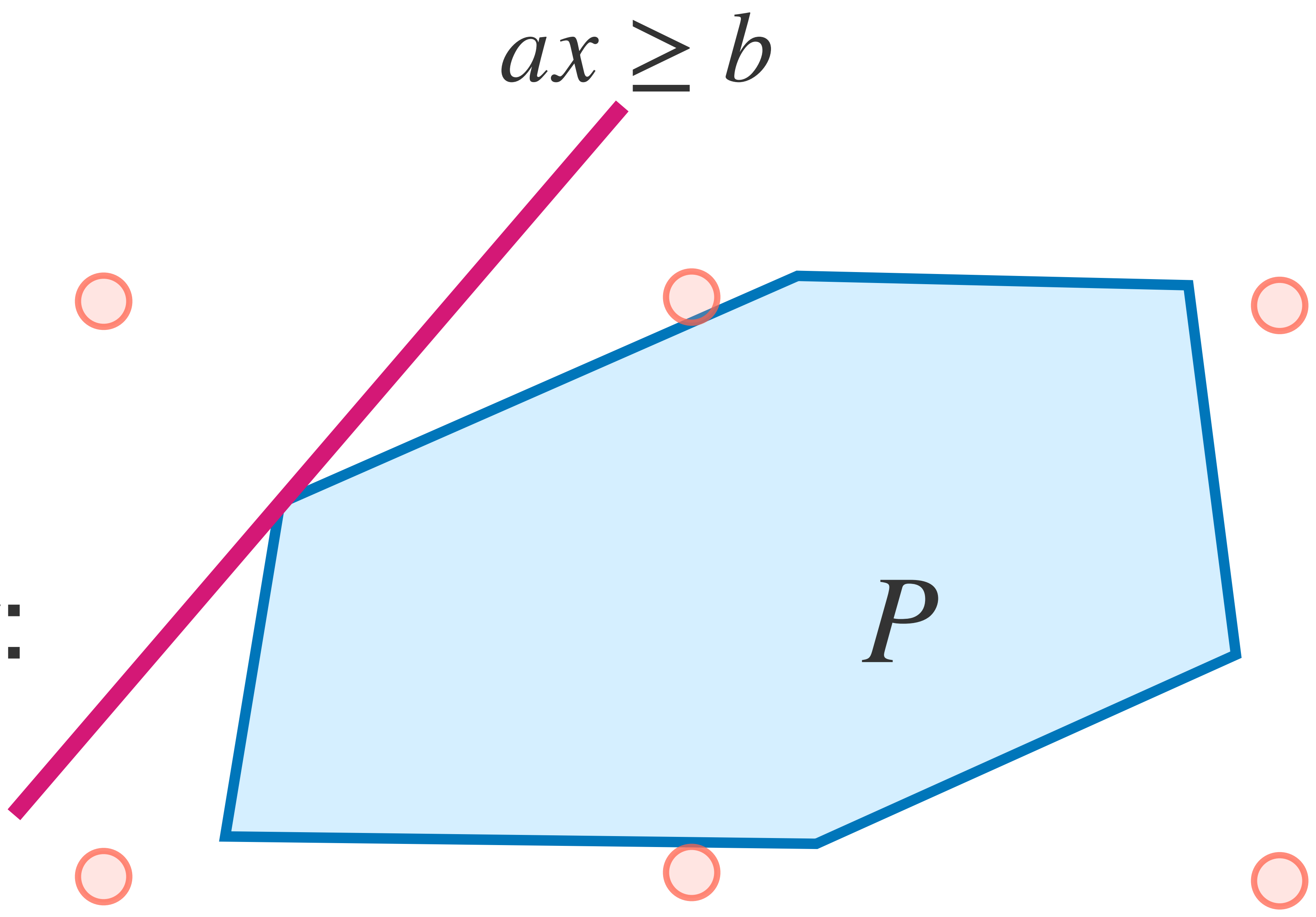

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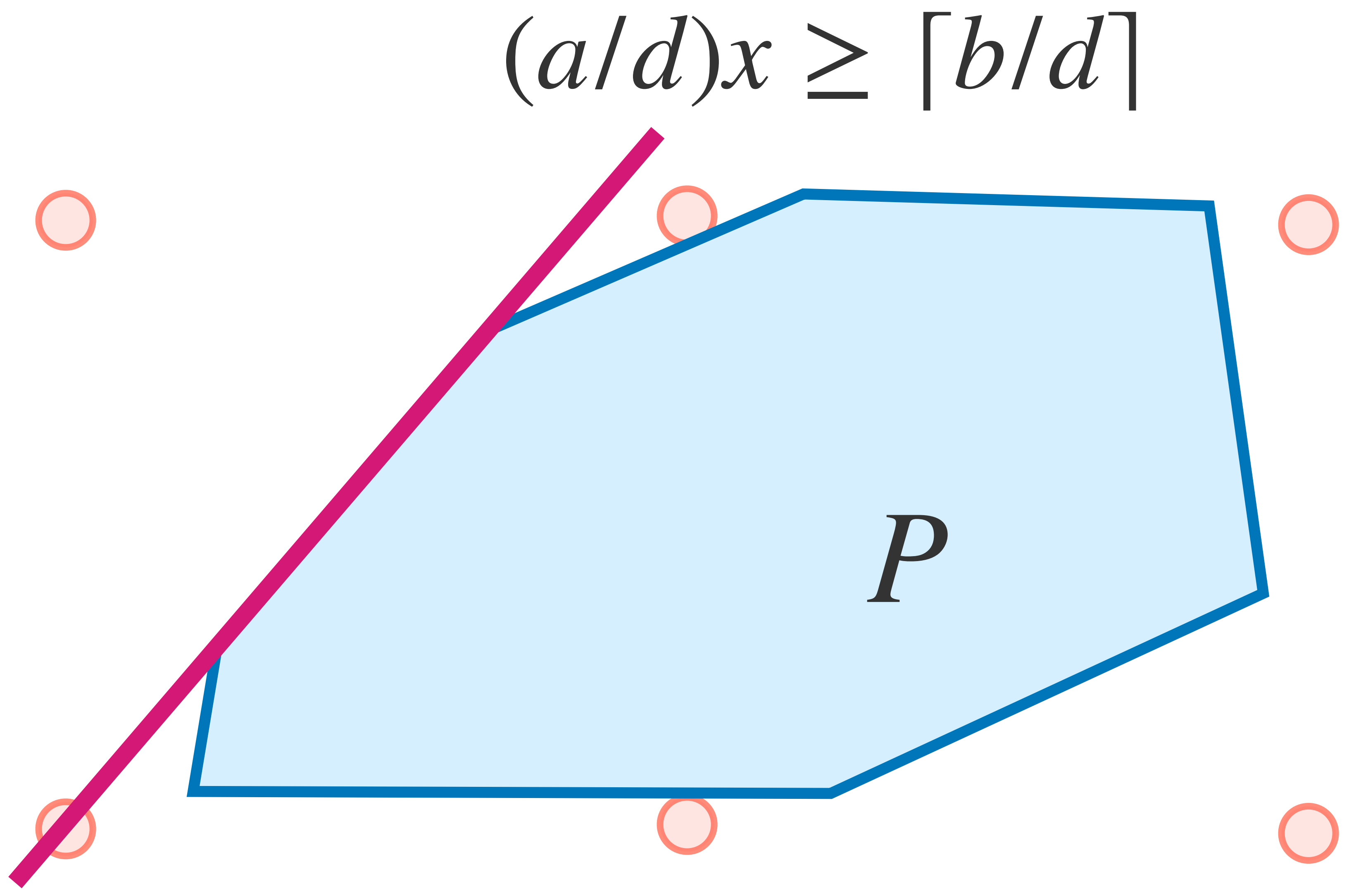

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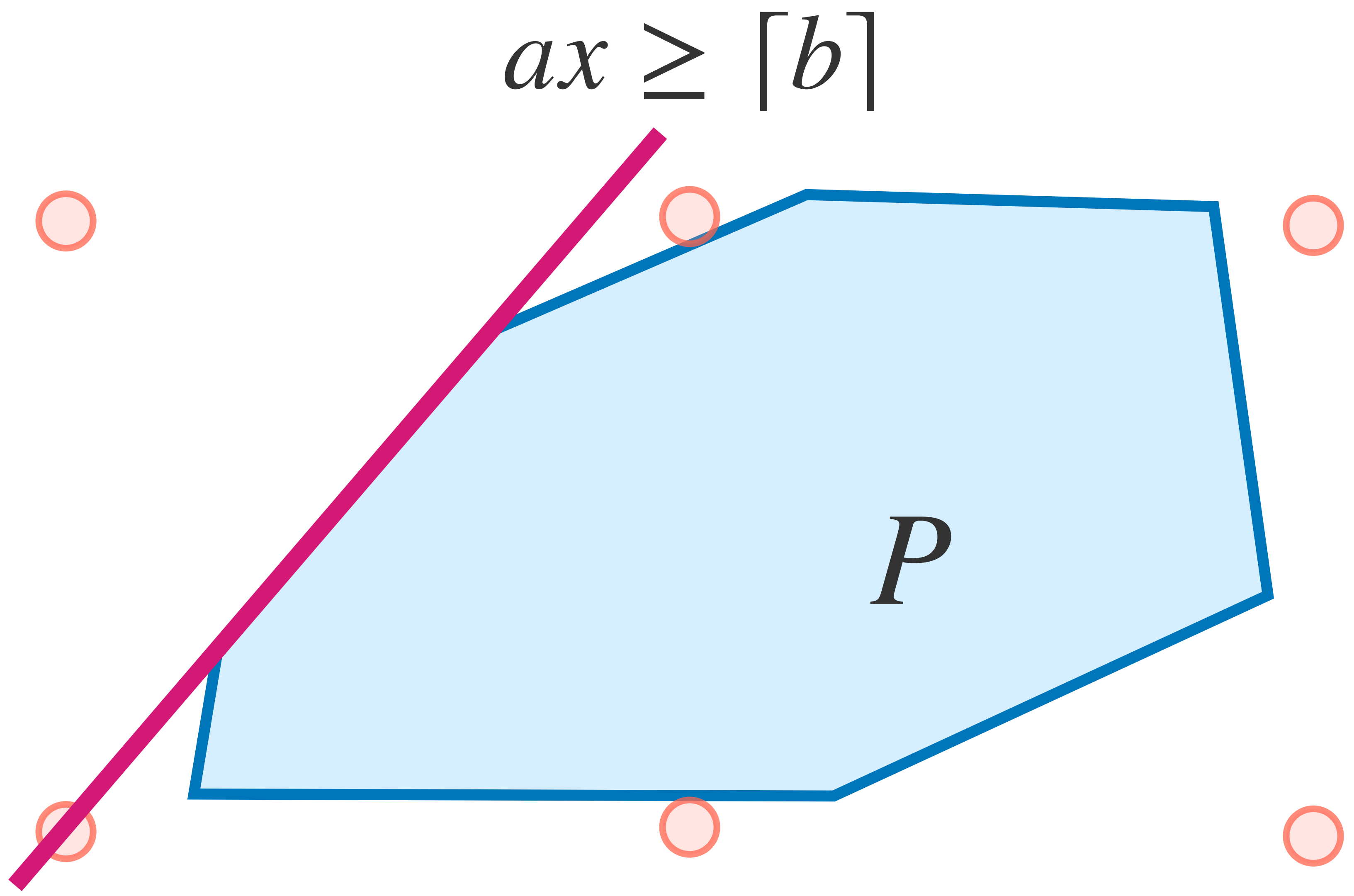

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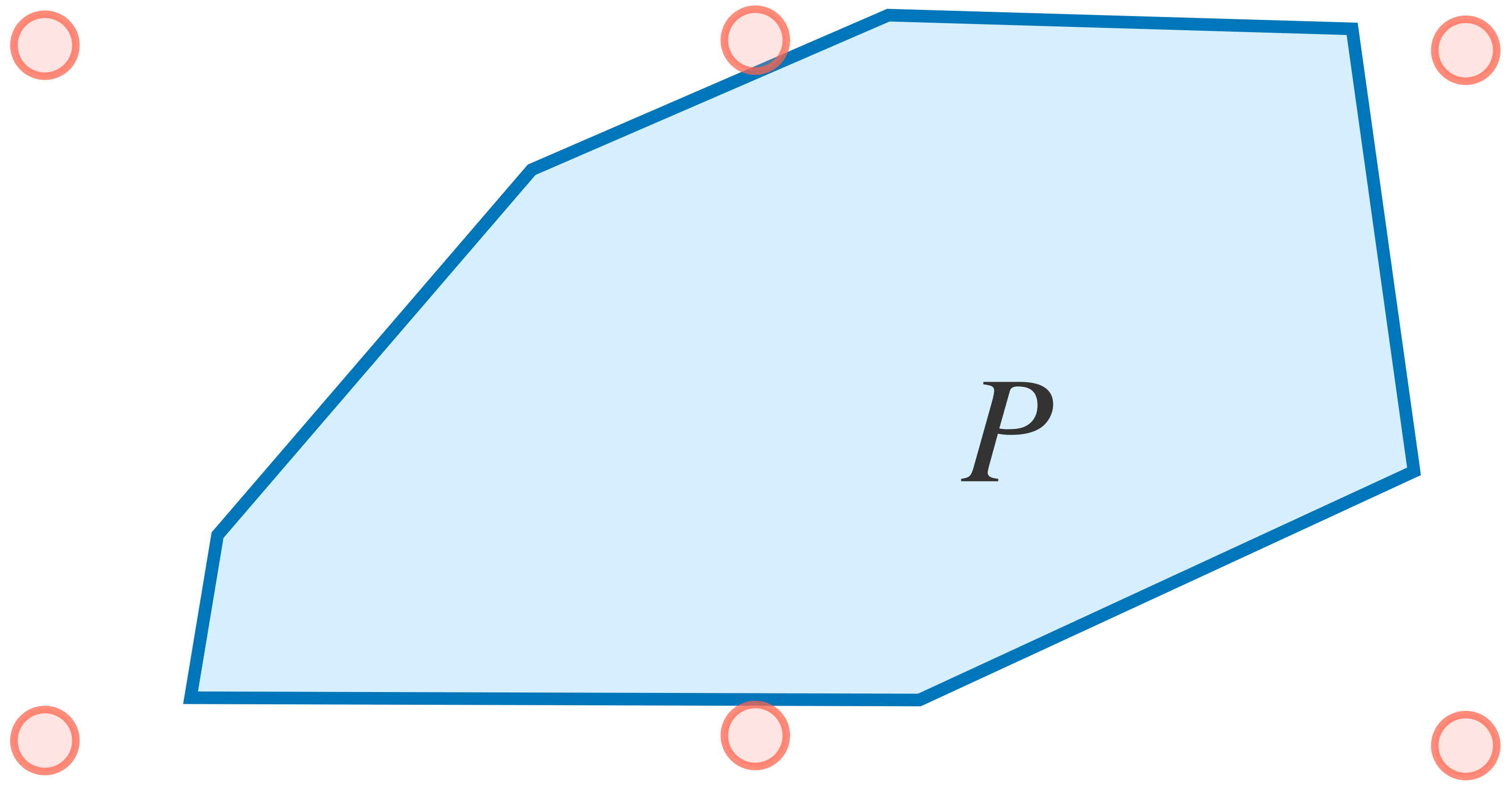

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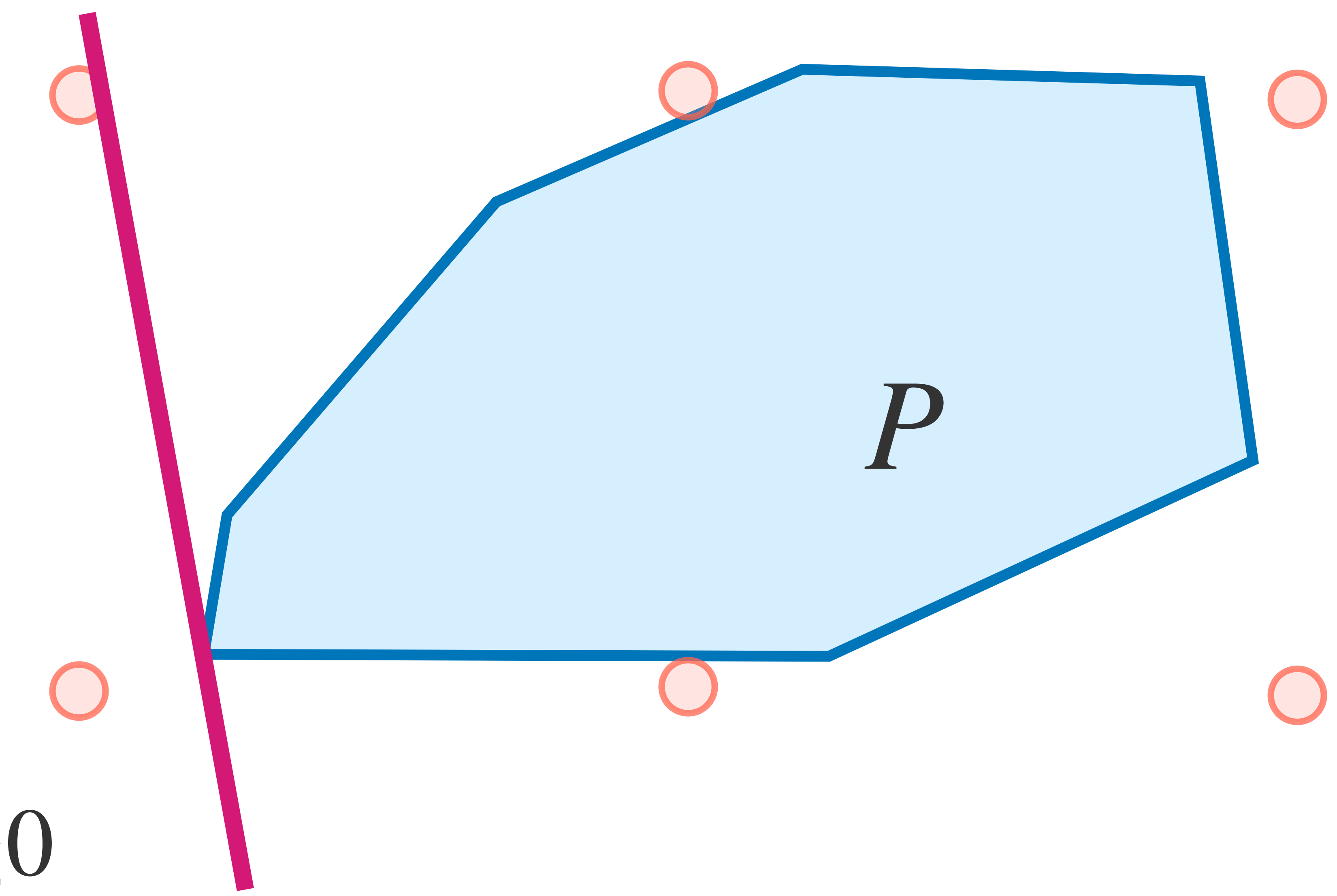

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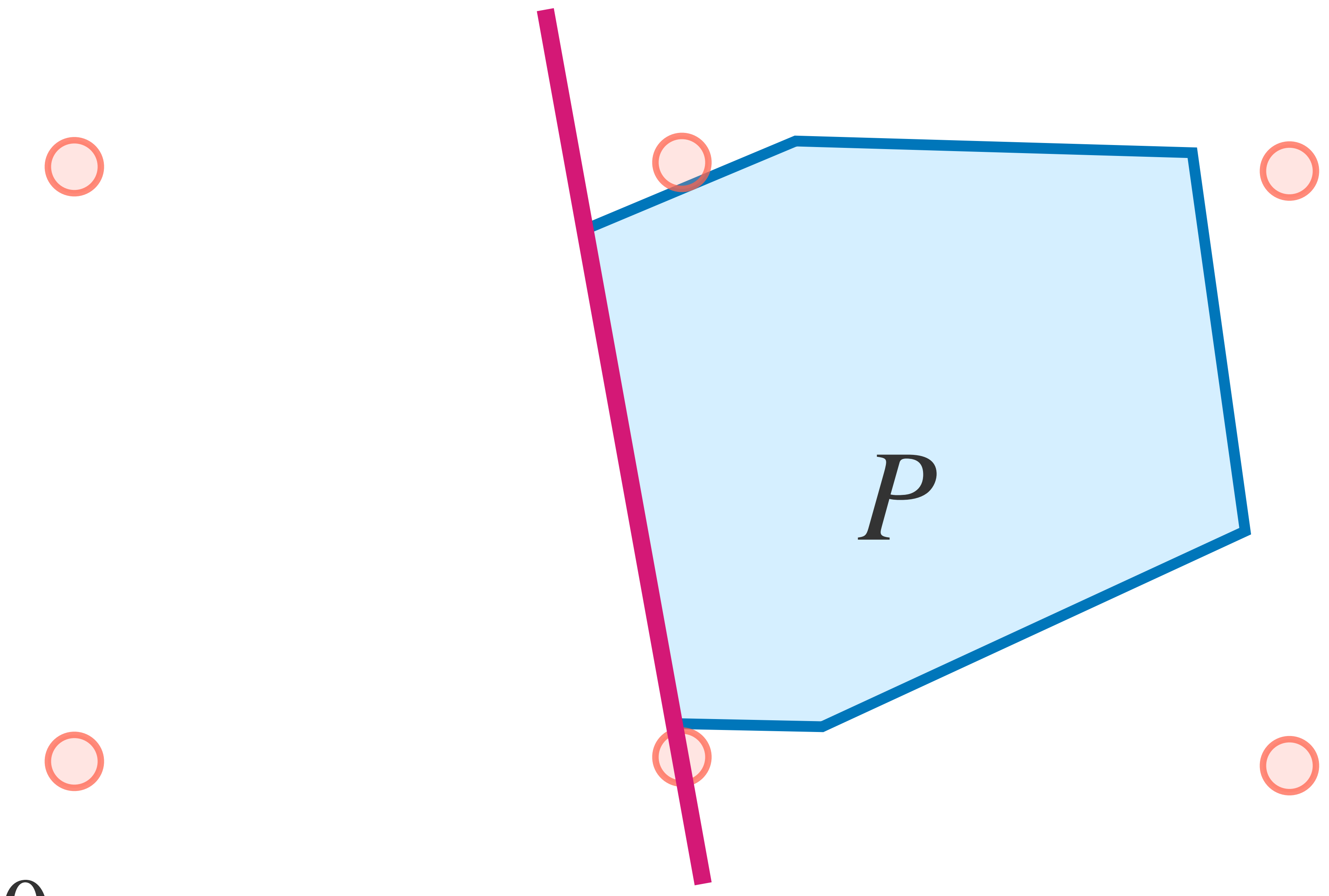

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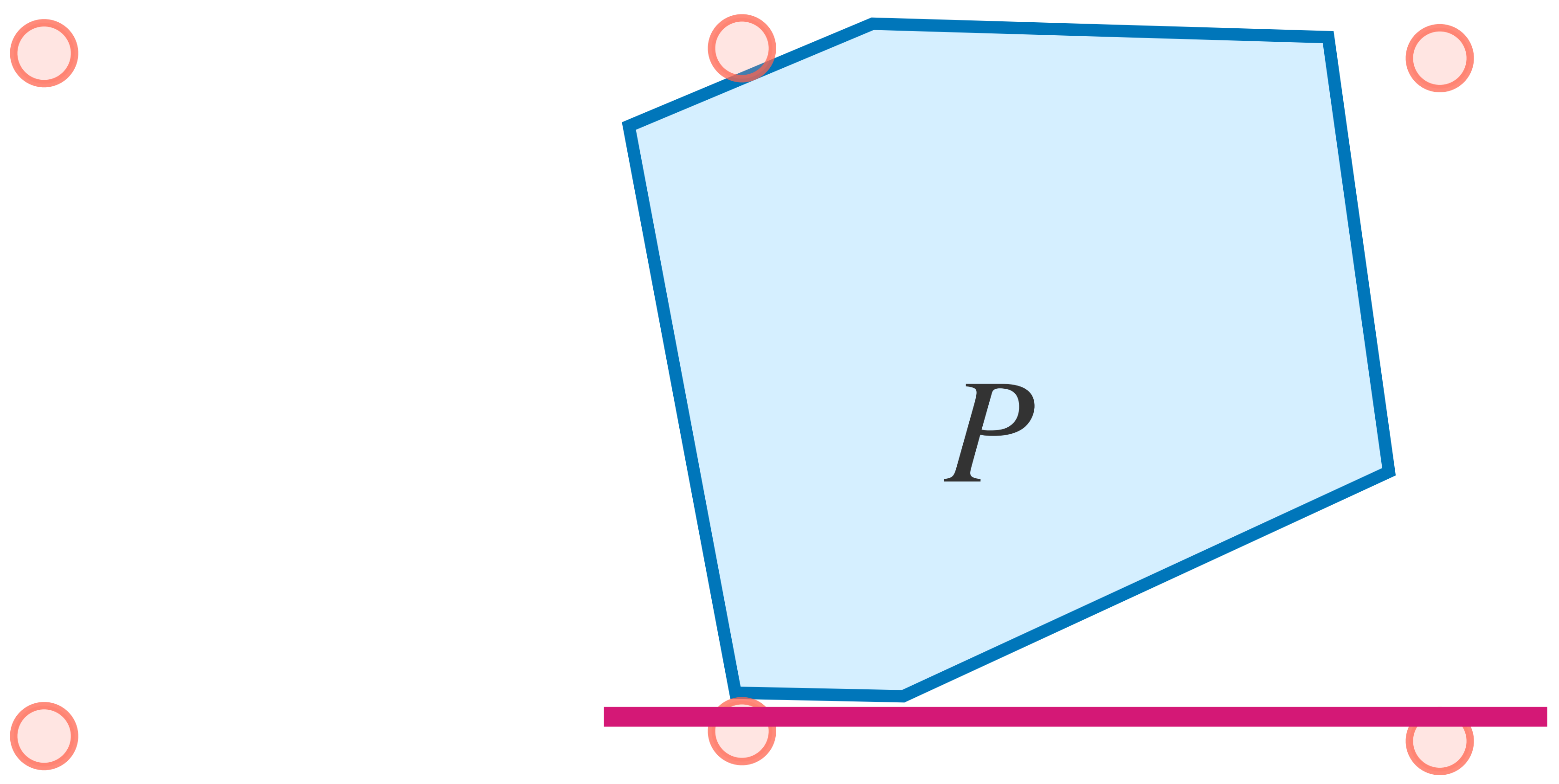

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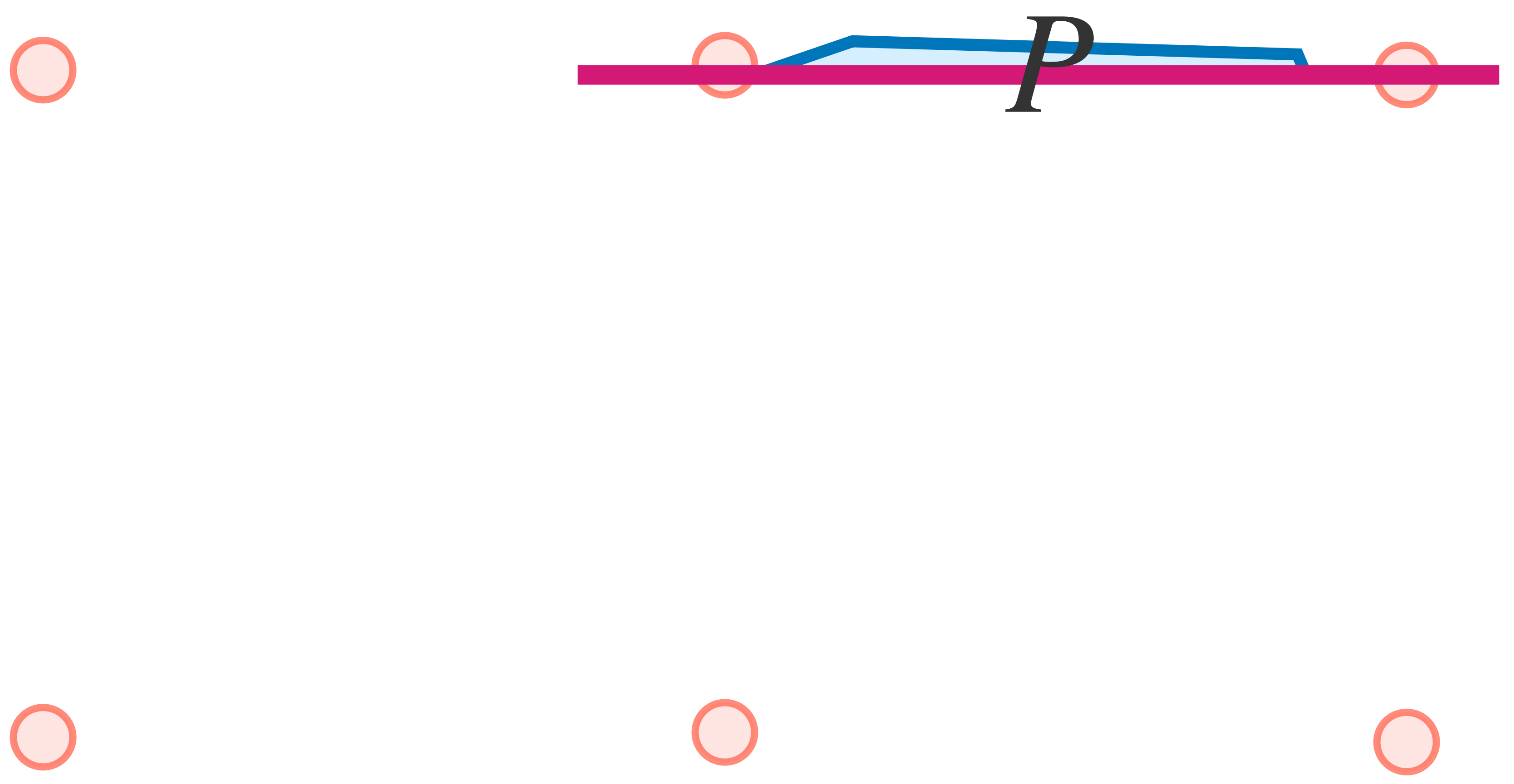

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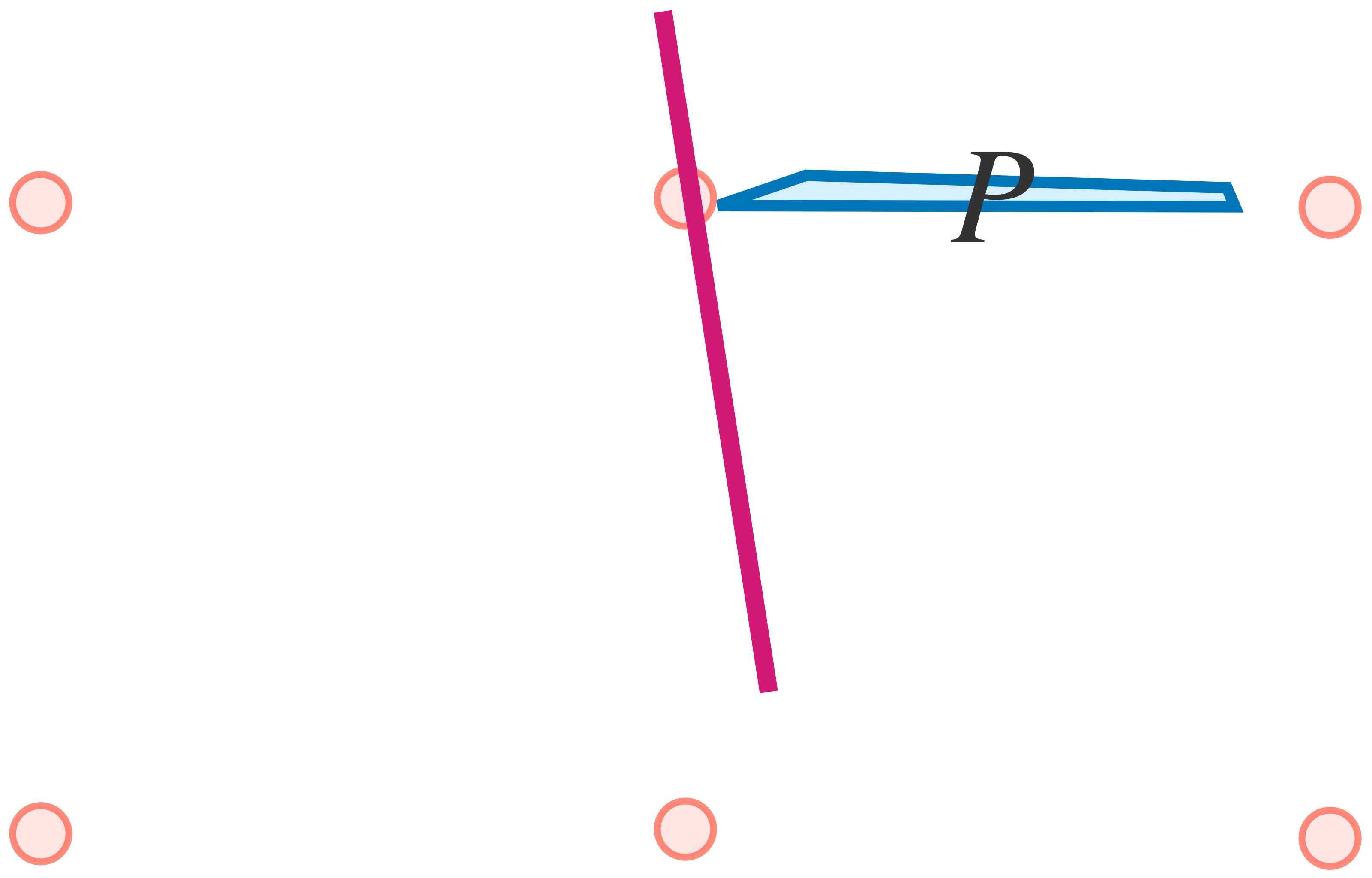

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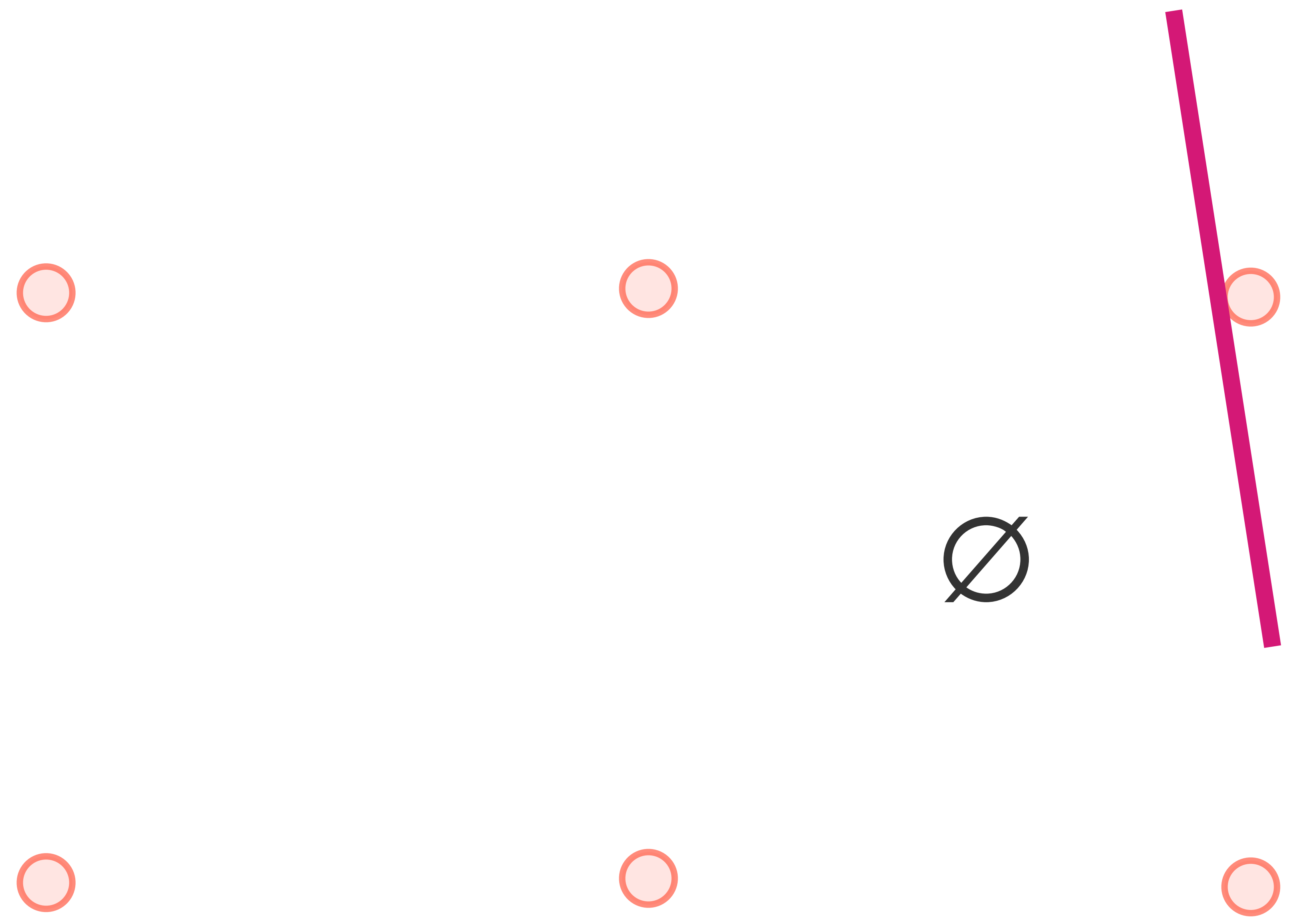

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However… Cutting Planes does not capture modern algorithms for IP (branch-and-cut)

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- DPLL querying integer linear inequalities!
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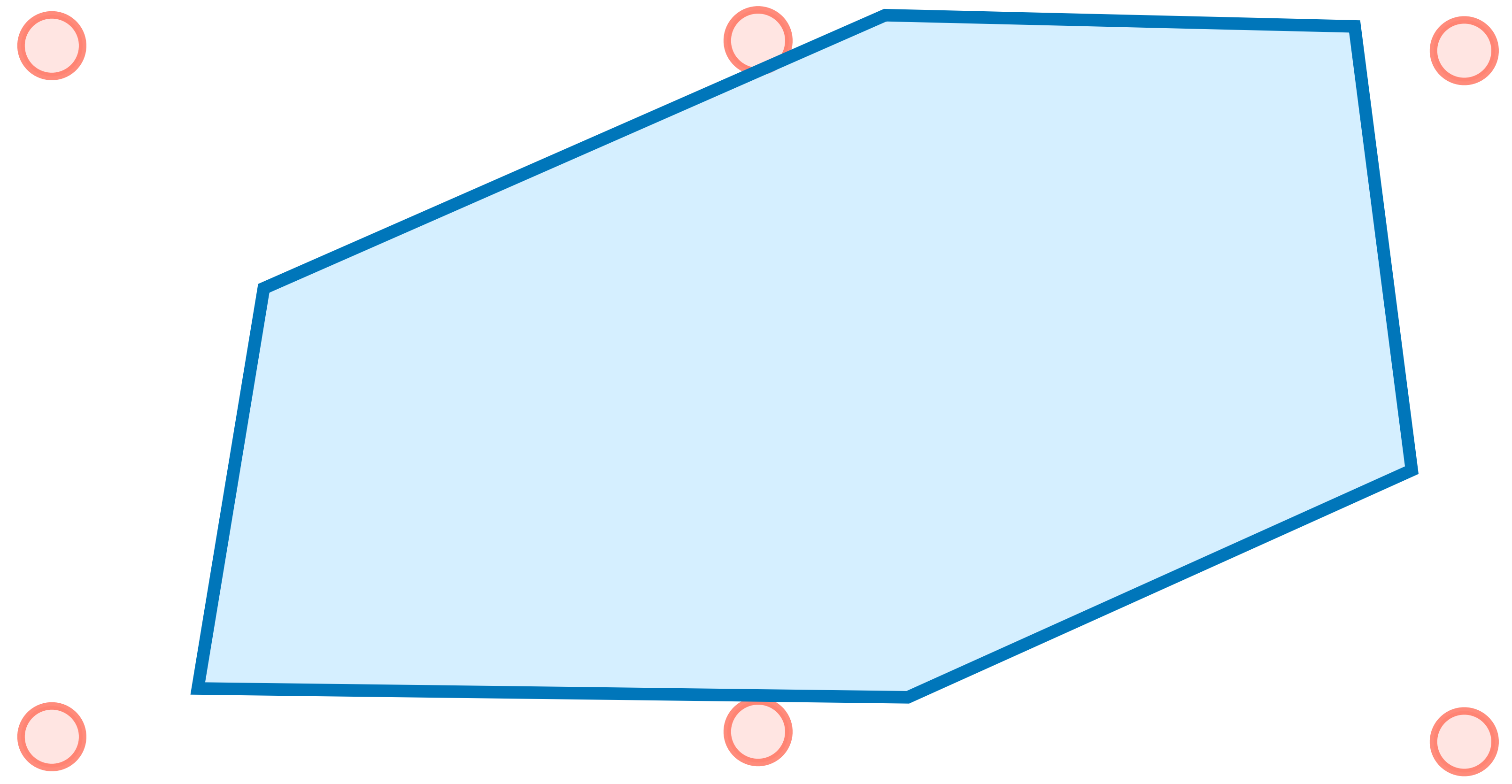

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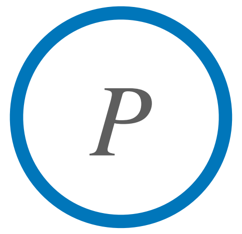

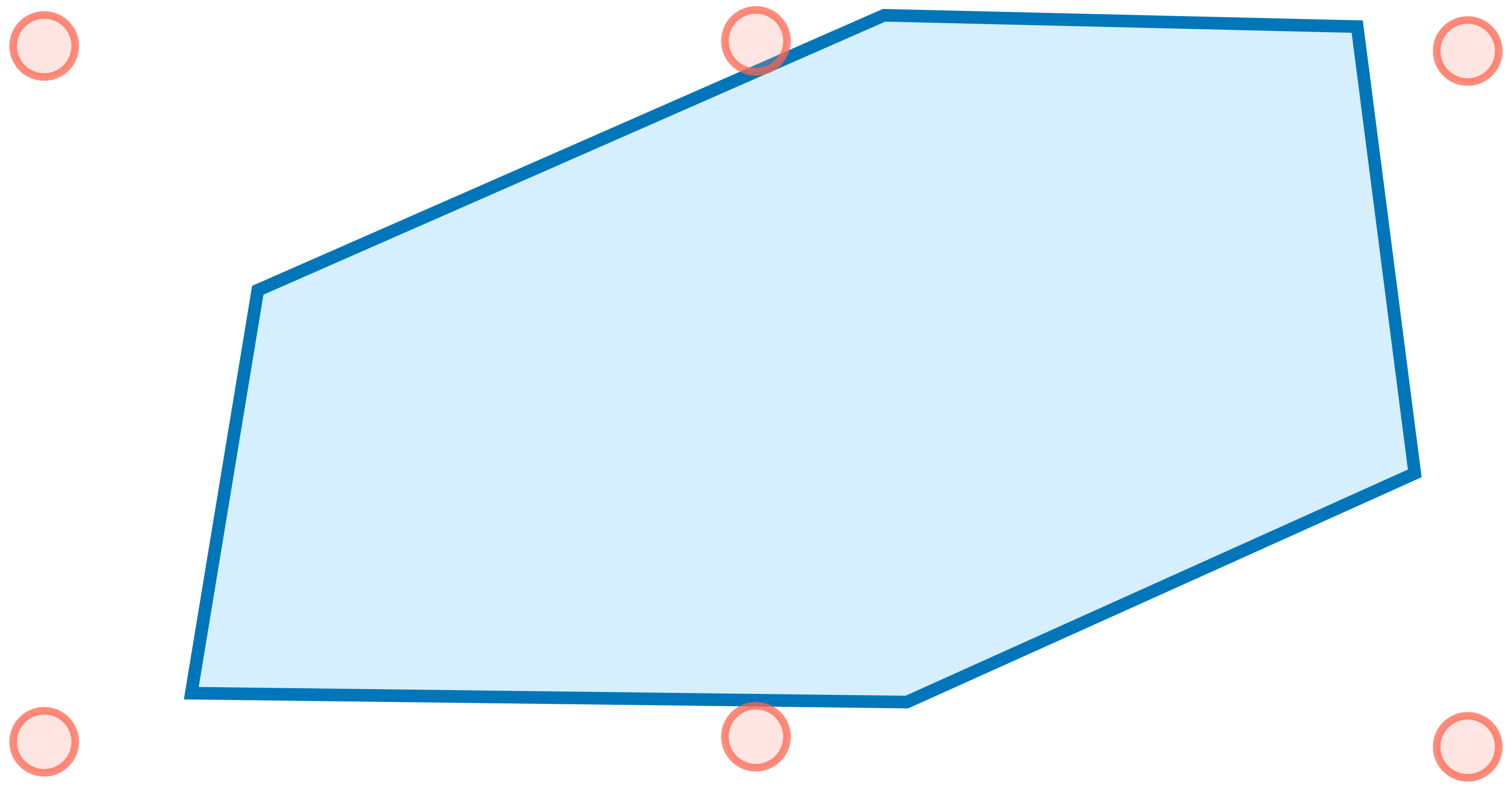

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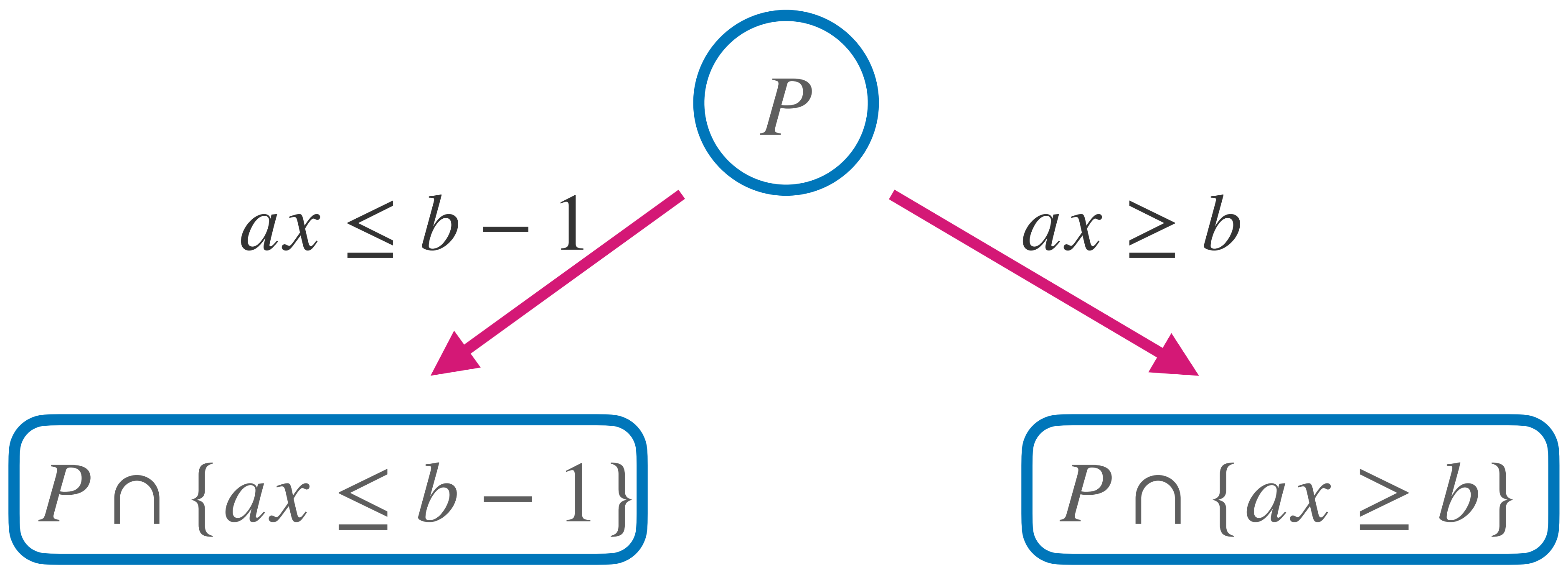

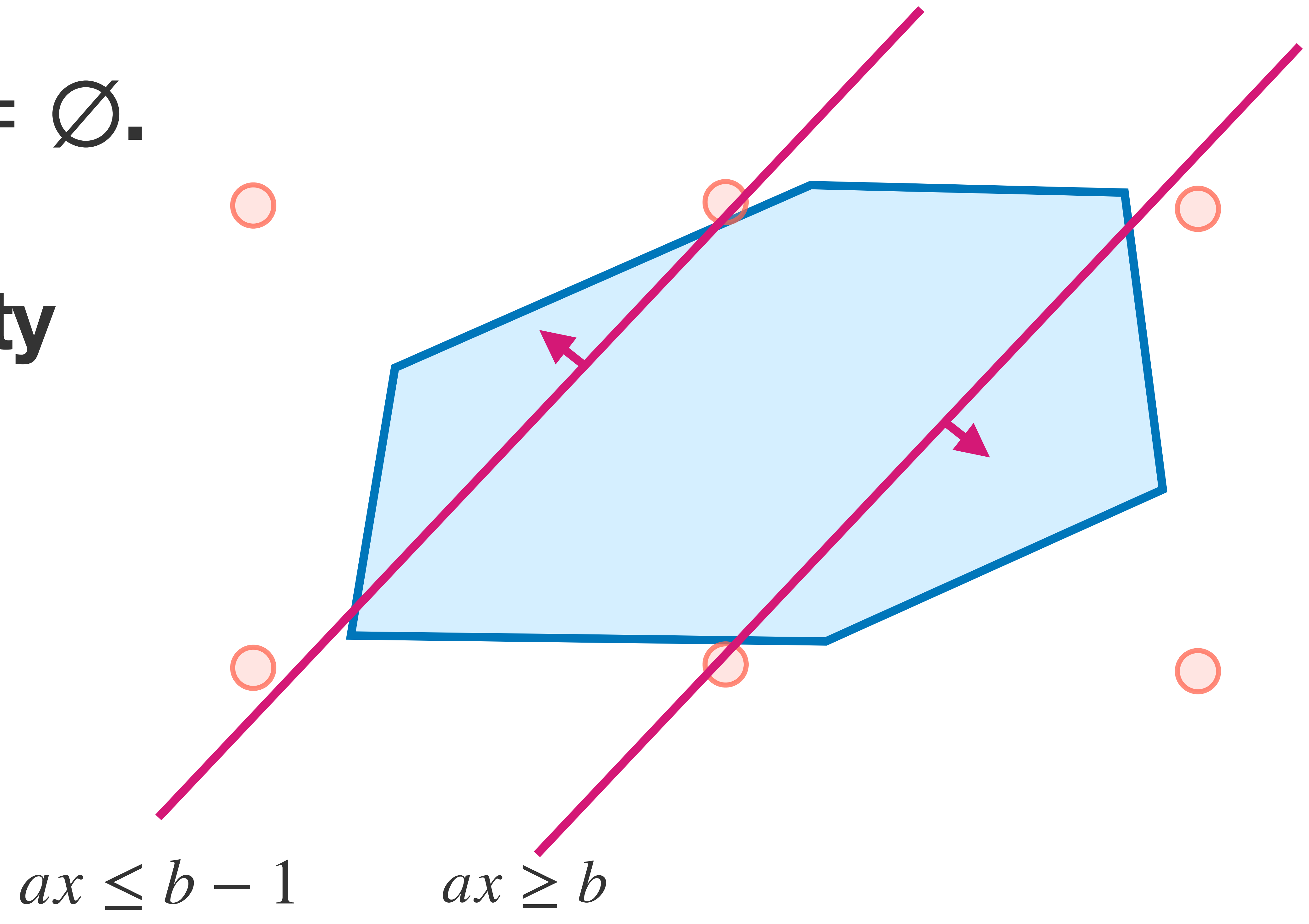

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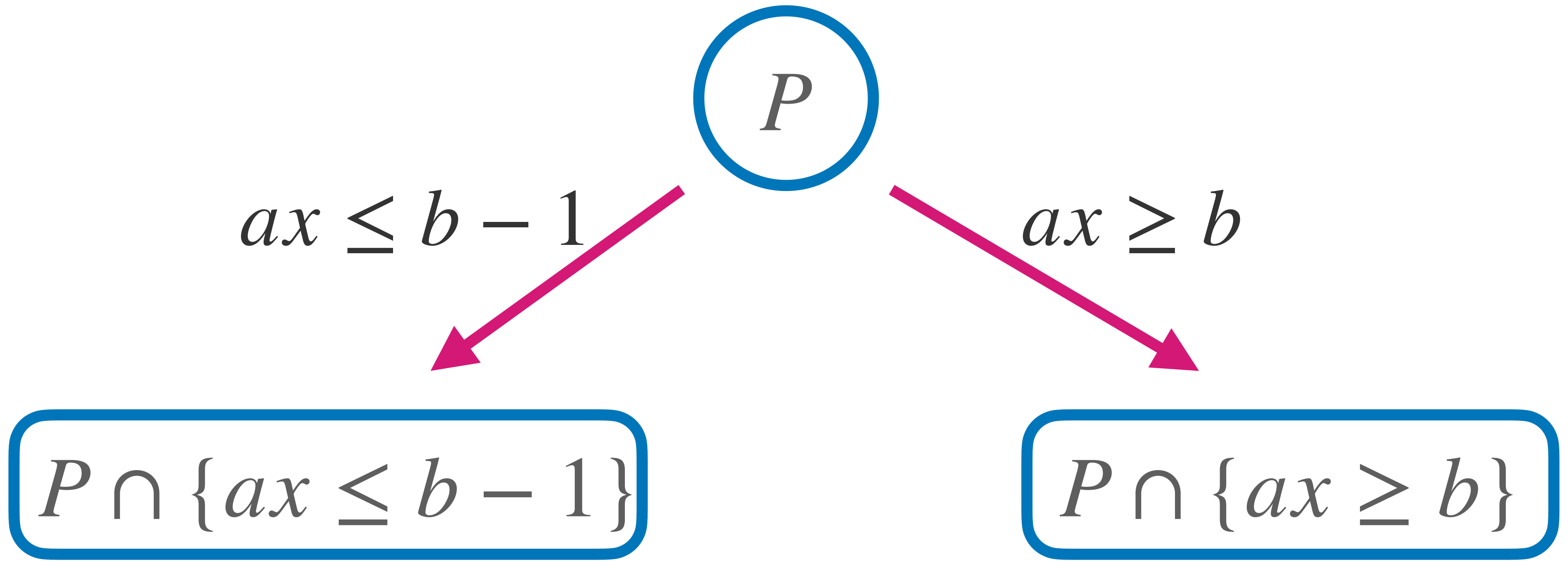

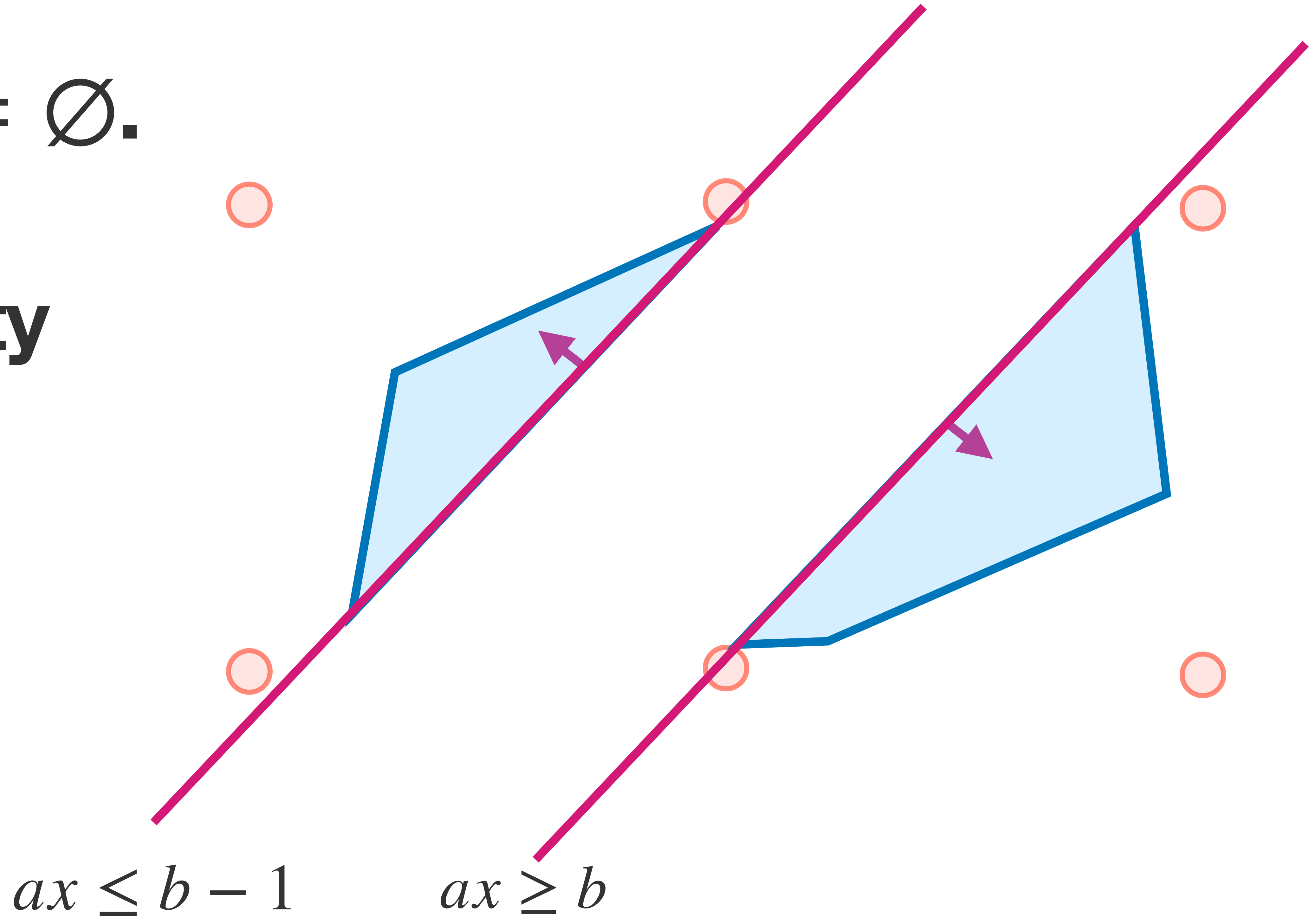

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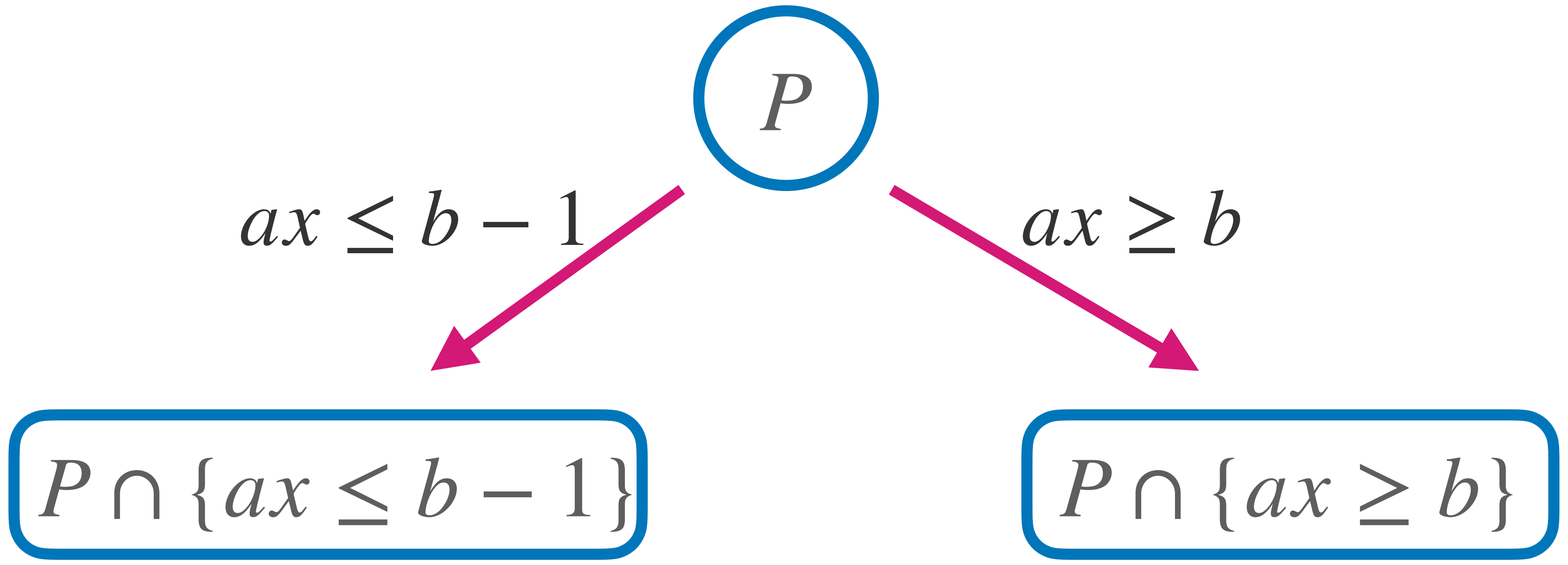

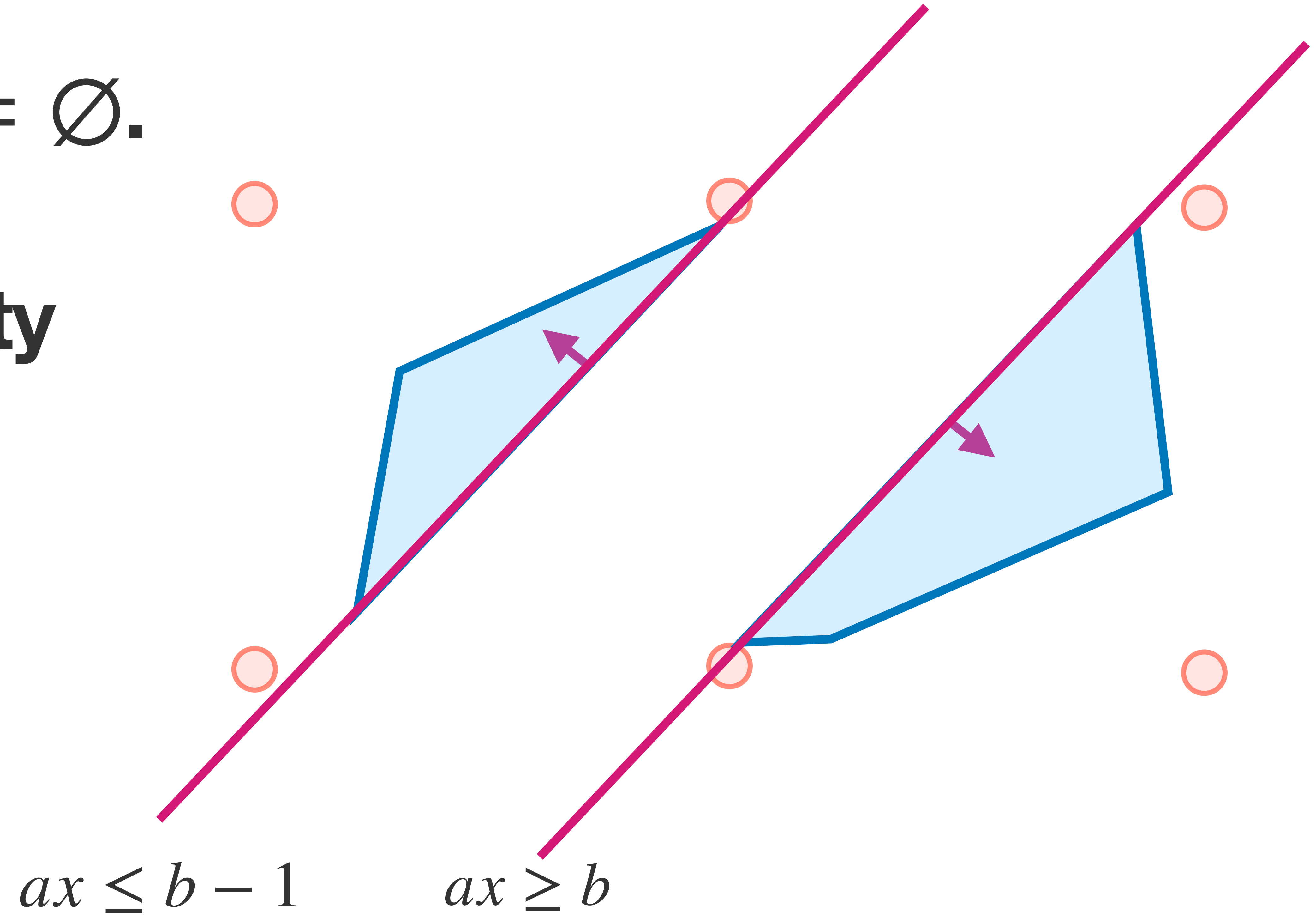

Because a, b are integral, **preserves** integer points!

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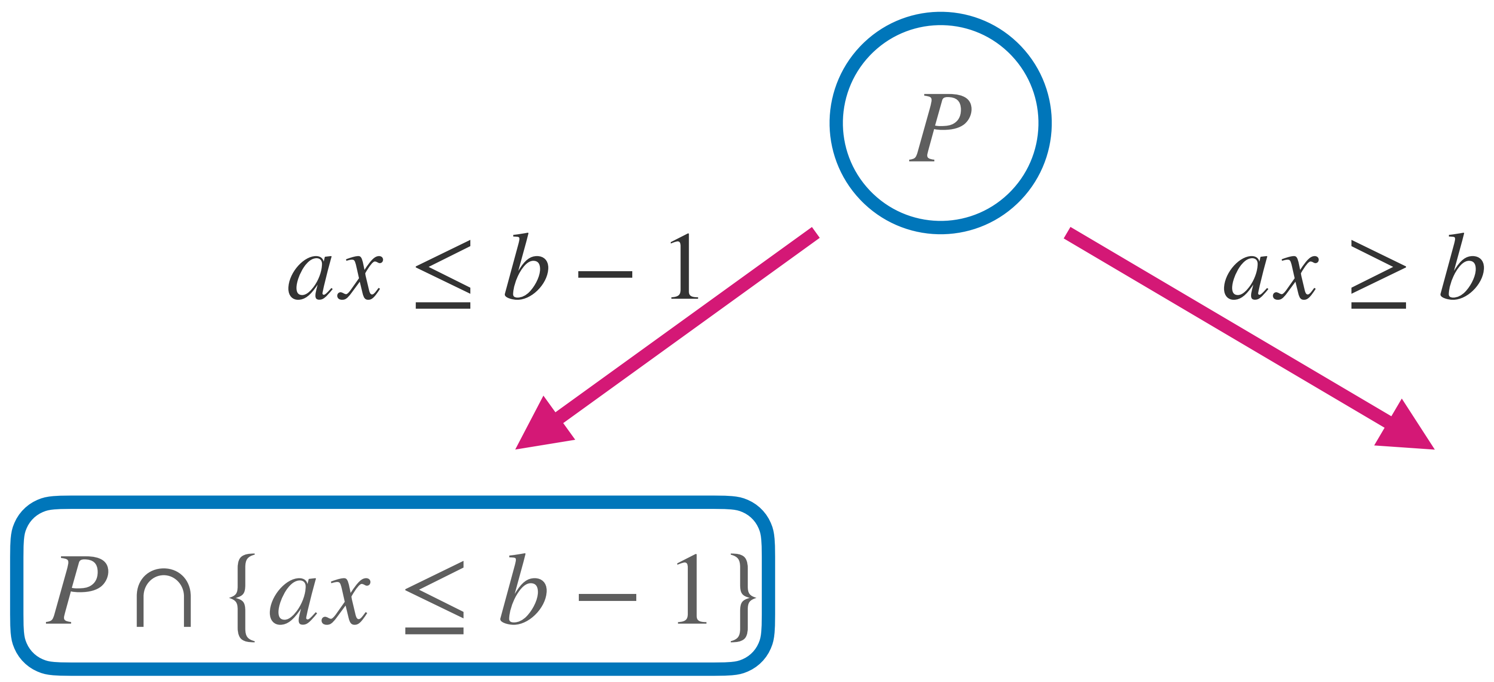

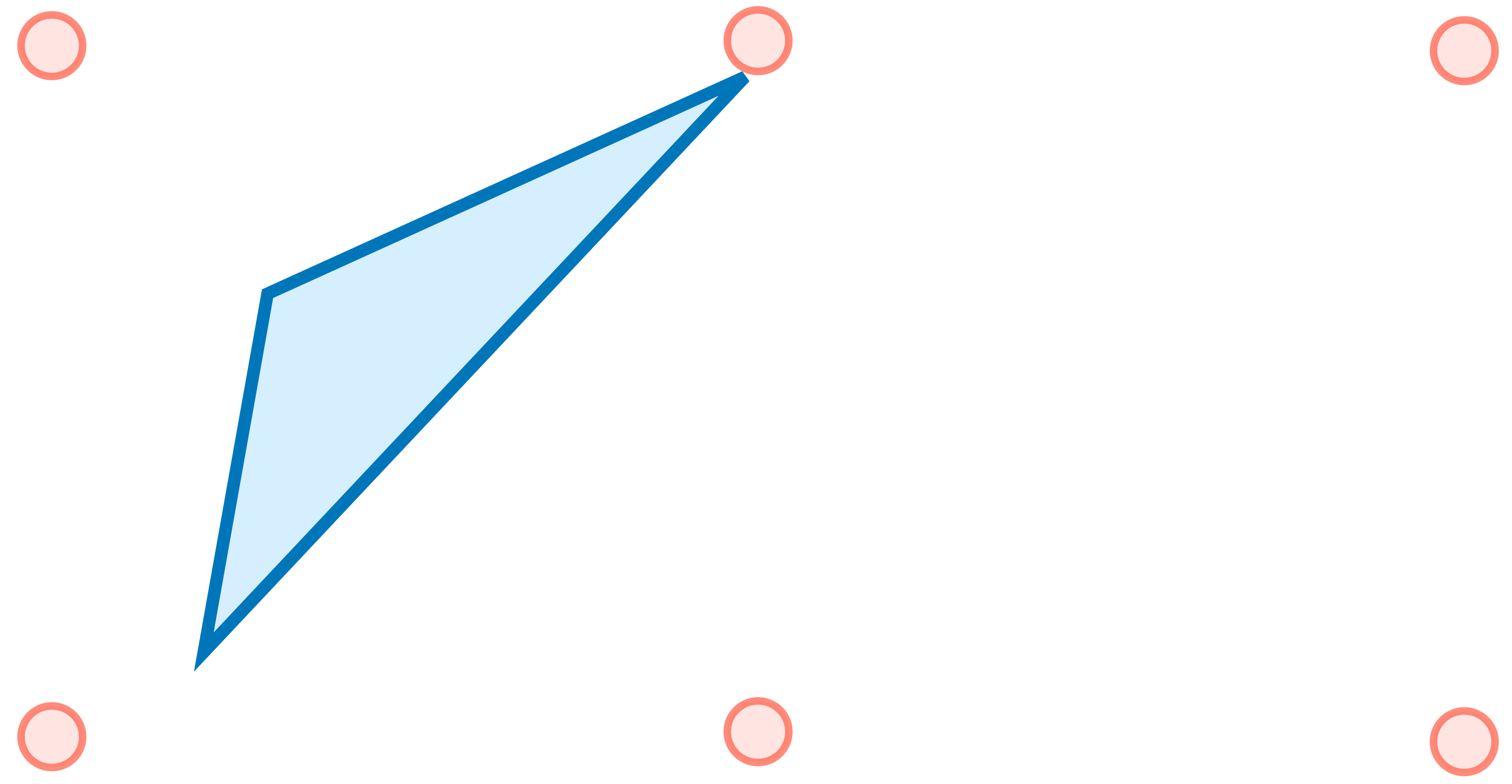

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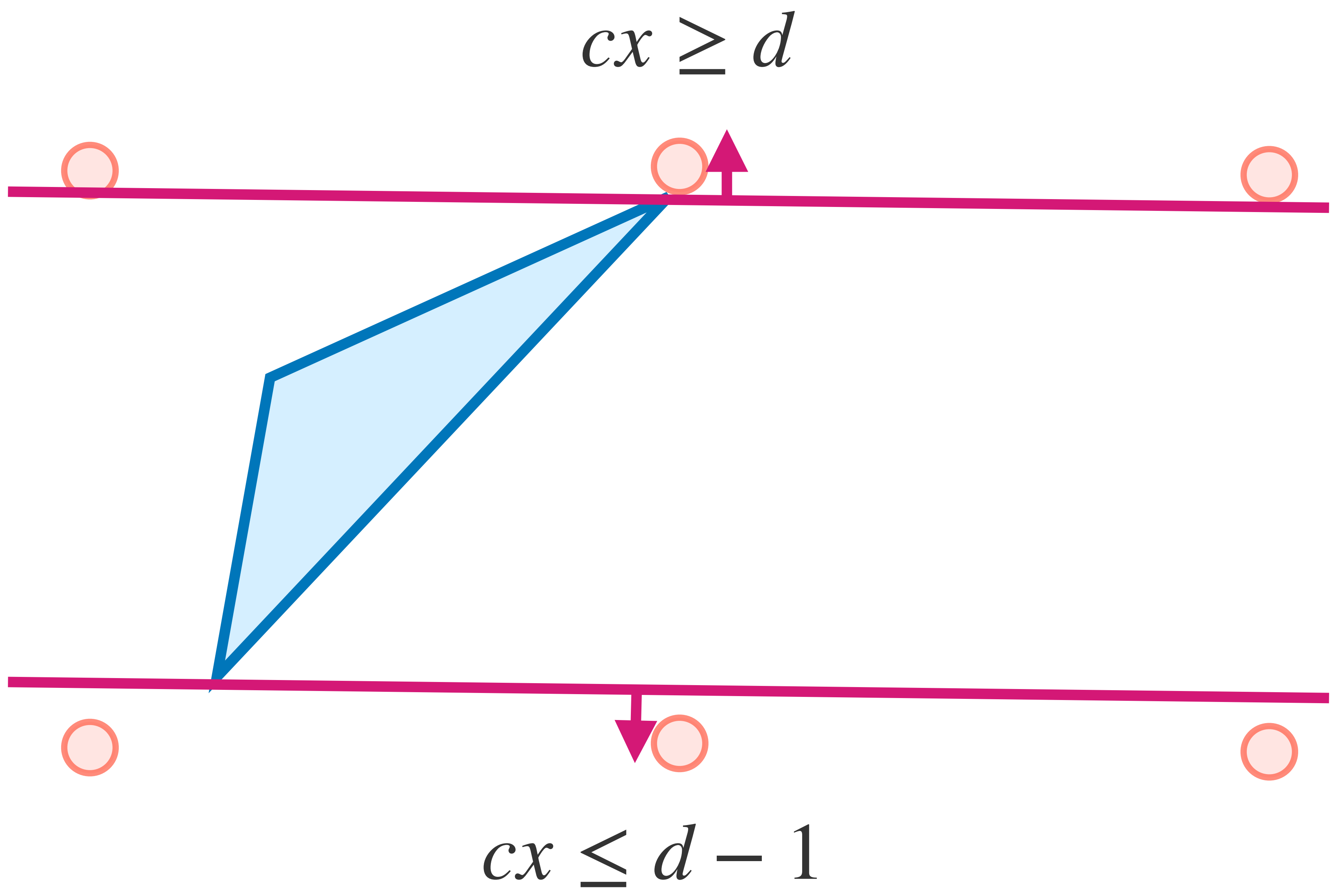

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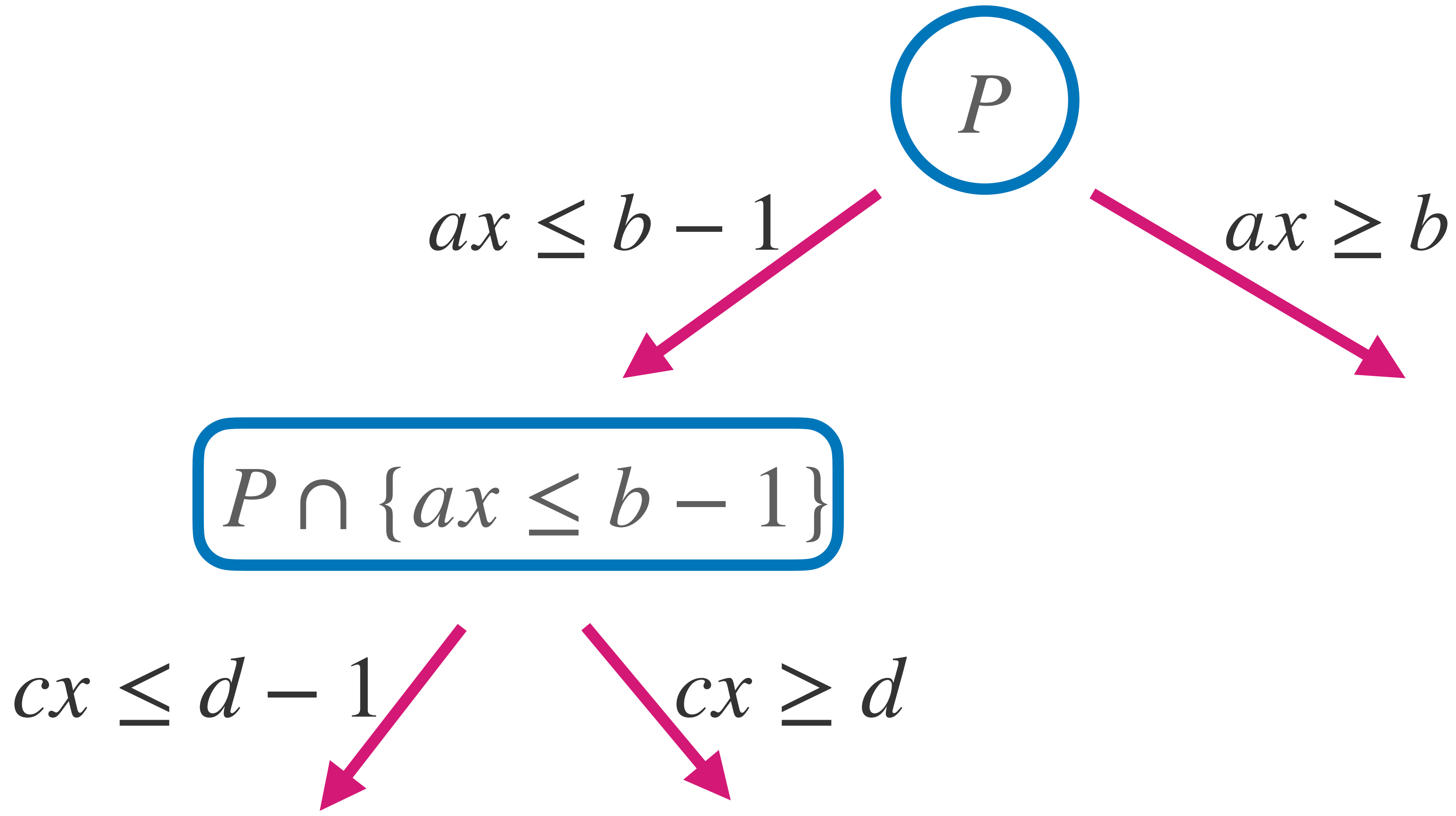

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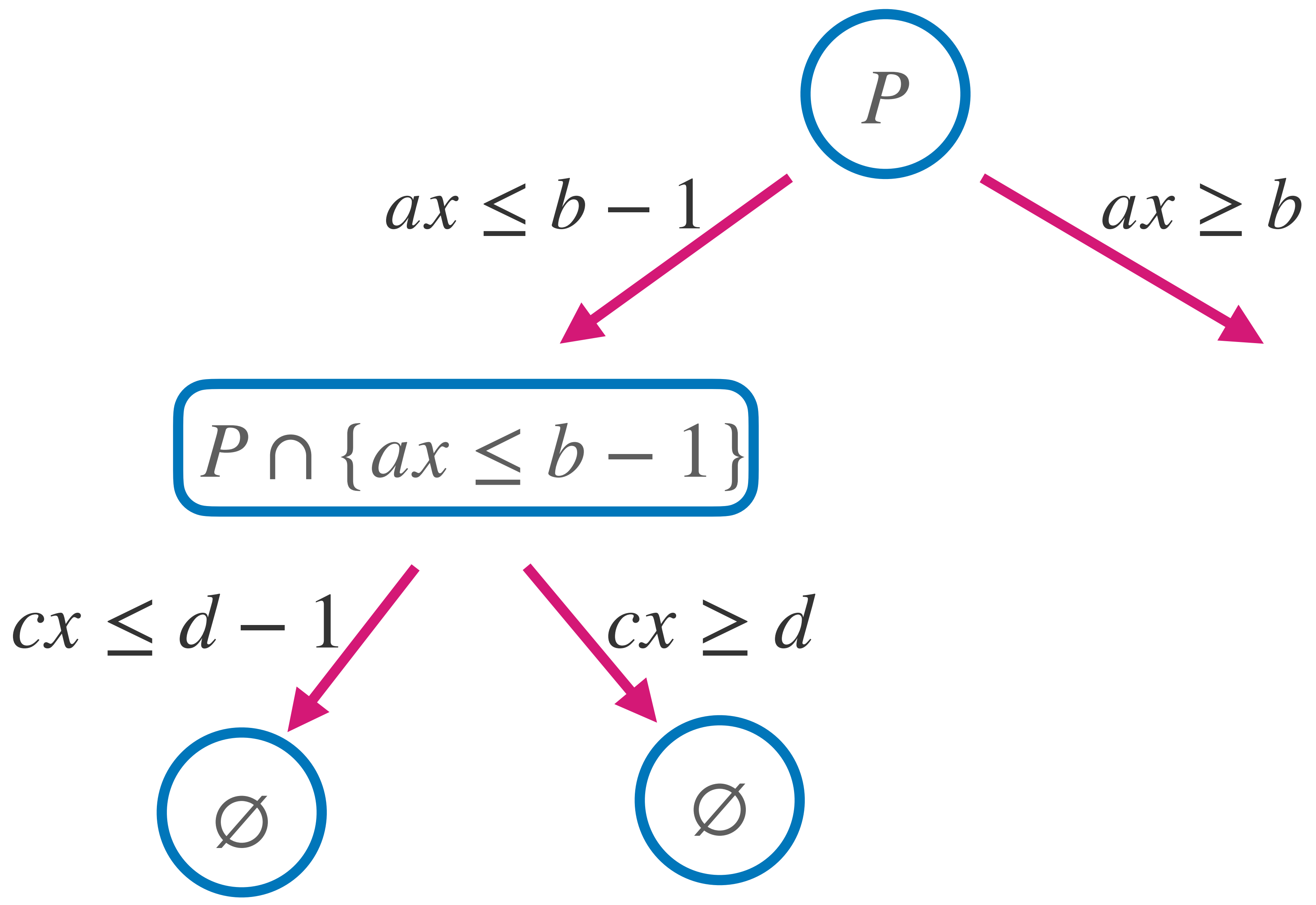

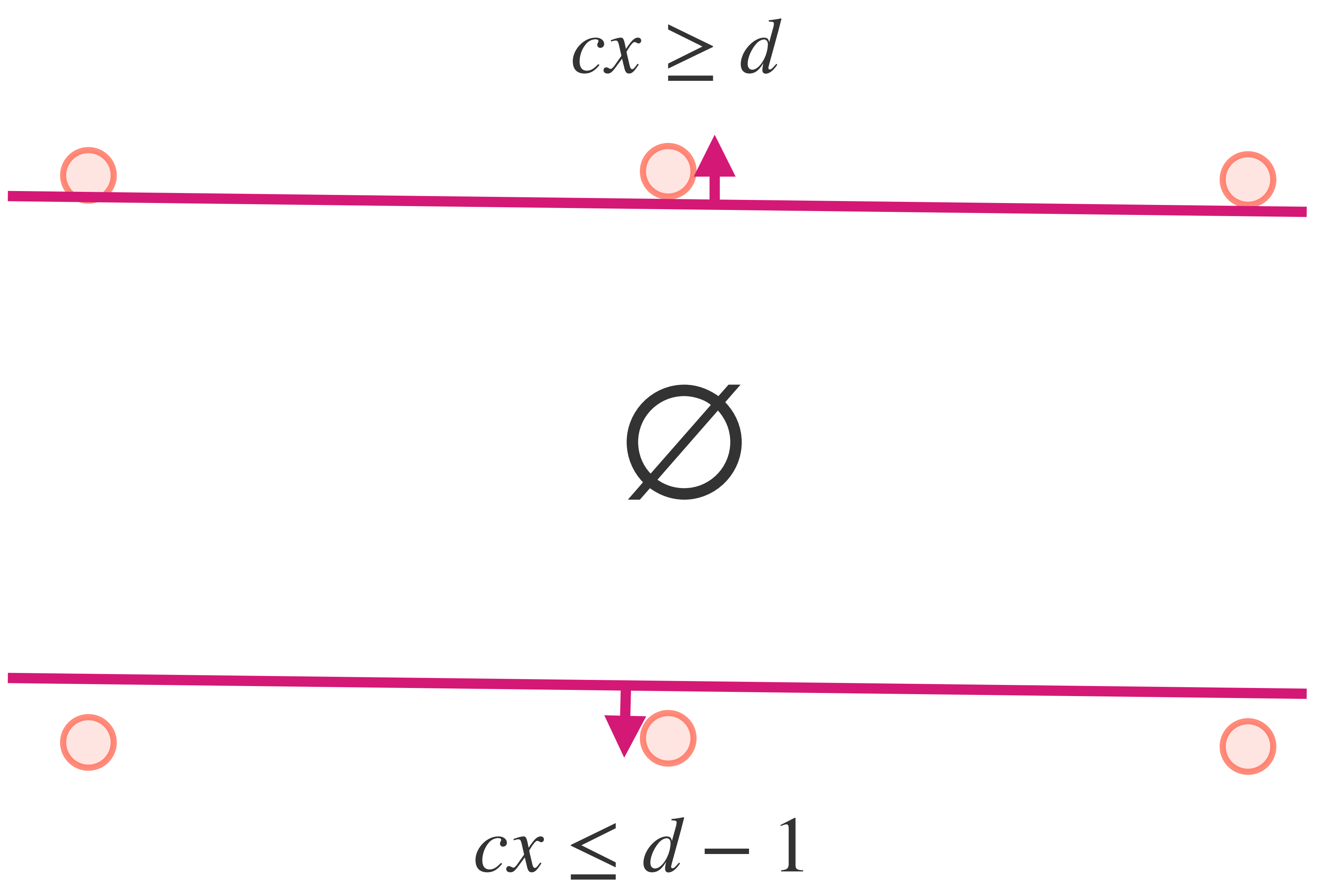

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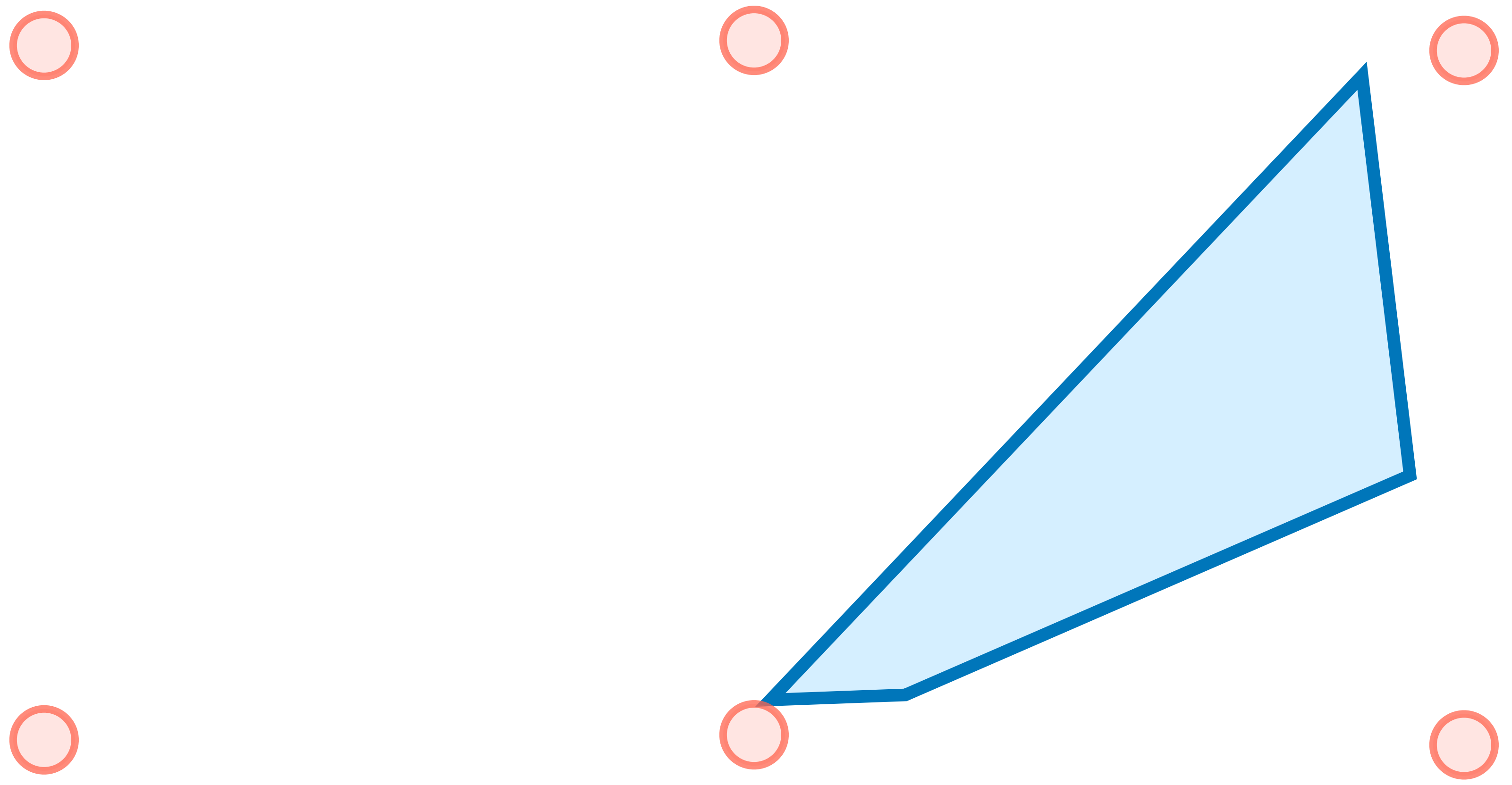

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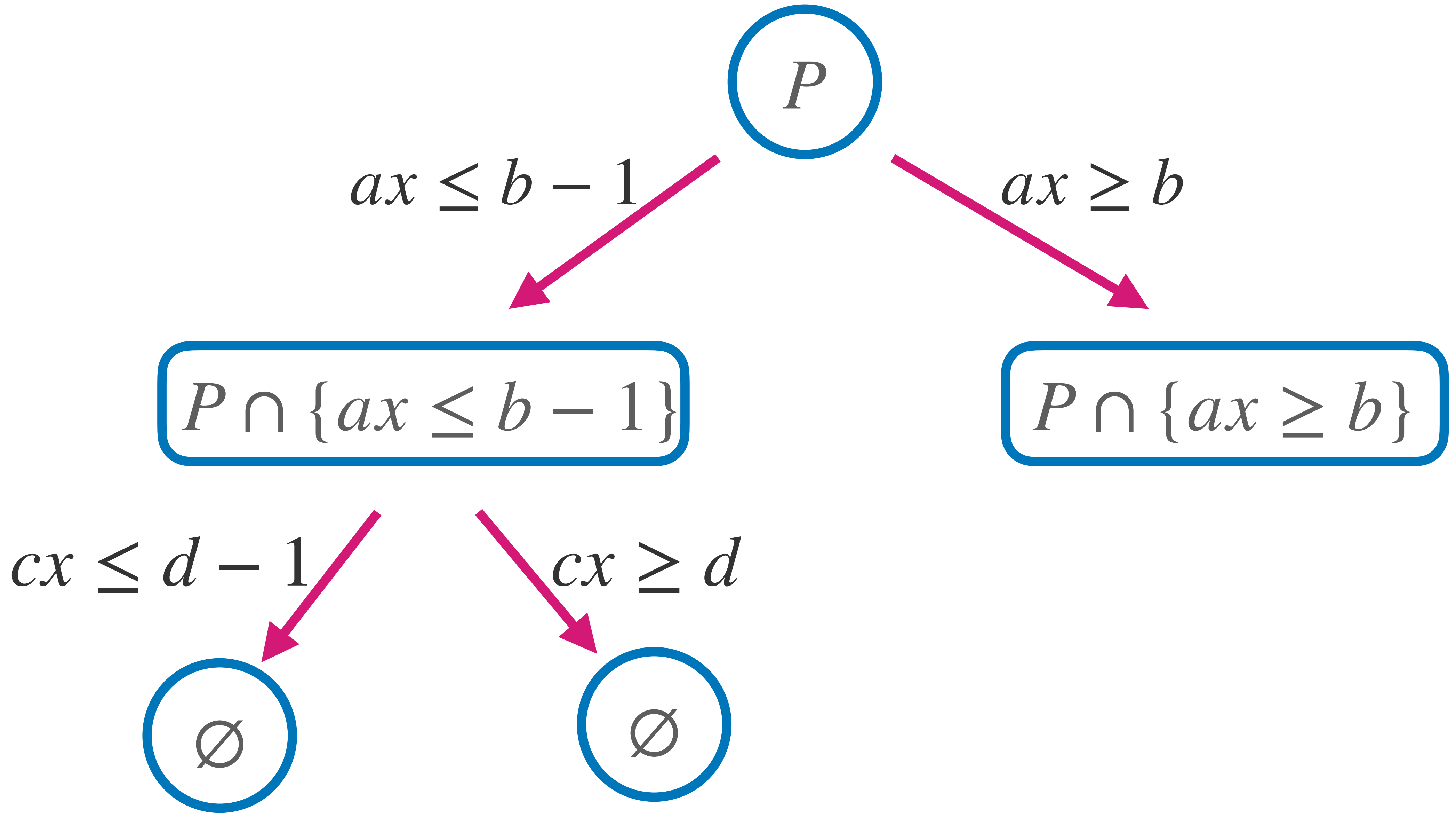

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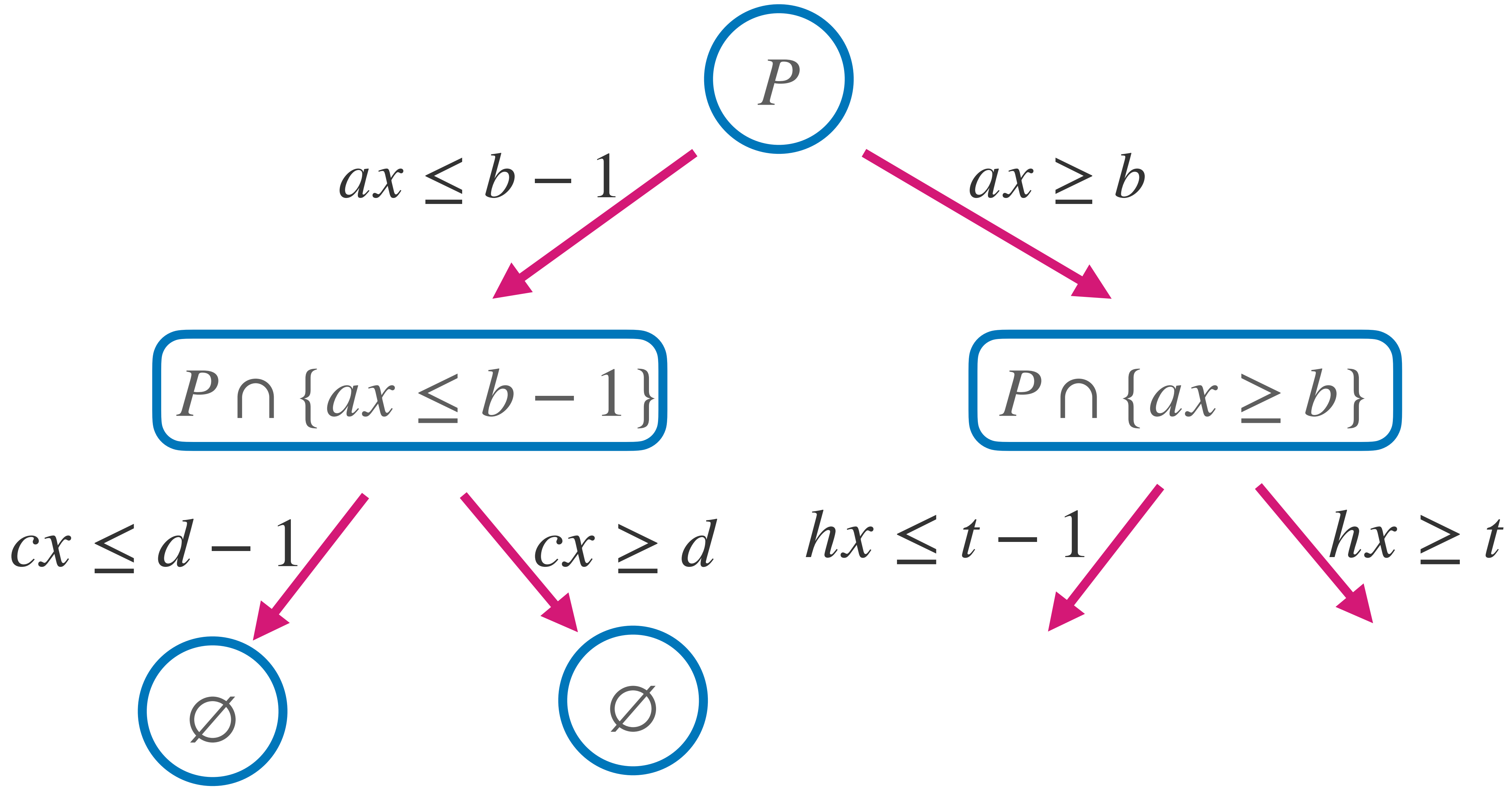

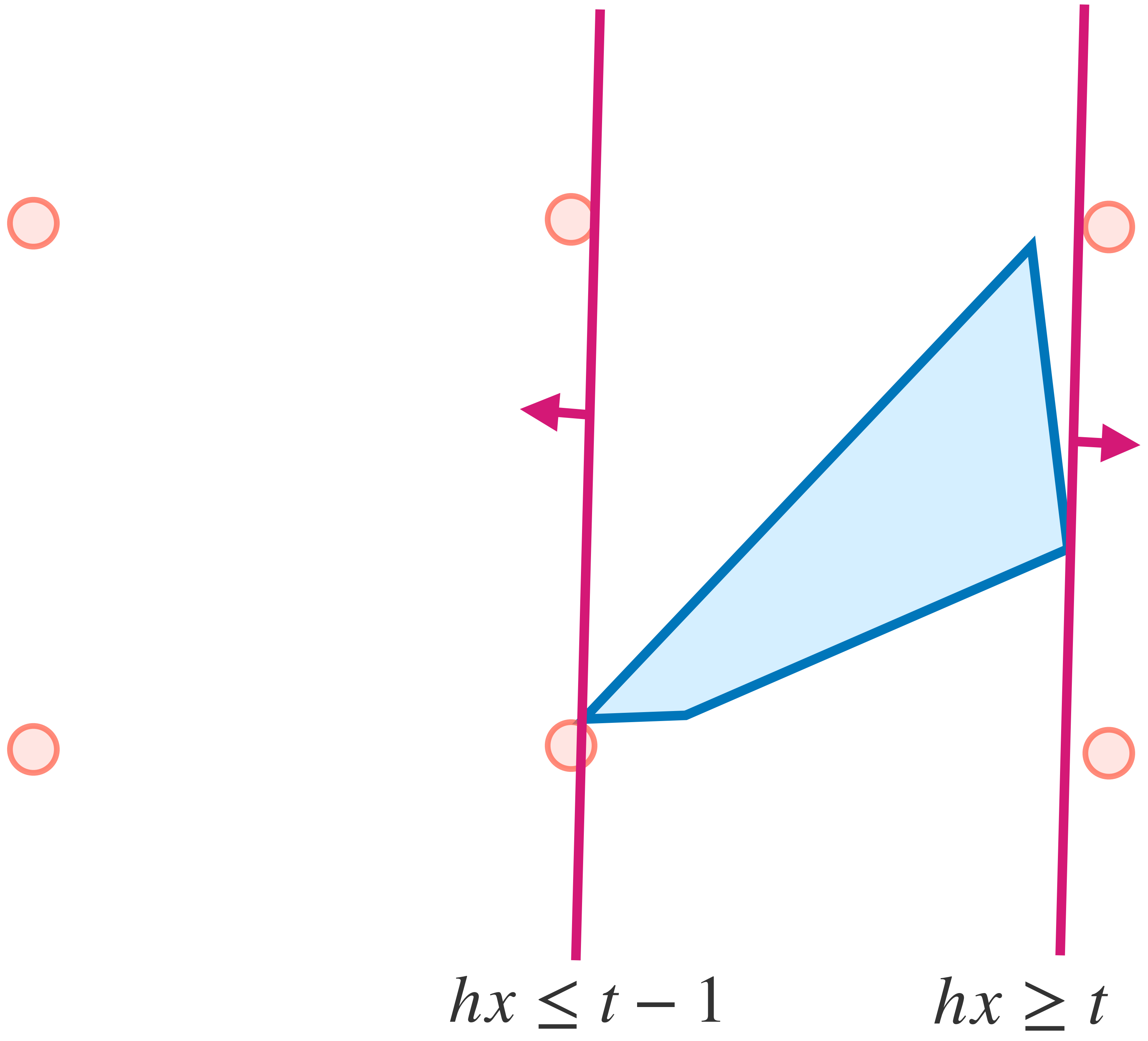

The Stabbing Planes Proof System

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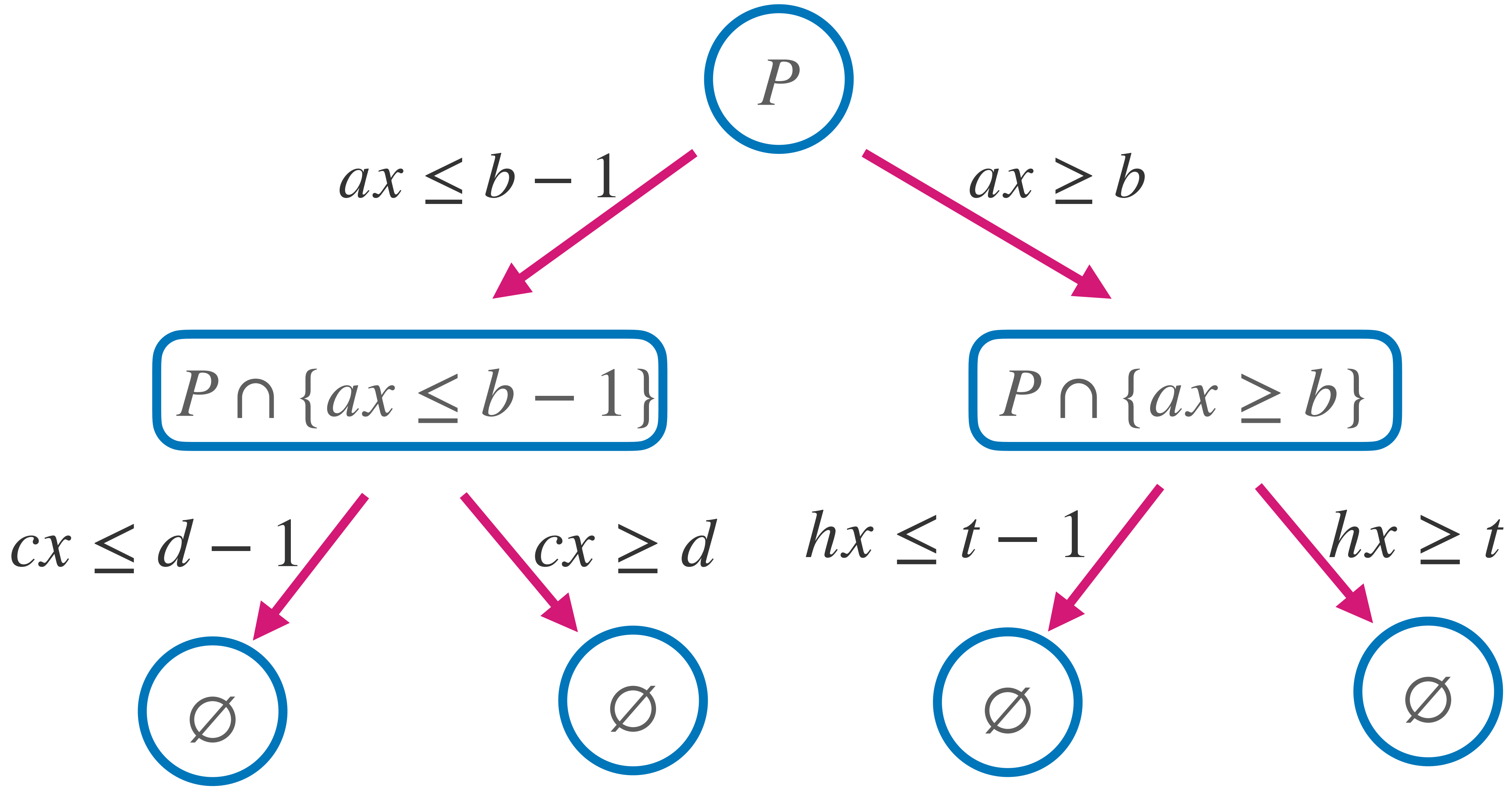

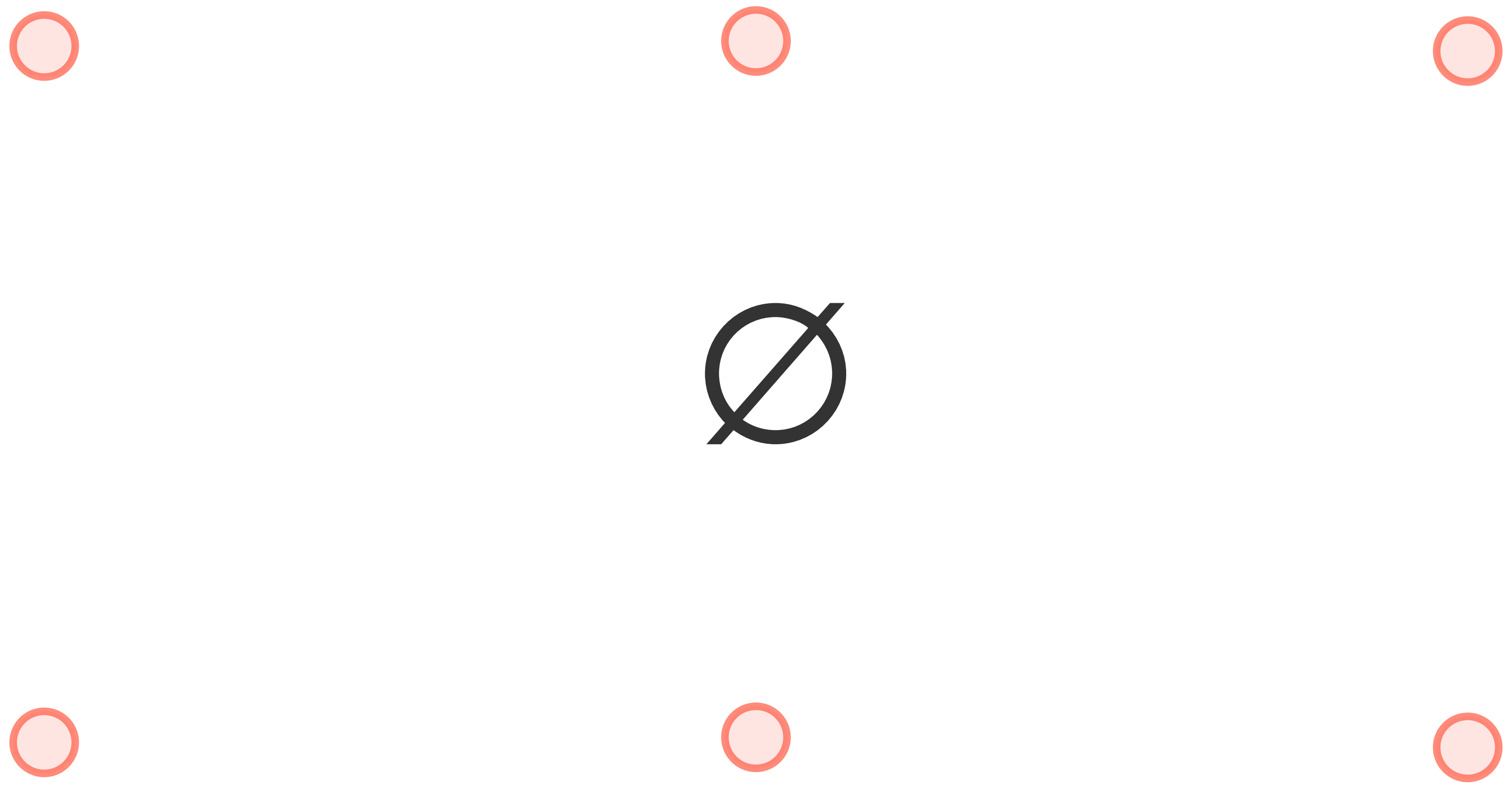

The Stabbing Planes Proof System

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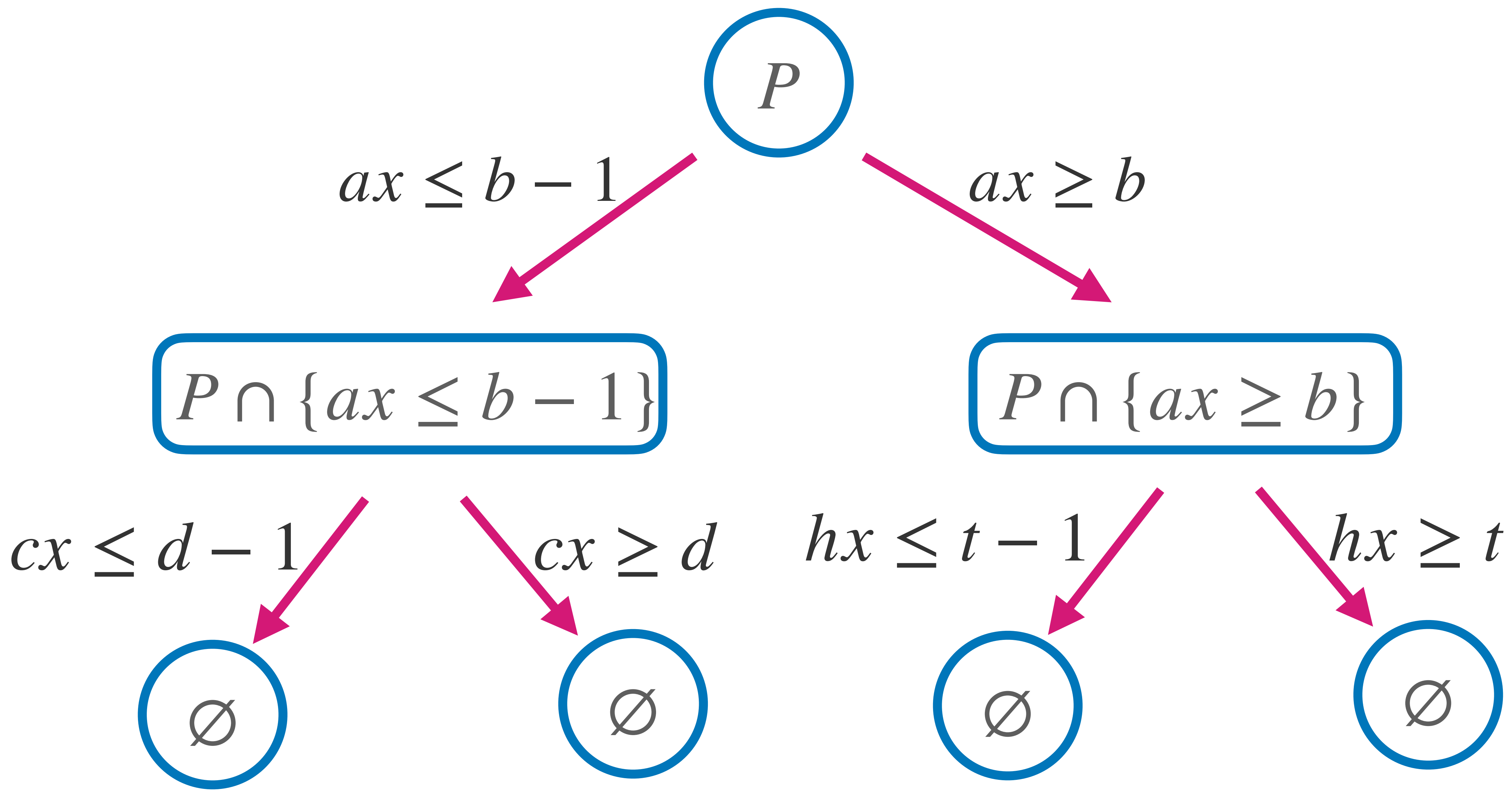

Proof that $P \cap \mathbb{Z}^n = \emptyset$!

Stabbing Planes

Claim

The execution of a **branch-and-cut** solver produces a **Stabbing Planes proof**.

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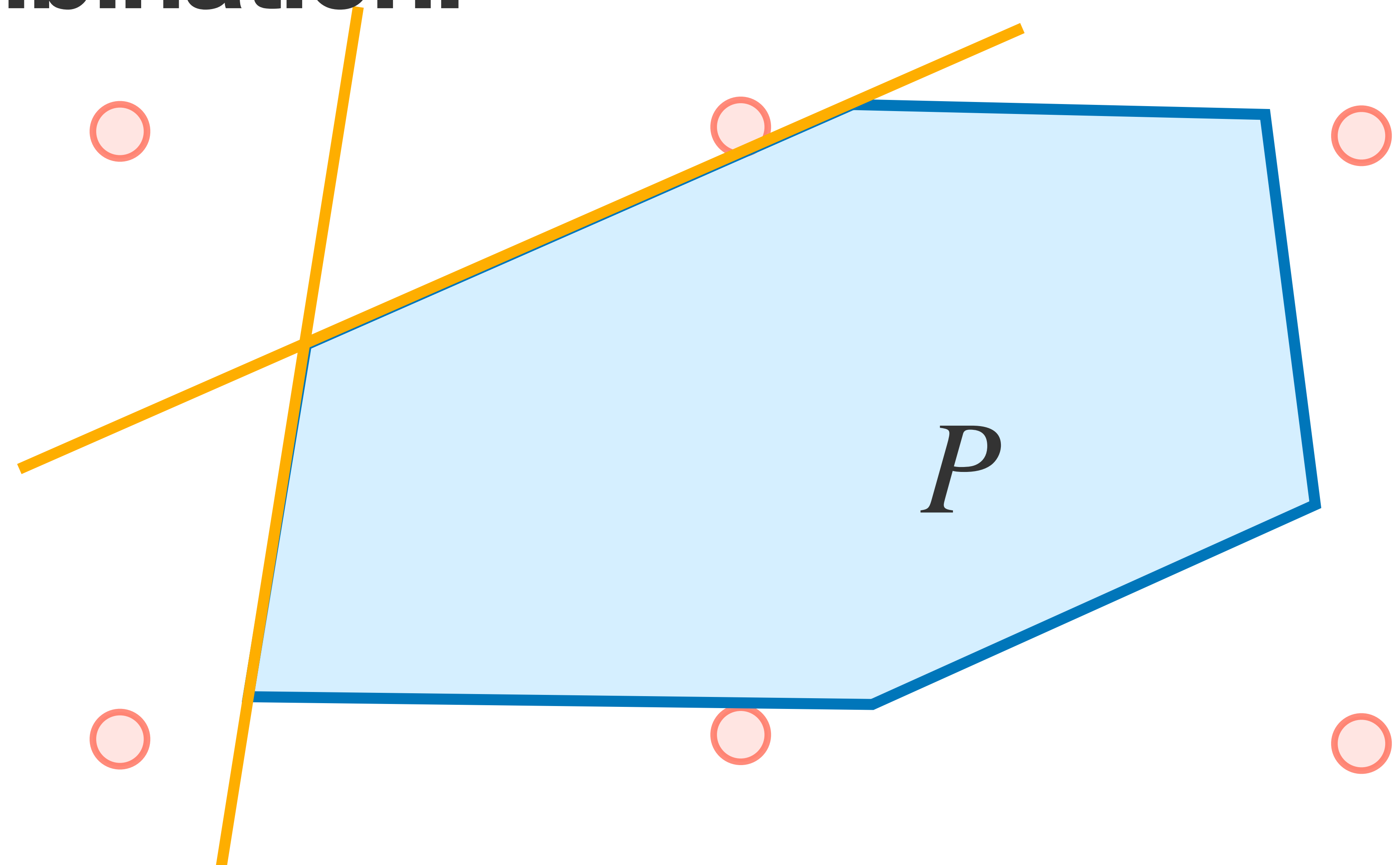

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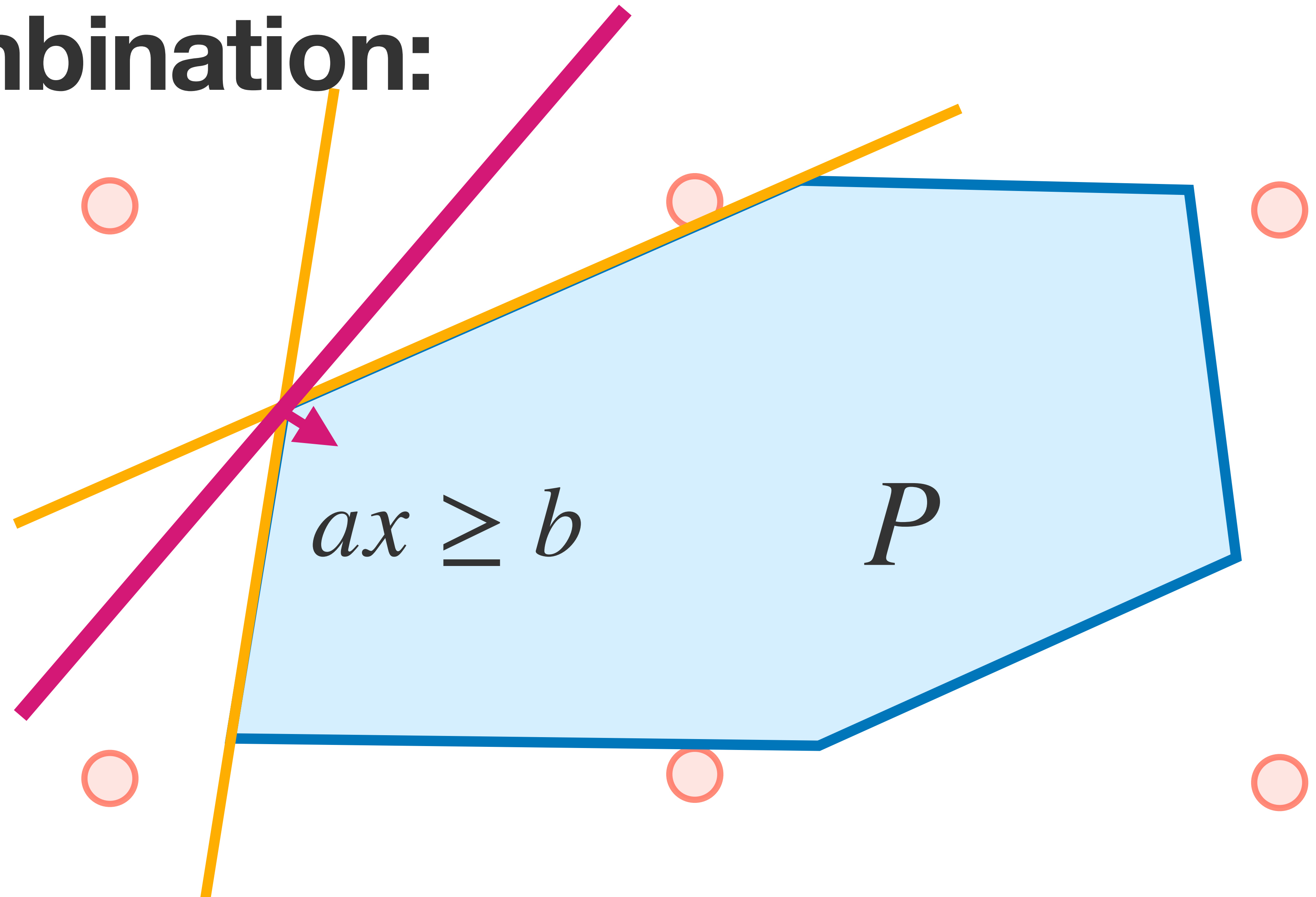

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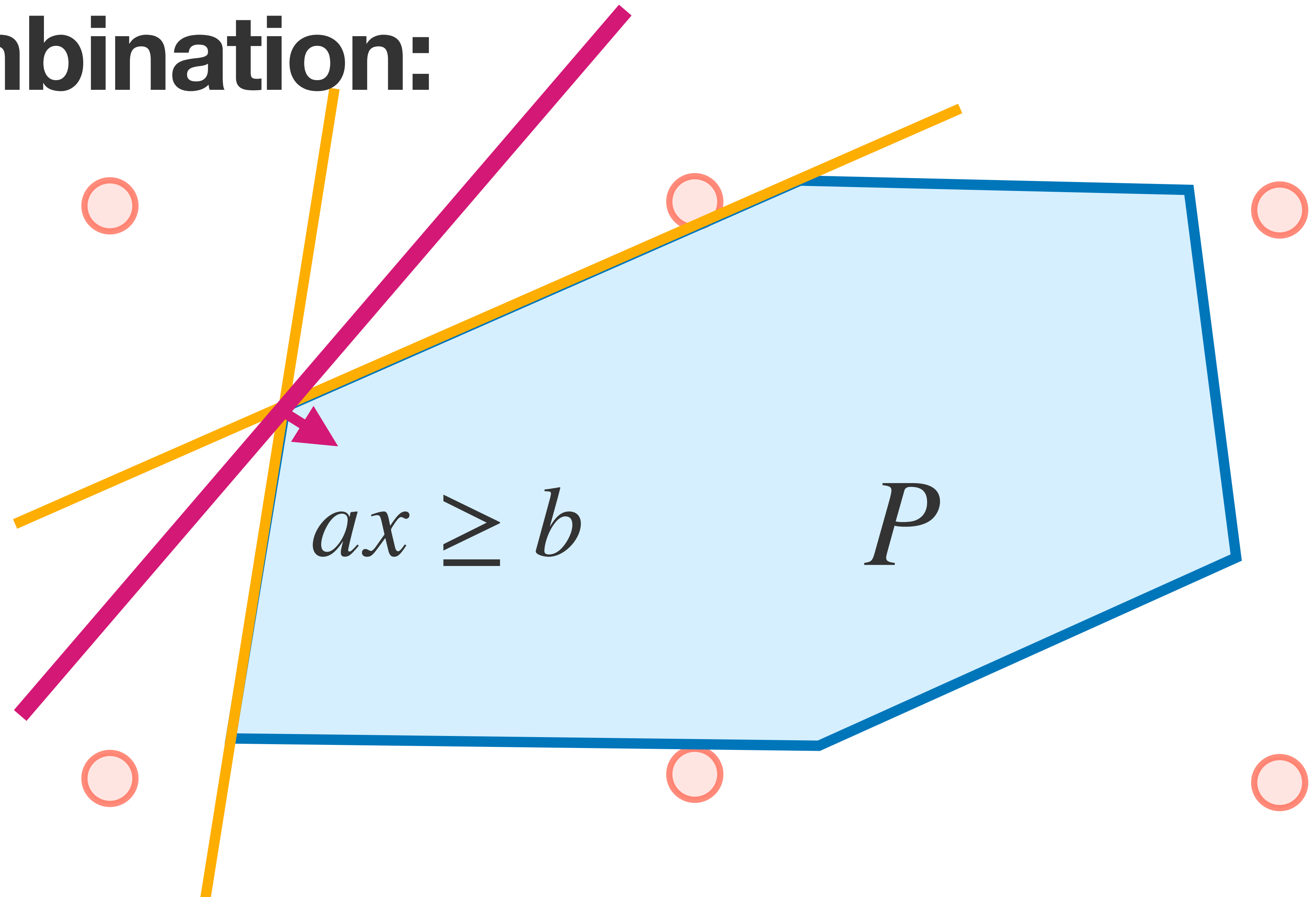

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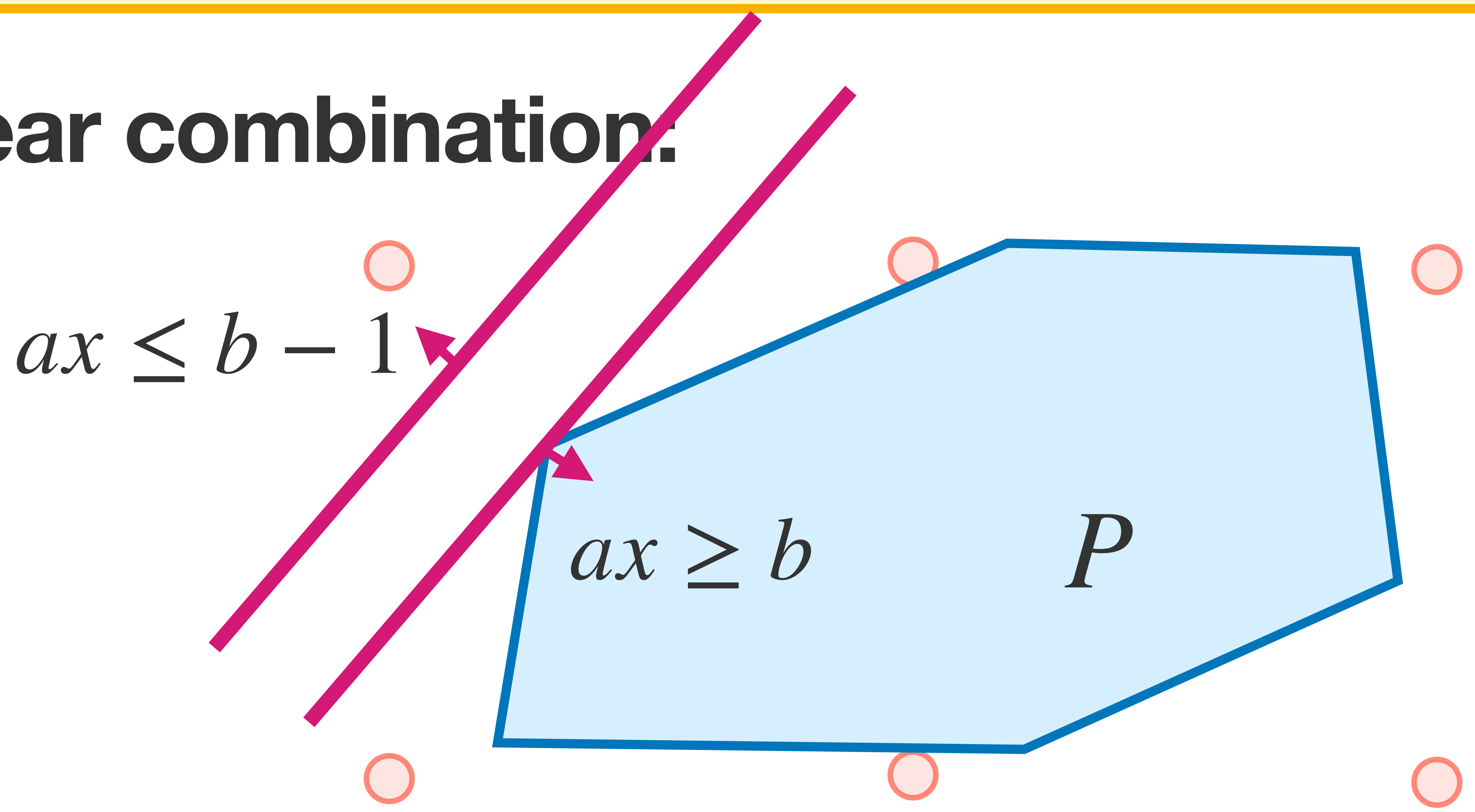

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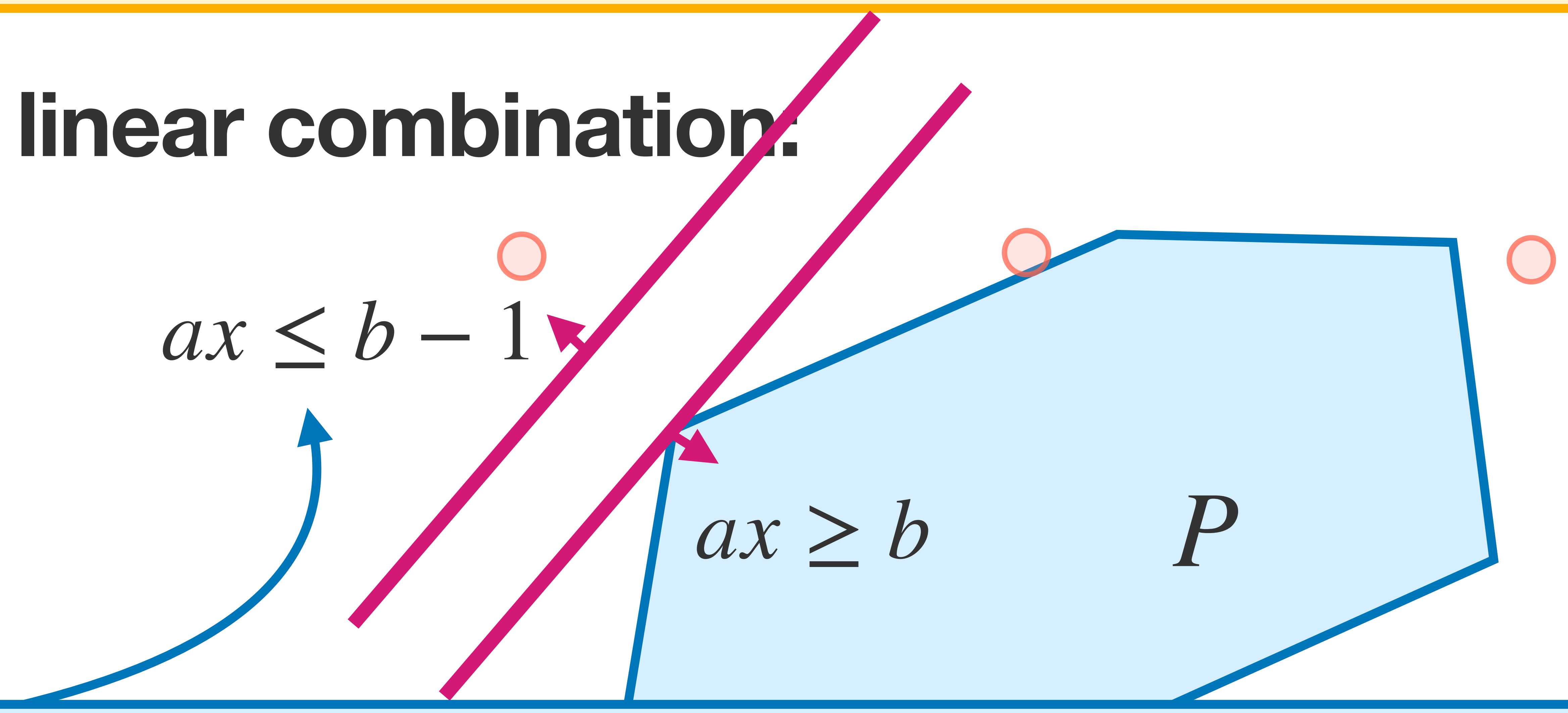

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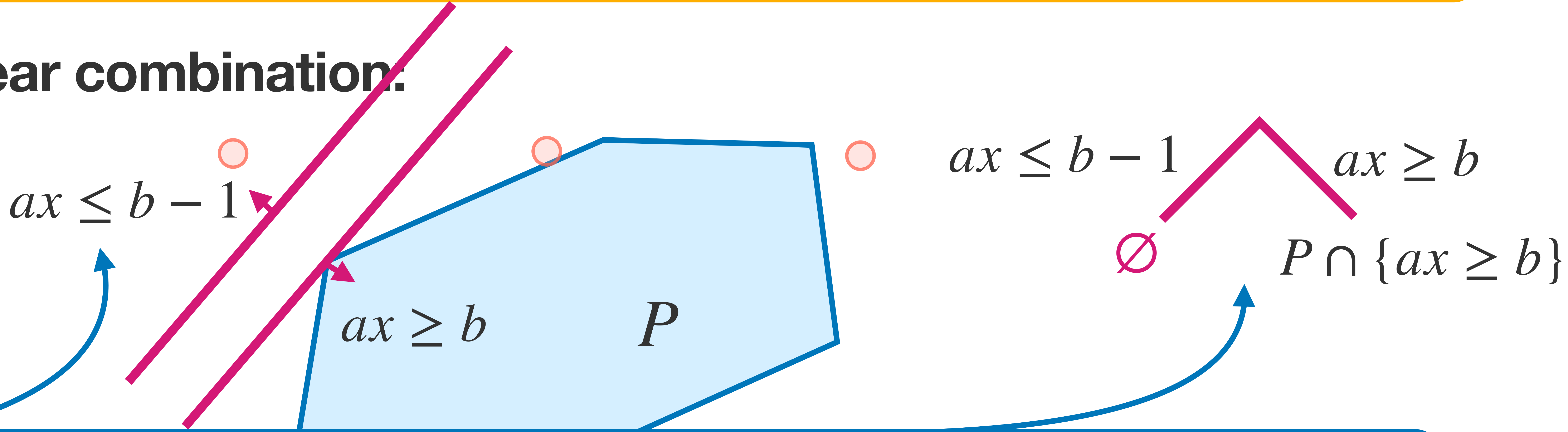

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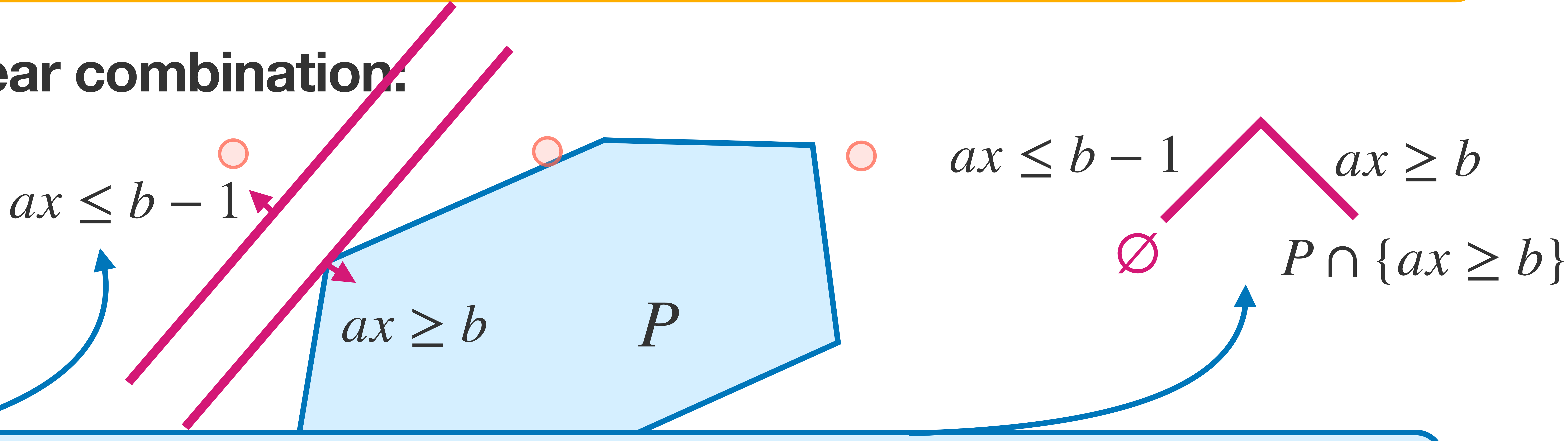

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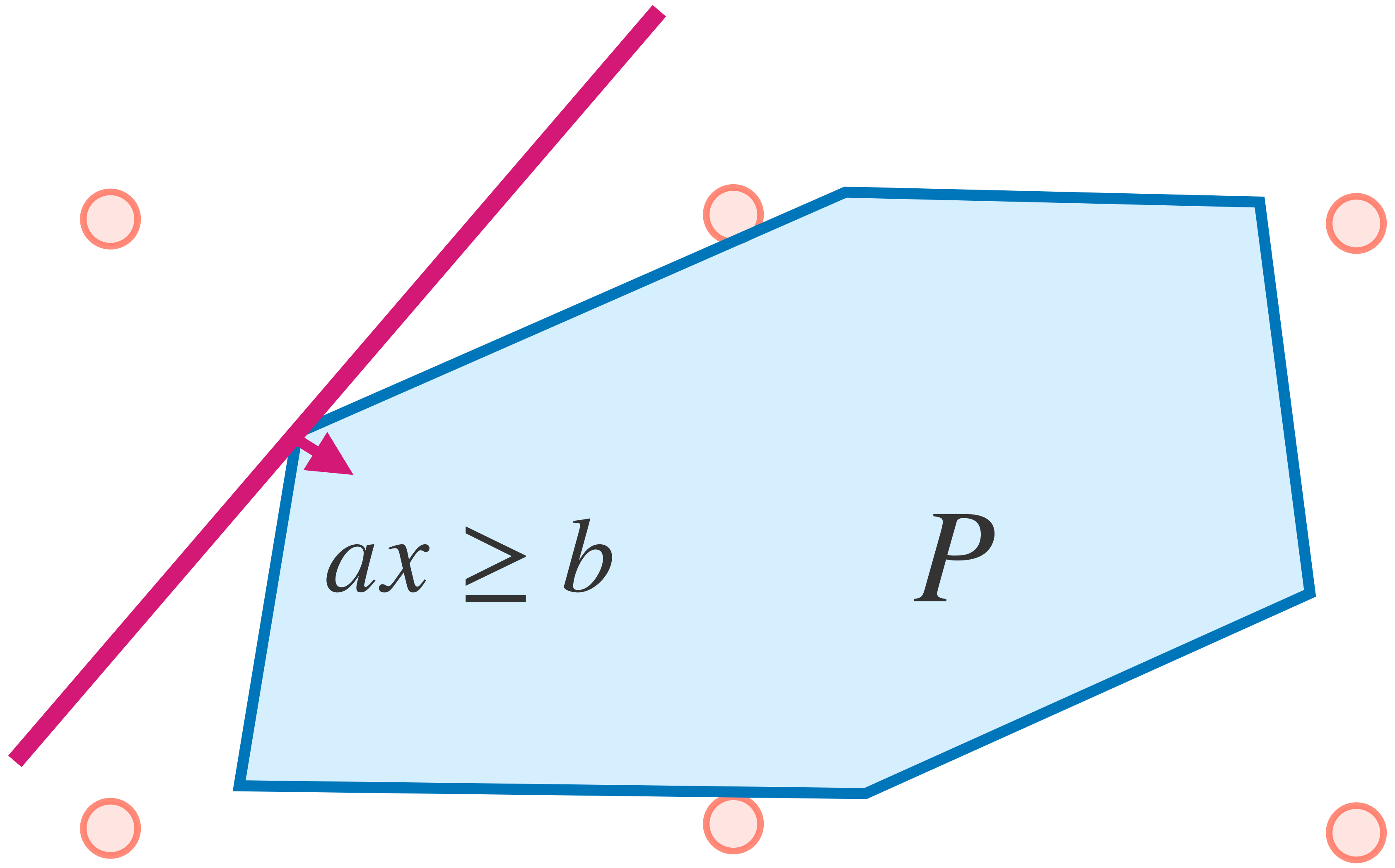

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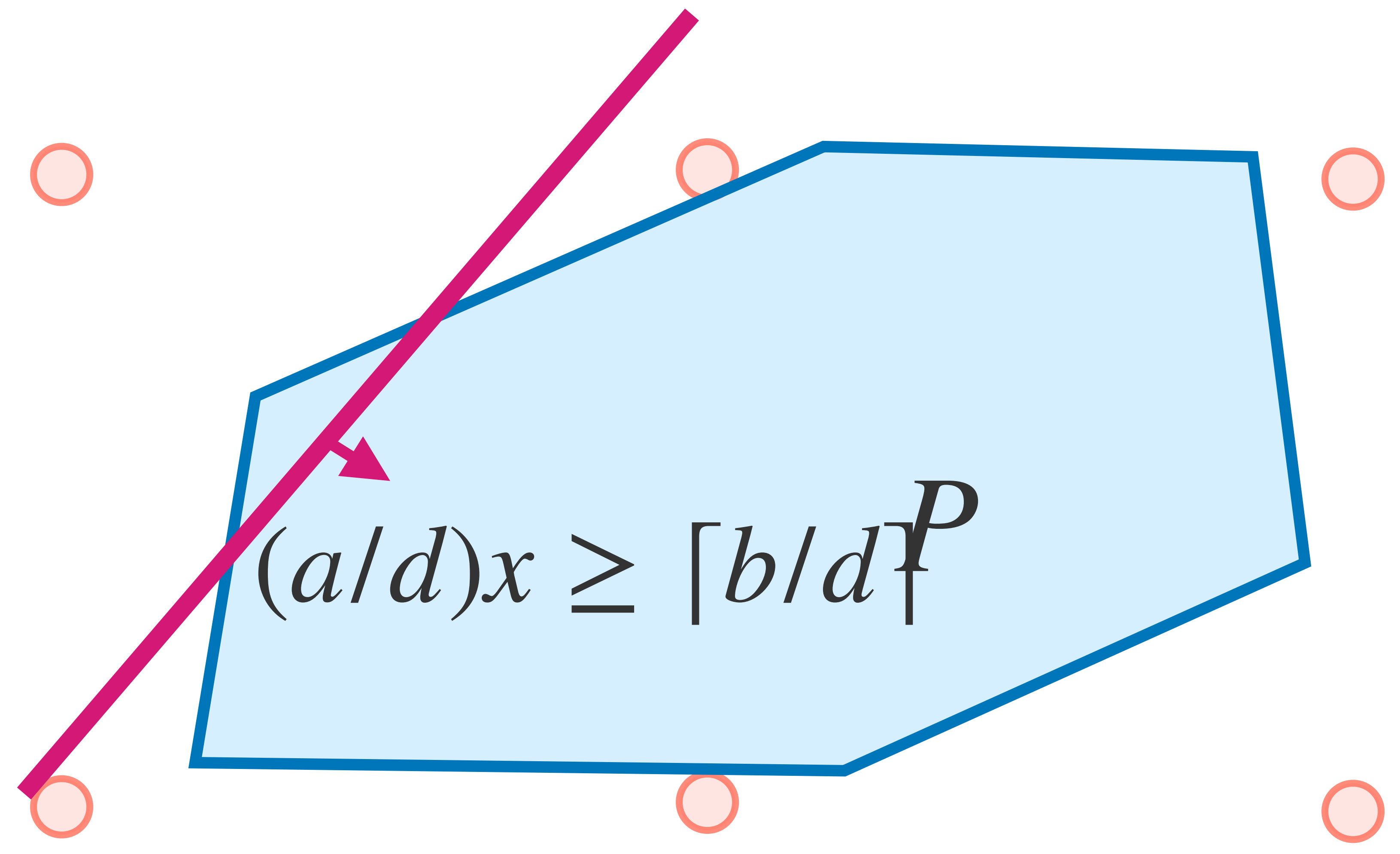

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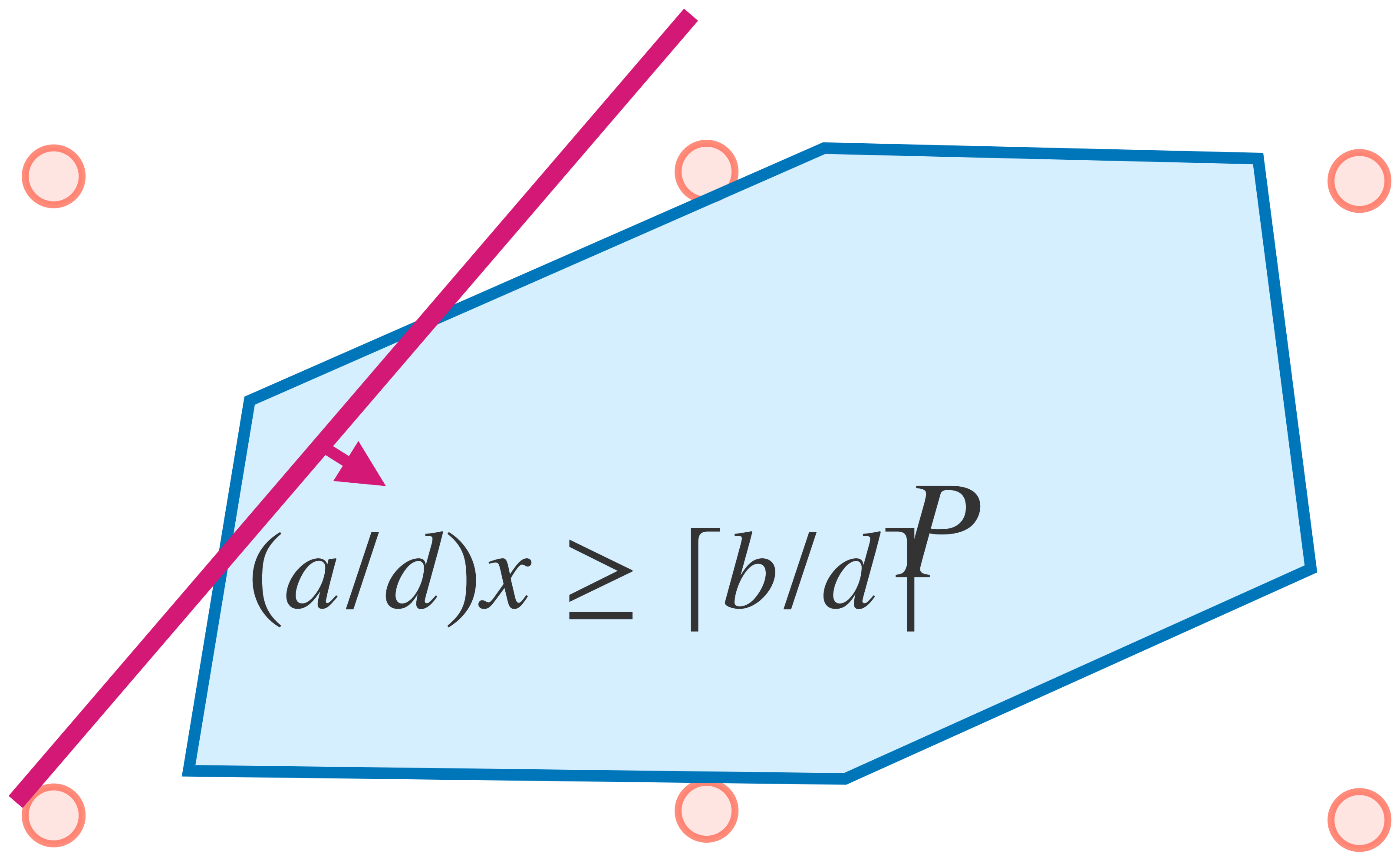

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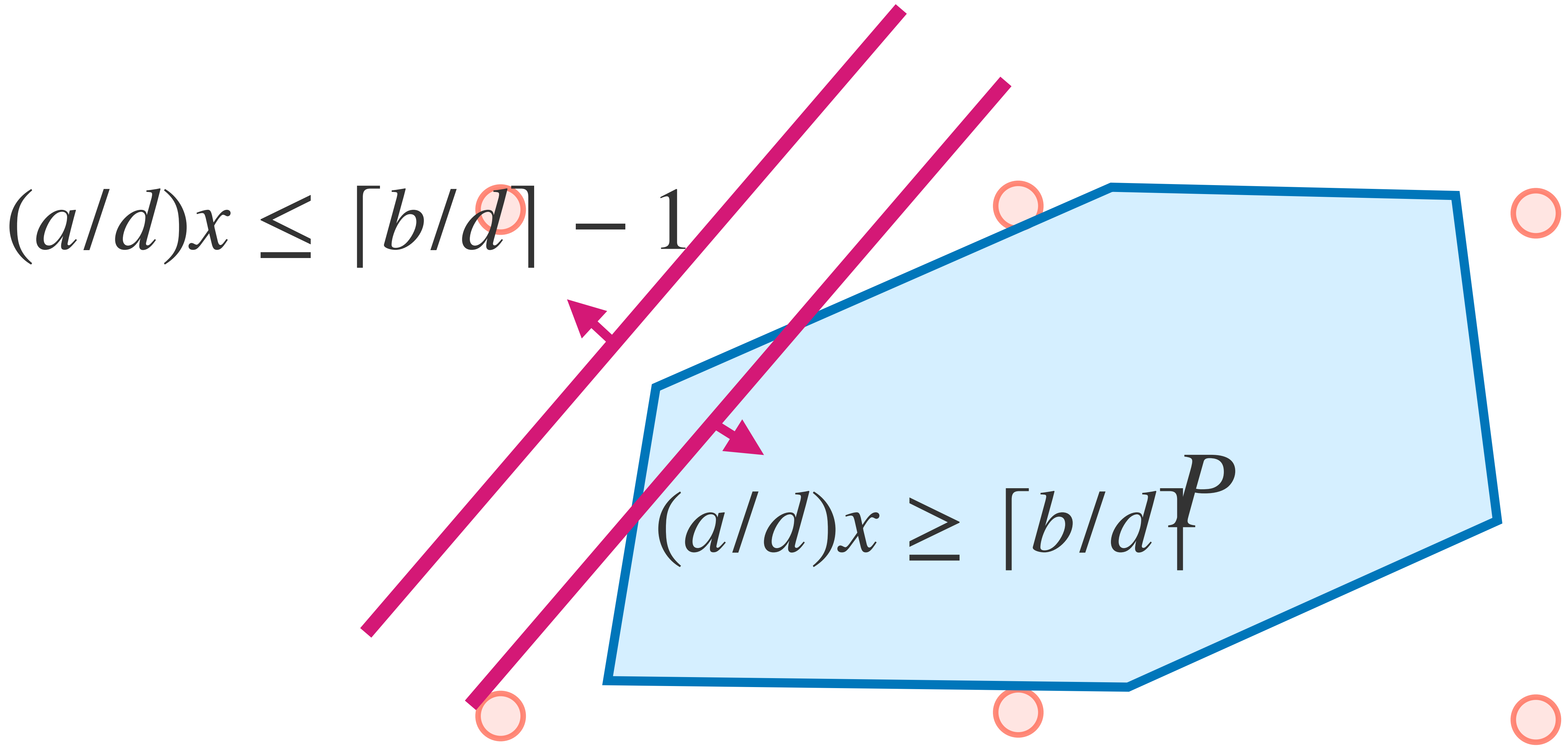

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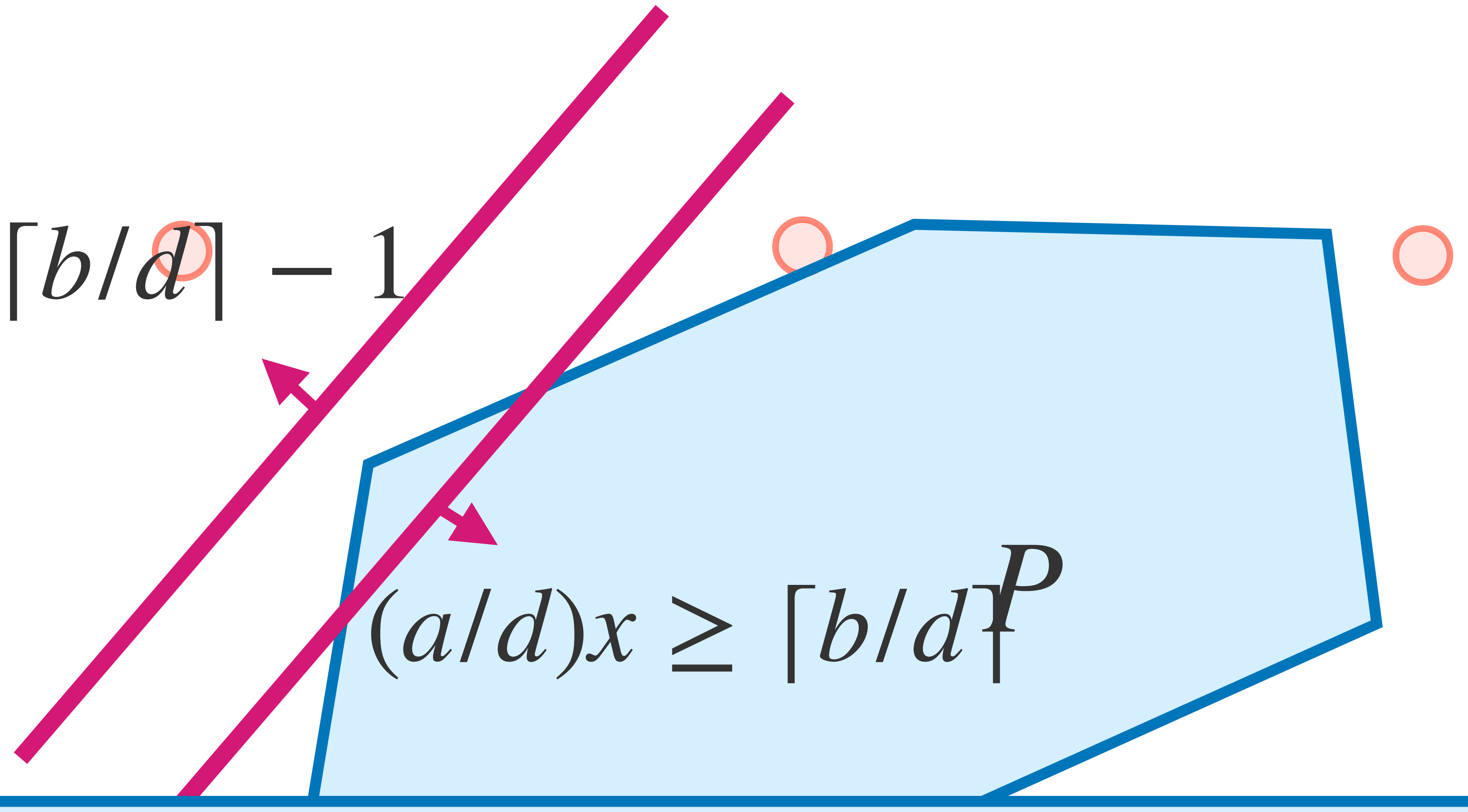

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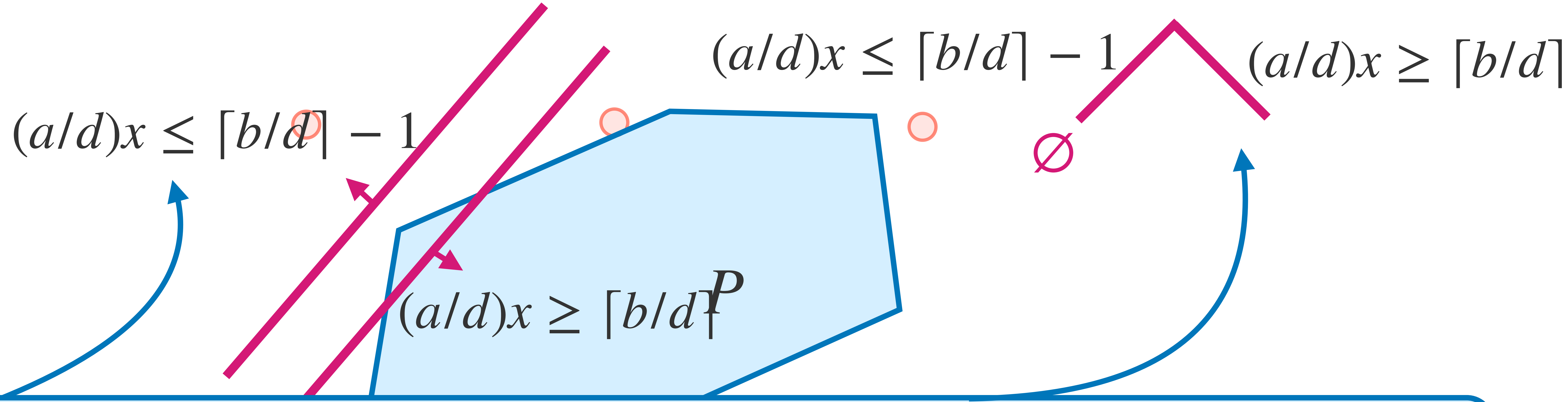

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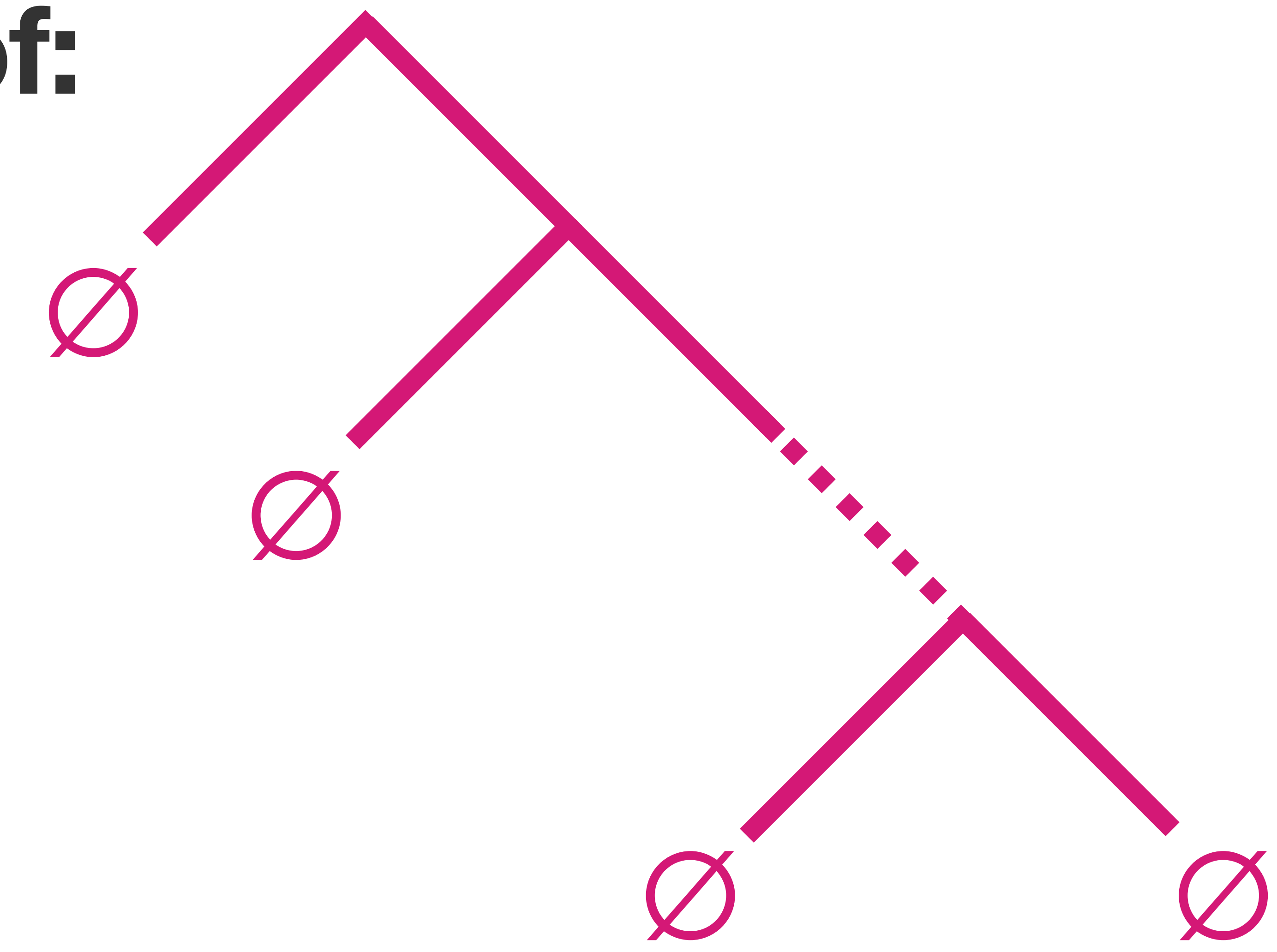

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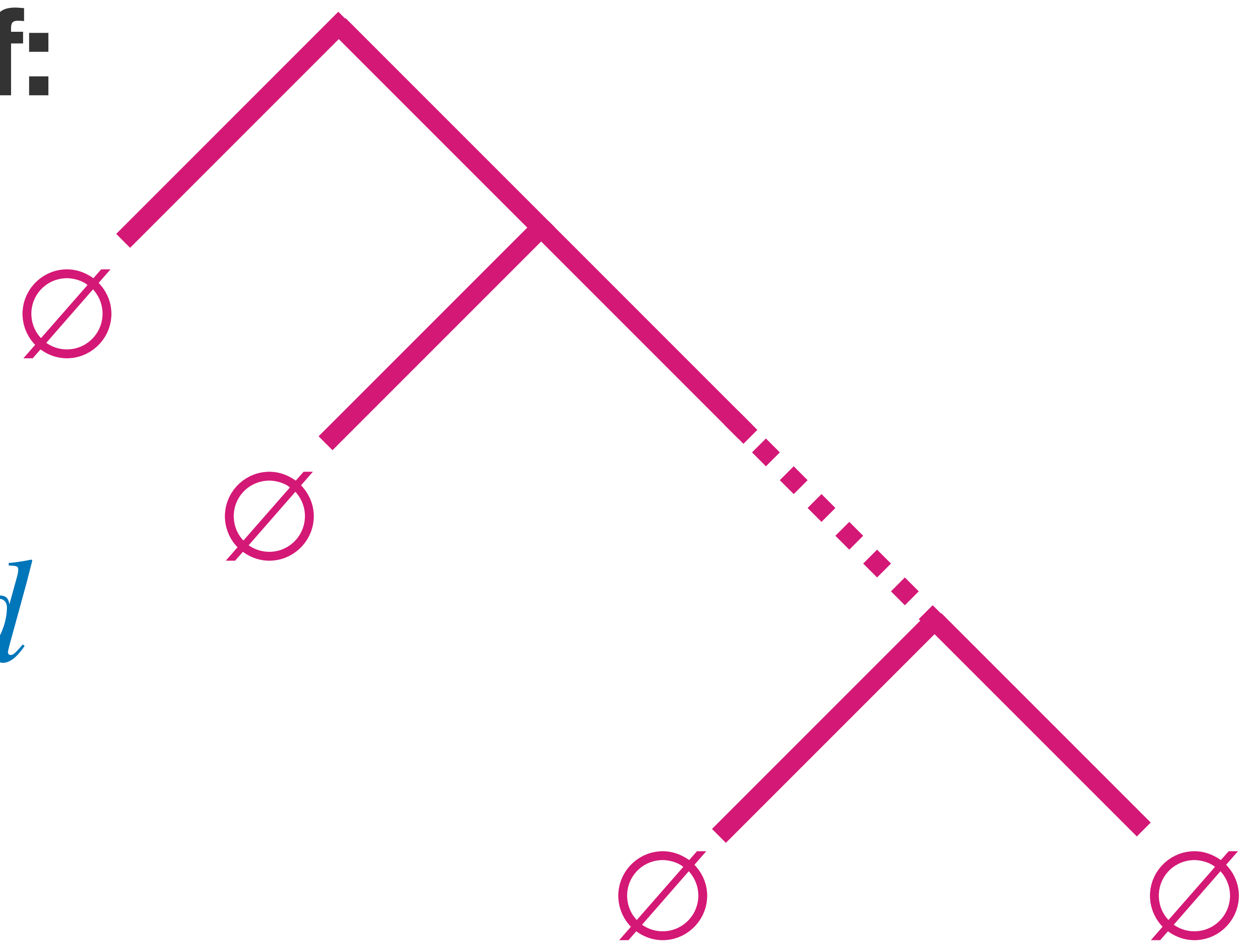

If the CP proof had size s, depth d

→ SP proof has size s, depth s

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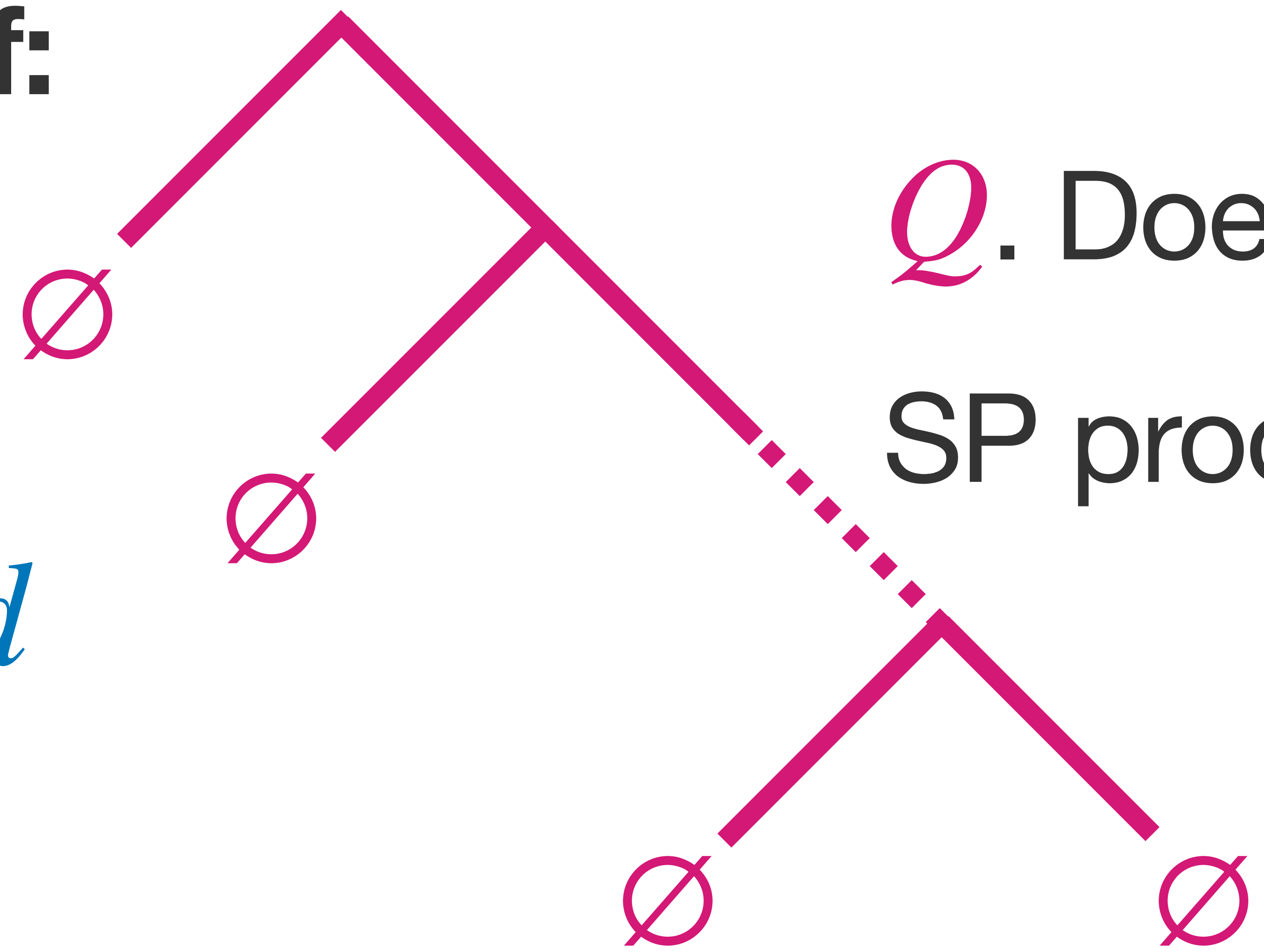
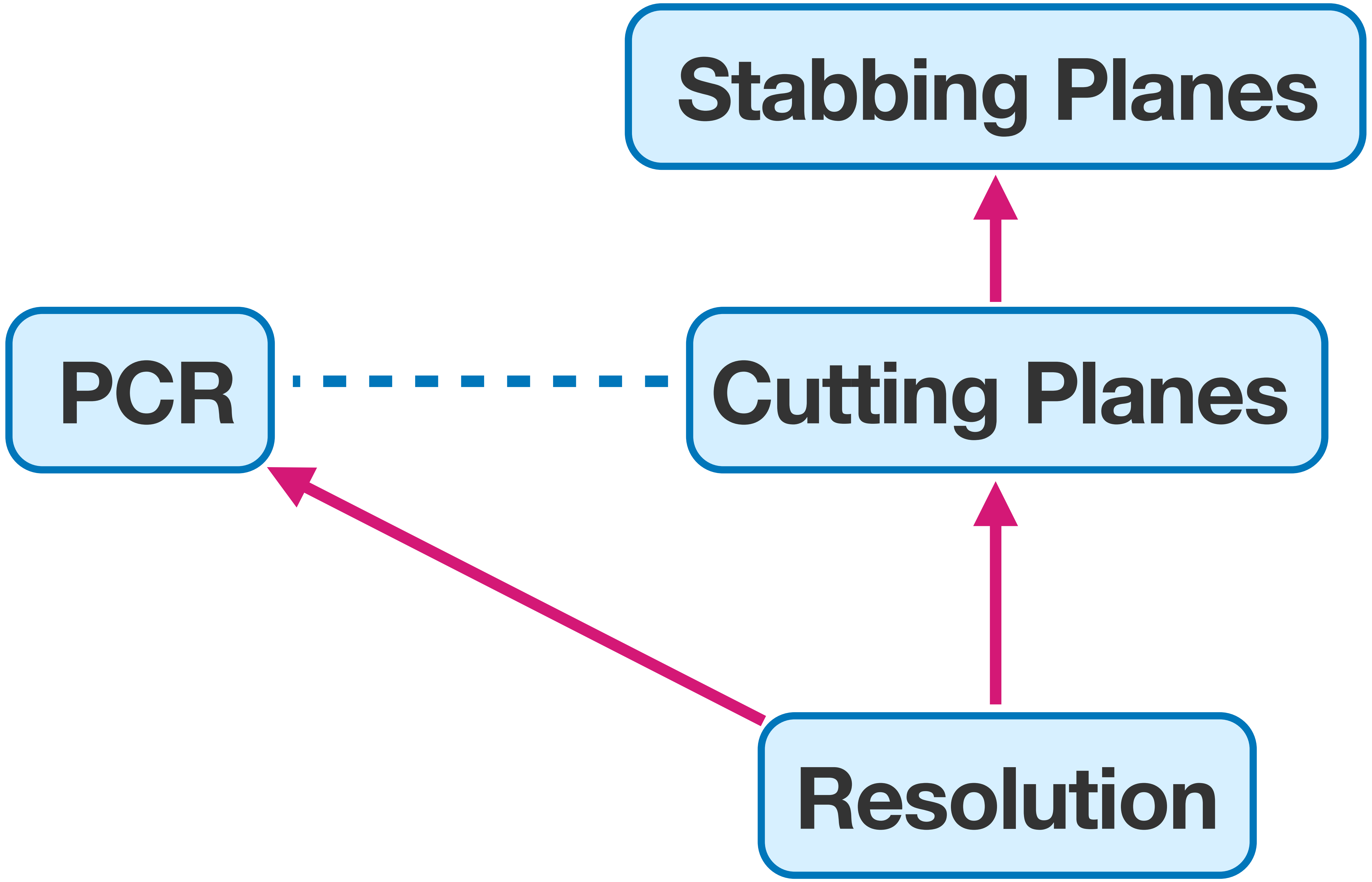


Q. Does there always exist an SP proof of size s and depth d?

If the CP proof had size s, depth d

→ SP proof has size s, depth s

Comparison of Proof Systems



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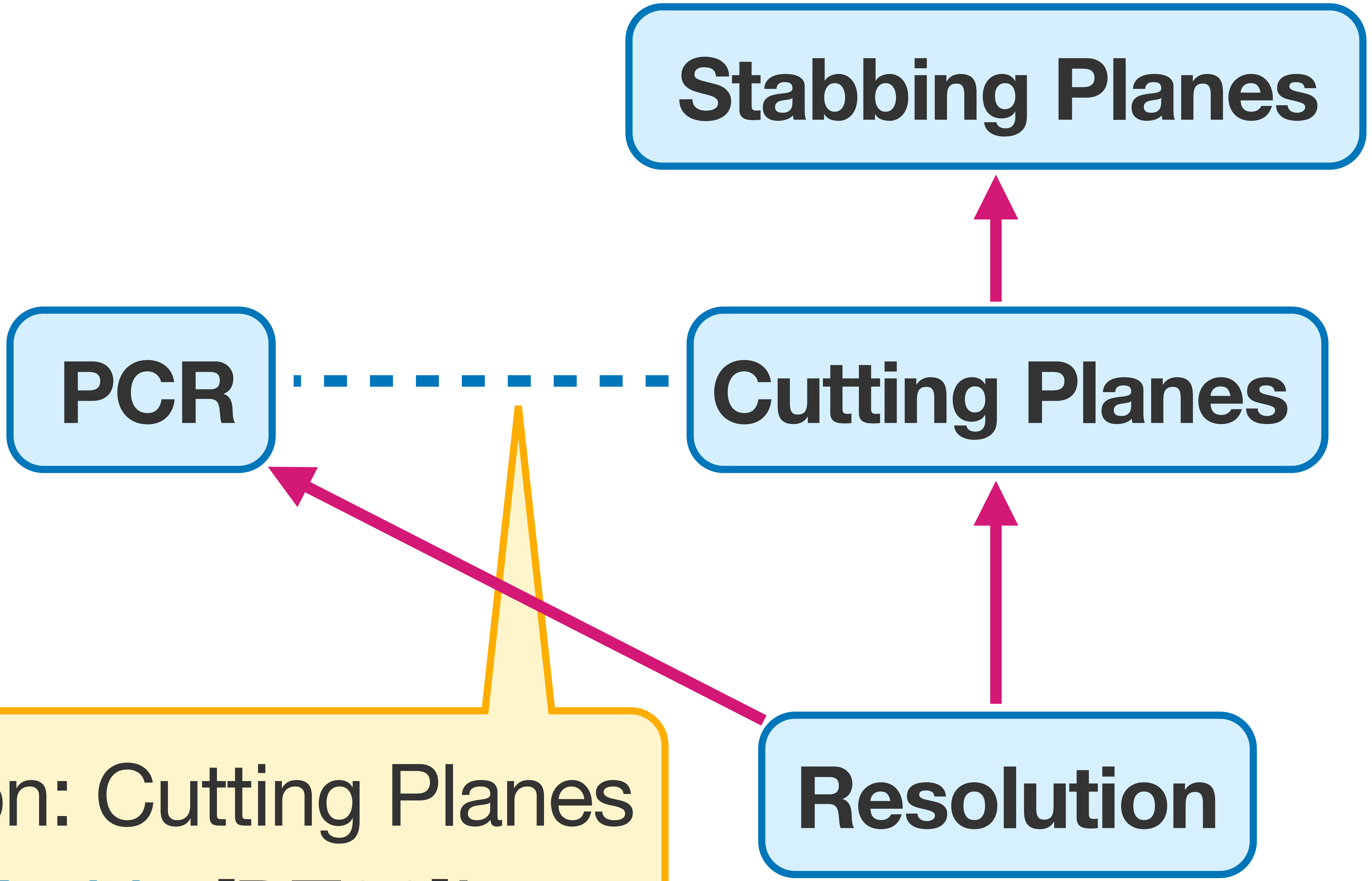

One direction: Cutting Planes
Can prove Tseitin [DT20]!

Cutting Planes Proves Tseitin!

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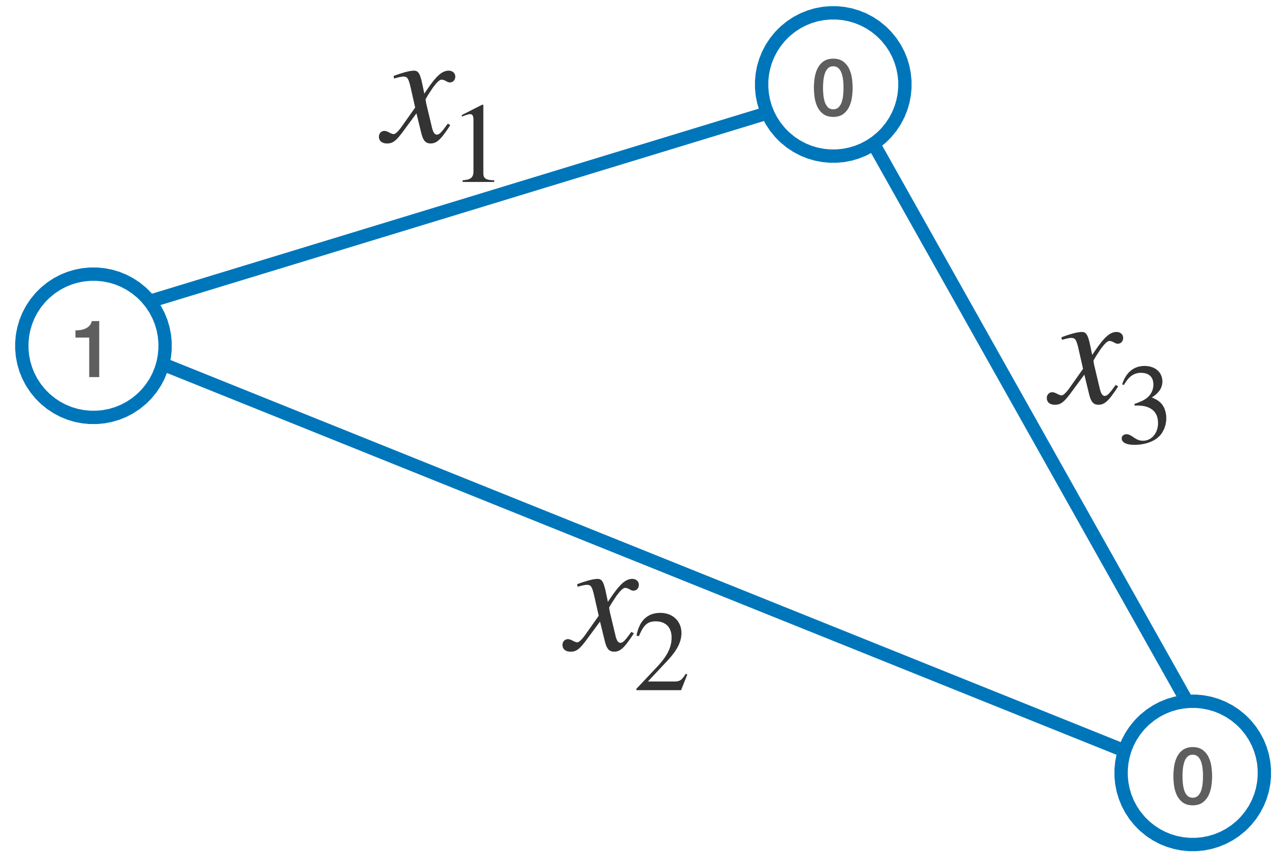

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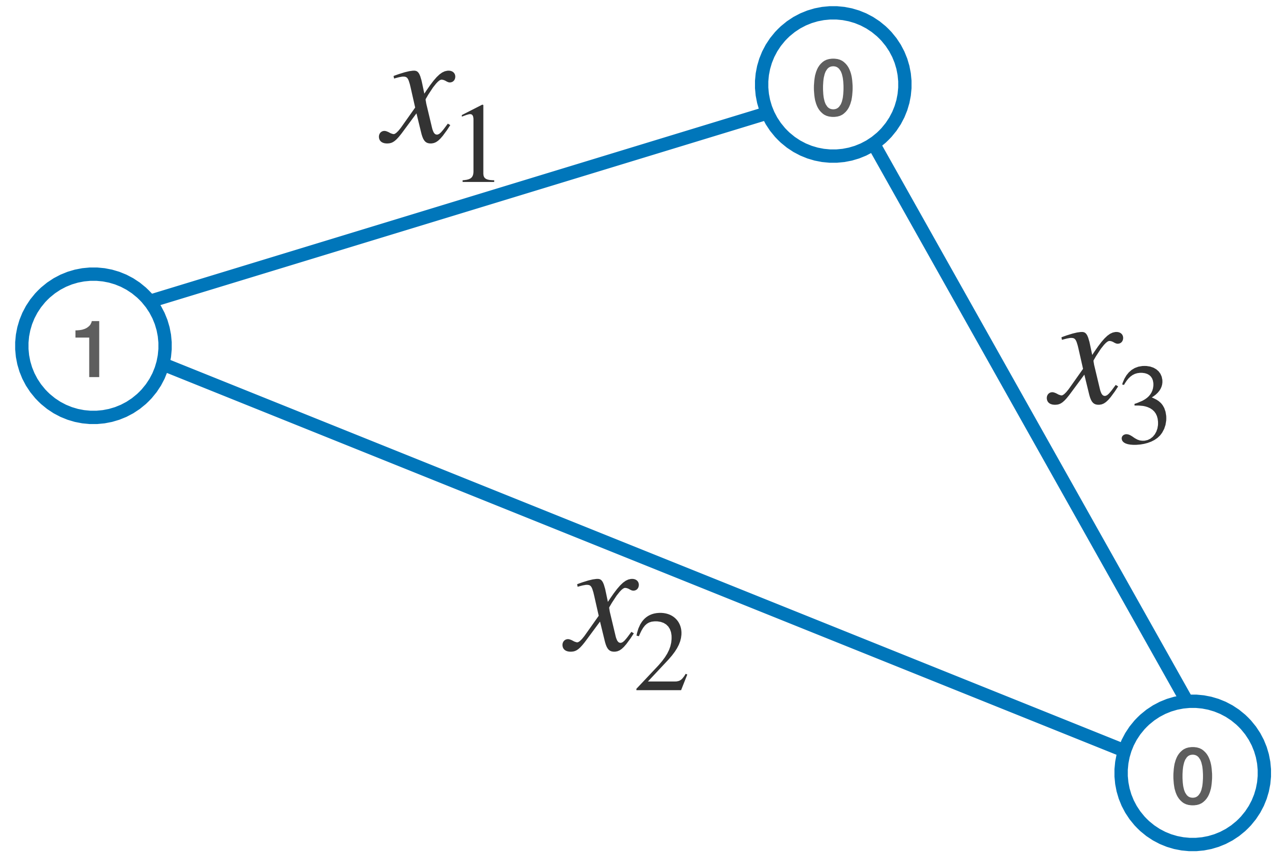

$$x_1 \oplus x_2 = 1$$
$$x_1 \oplus x_3 = 0$$
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Cutting Planes Proves Tseitin!

Thm[DT20]**:** There are quasipolynomial size Cutting Planes proofs of $\text{Tseitin}_{G,\sigma}$

High Level:

1. Exhibit a quasipolynomial size Stabbing Planes proof of Tseitin
2. Translate that proof into Cutting Planes

Stabbing Planes Proves Tseitin

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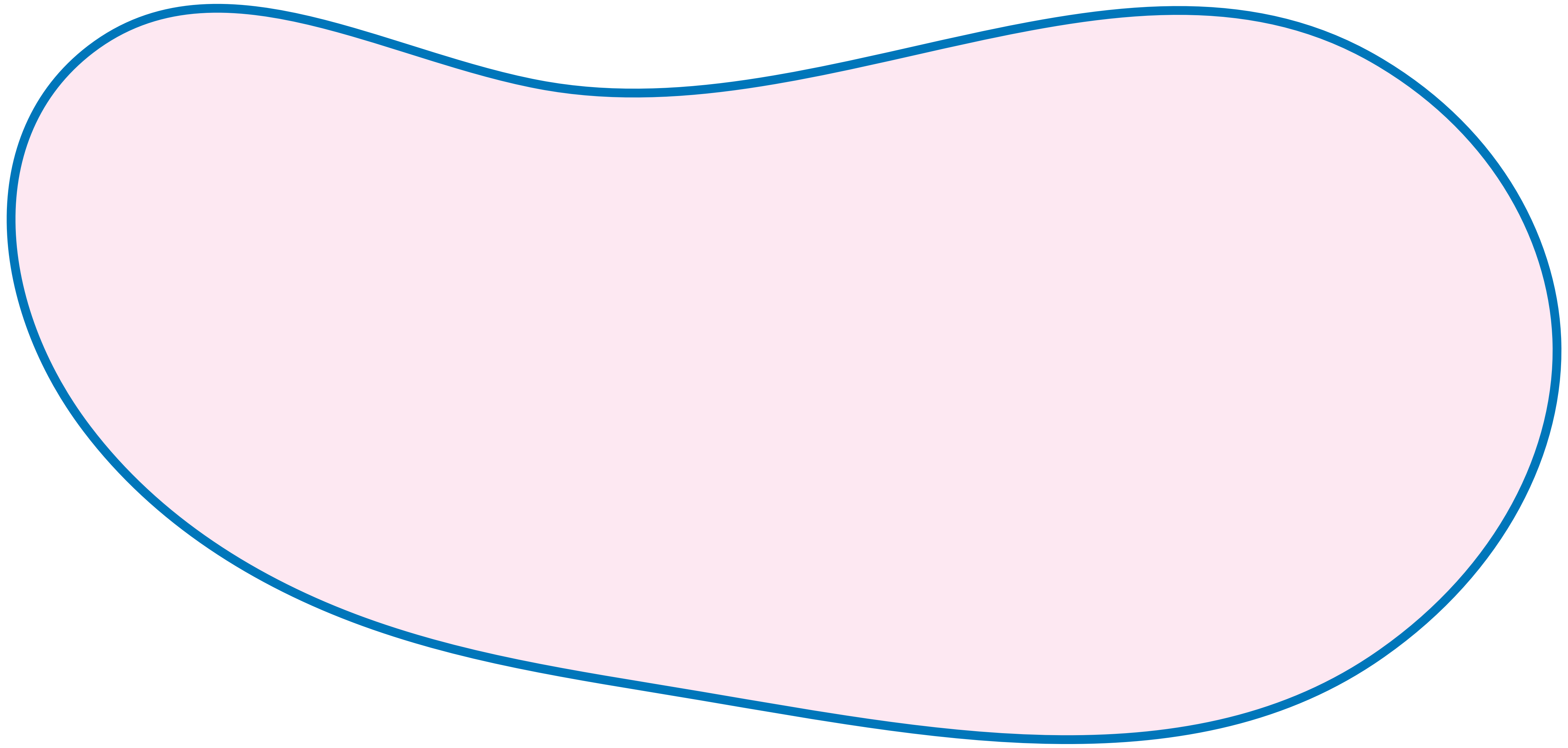
1. We describe an algorithm that, given an assignment $y \in \{0,1\}^n$, finds a falsified constraint of Tseitin$_{G,\sigma}(y)$
2. "Implement" the algorithm in Stabbing Planes

Algorithm for Finding Falsified Clause

Given: $y \in \{0,1\}^n$ to the variables of $\text{Tseitin}_{G,\sigma}$



$G = (V, E)$

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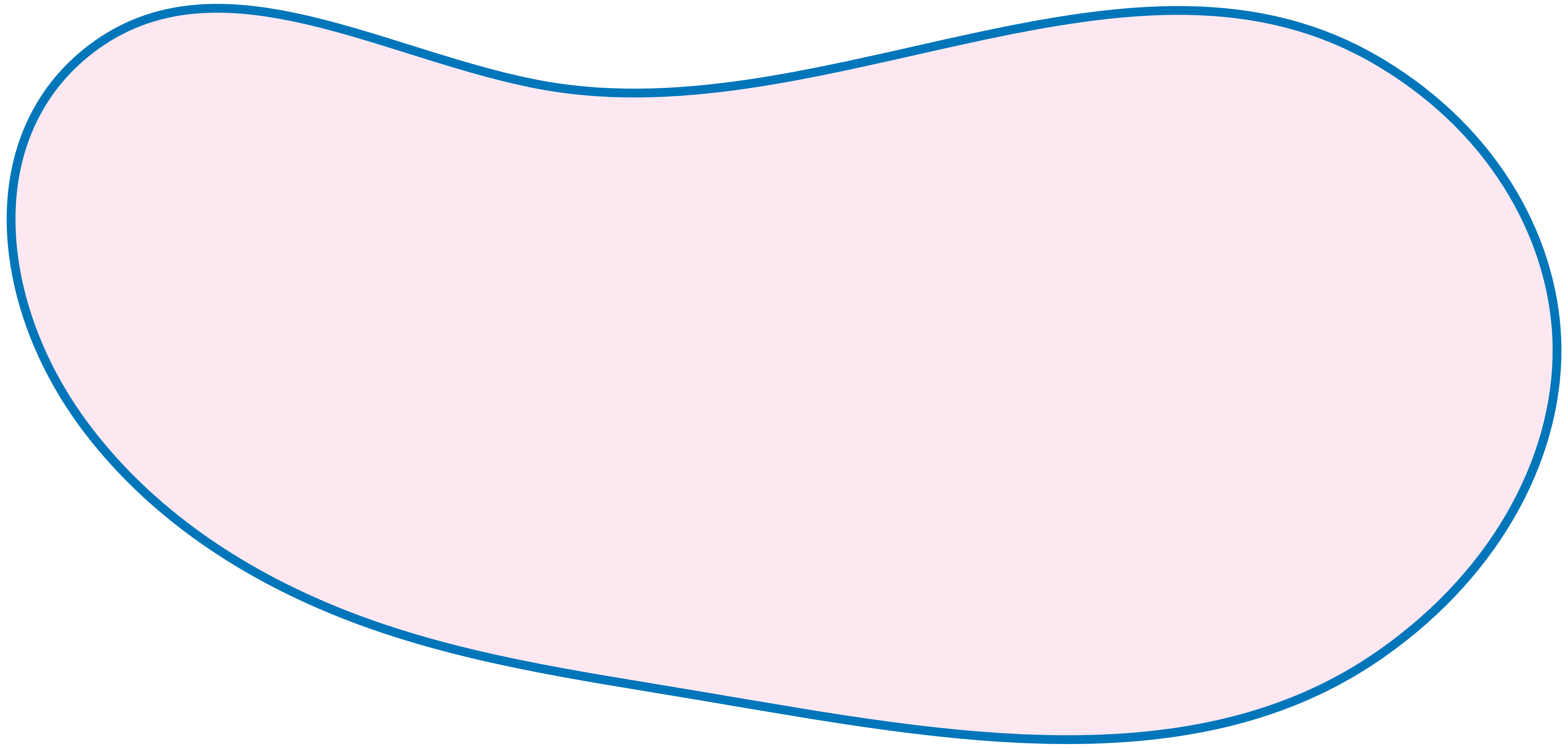

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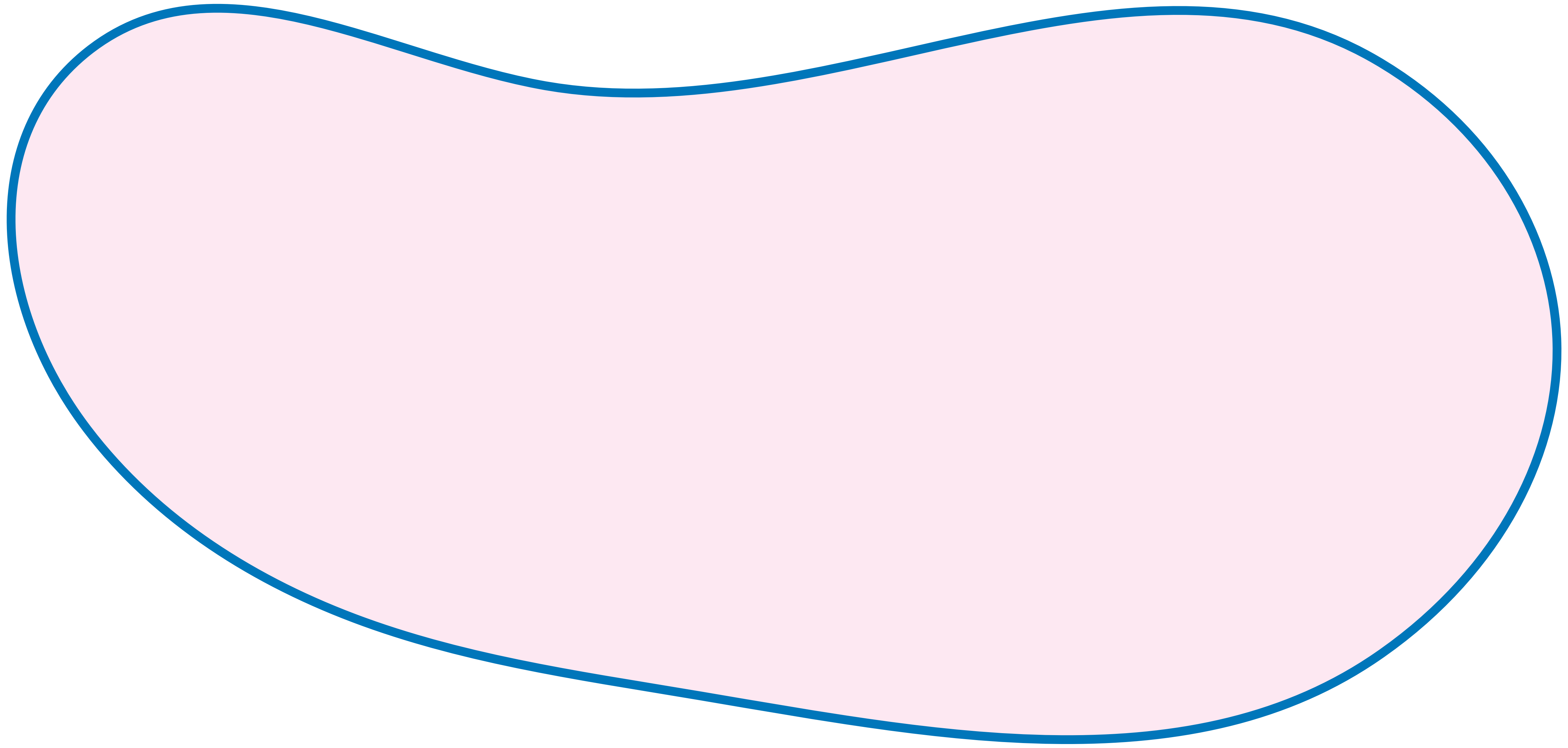
Algorithm proceeds in rounds:

- Each round maintains: $U \subseteq V$



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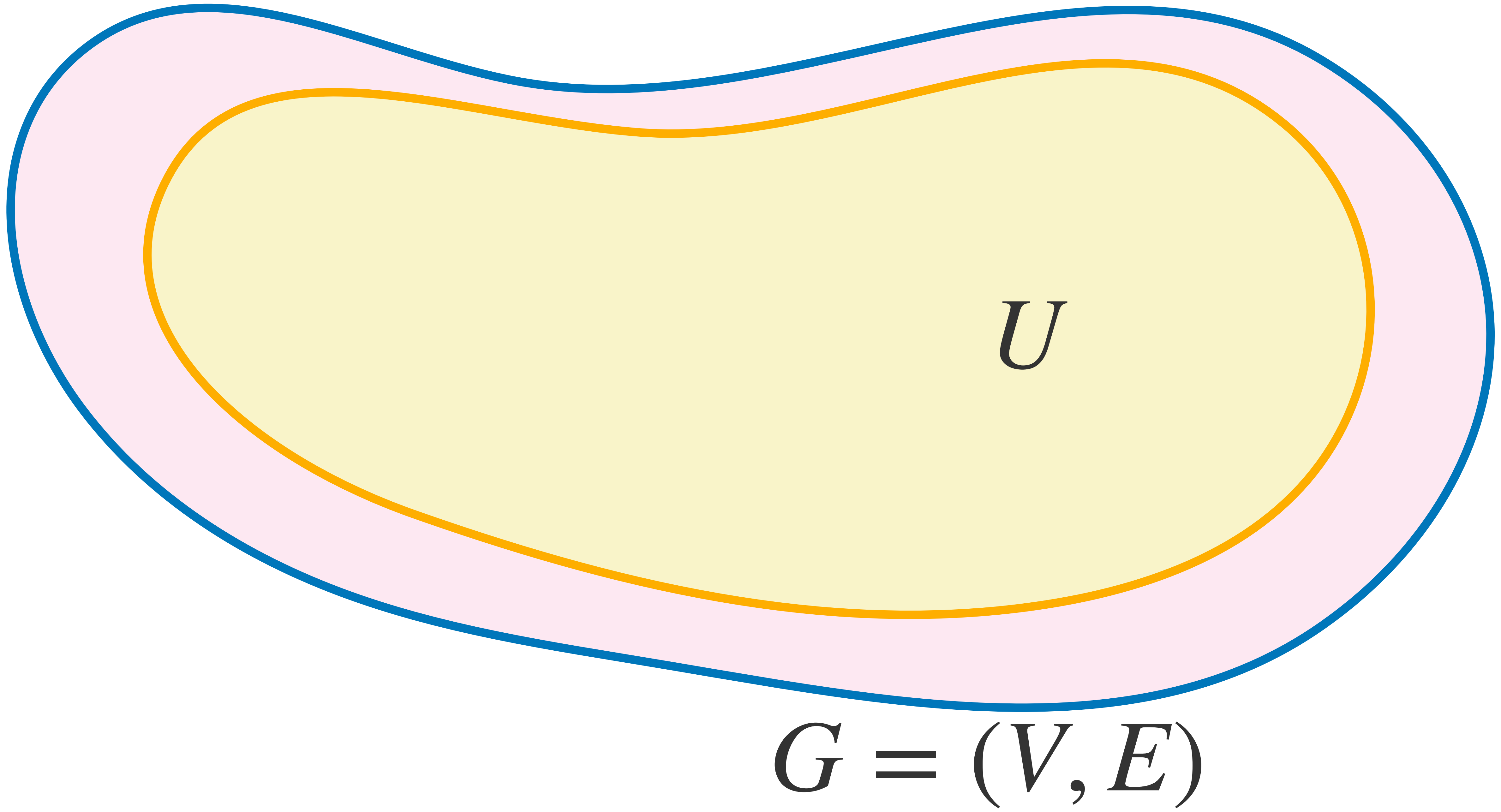

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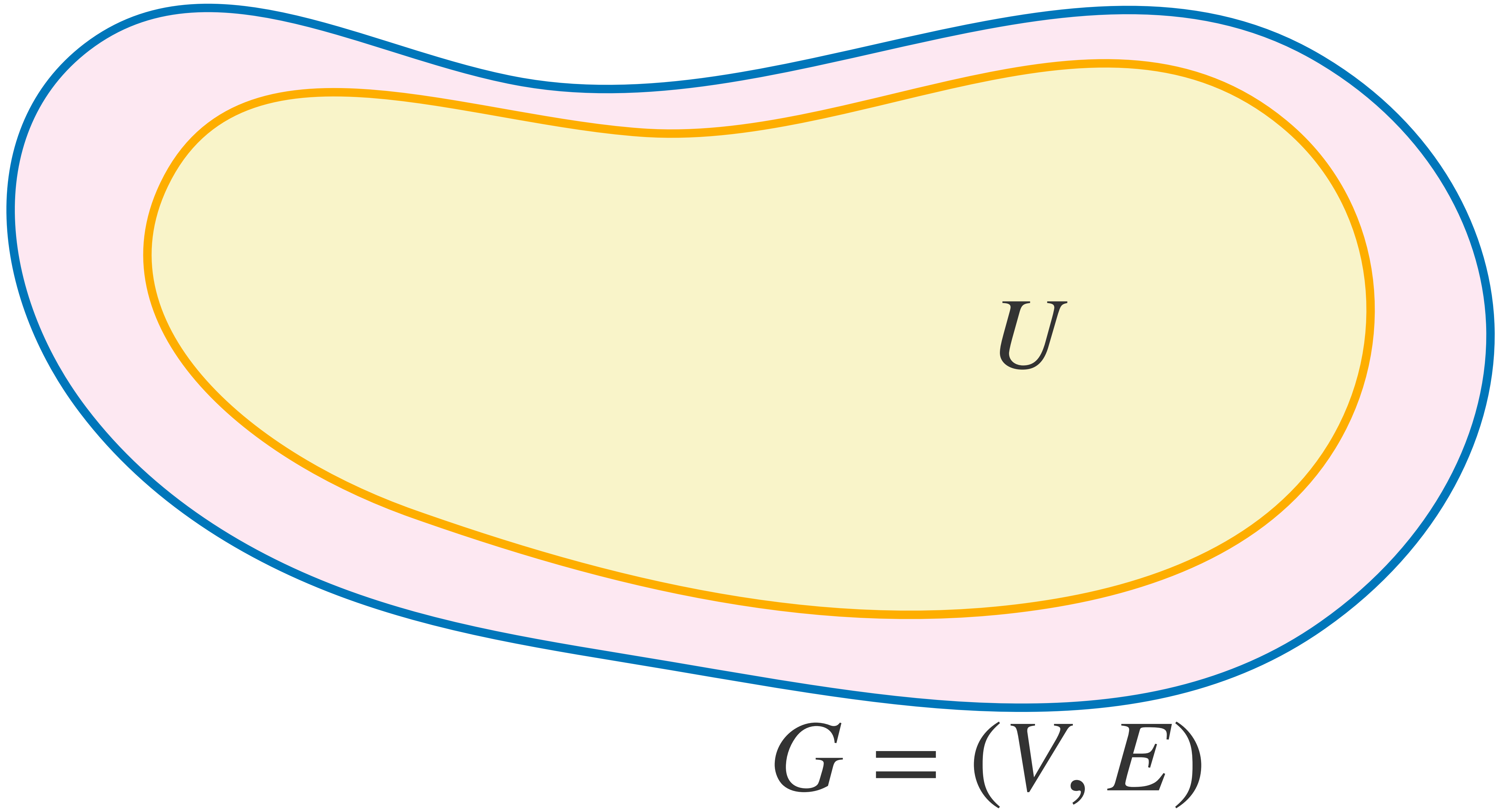

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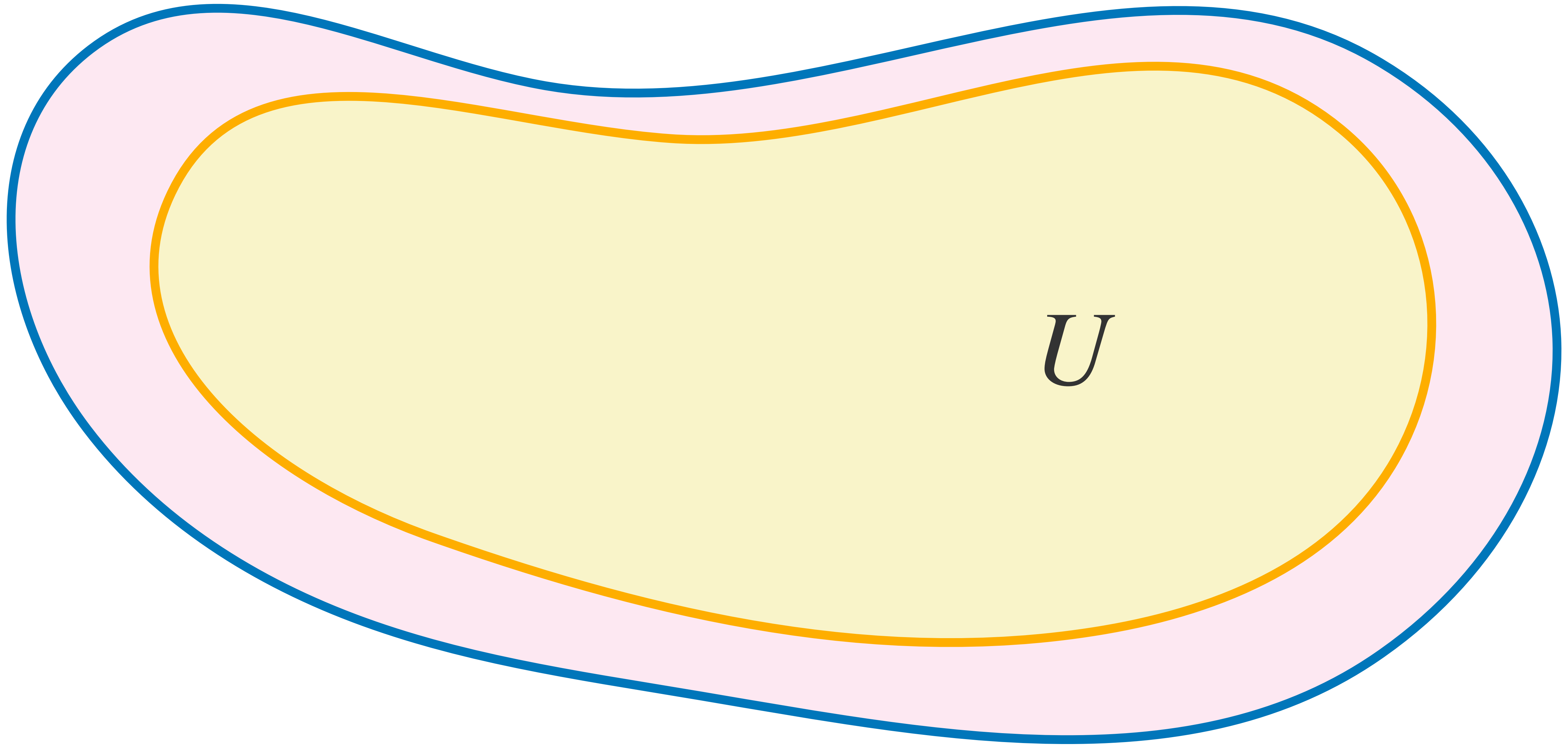

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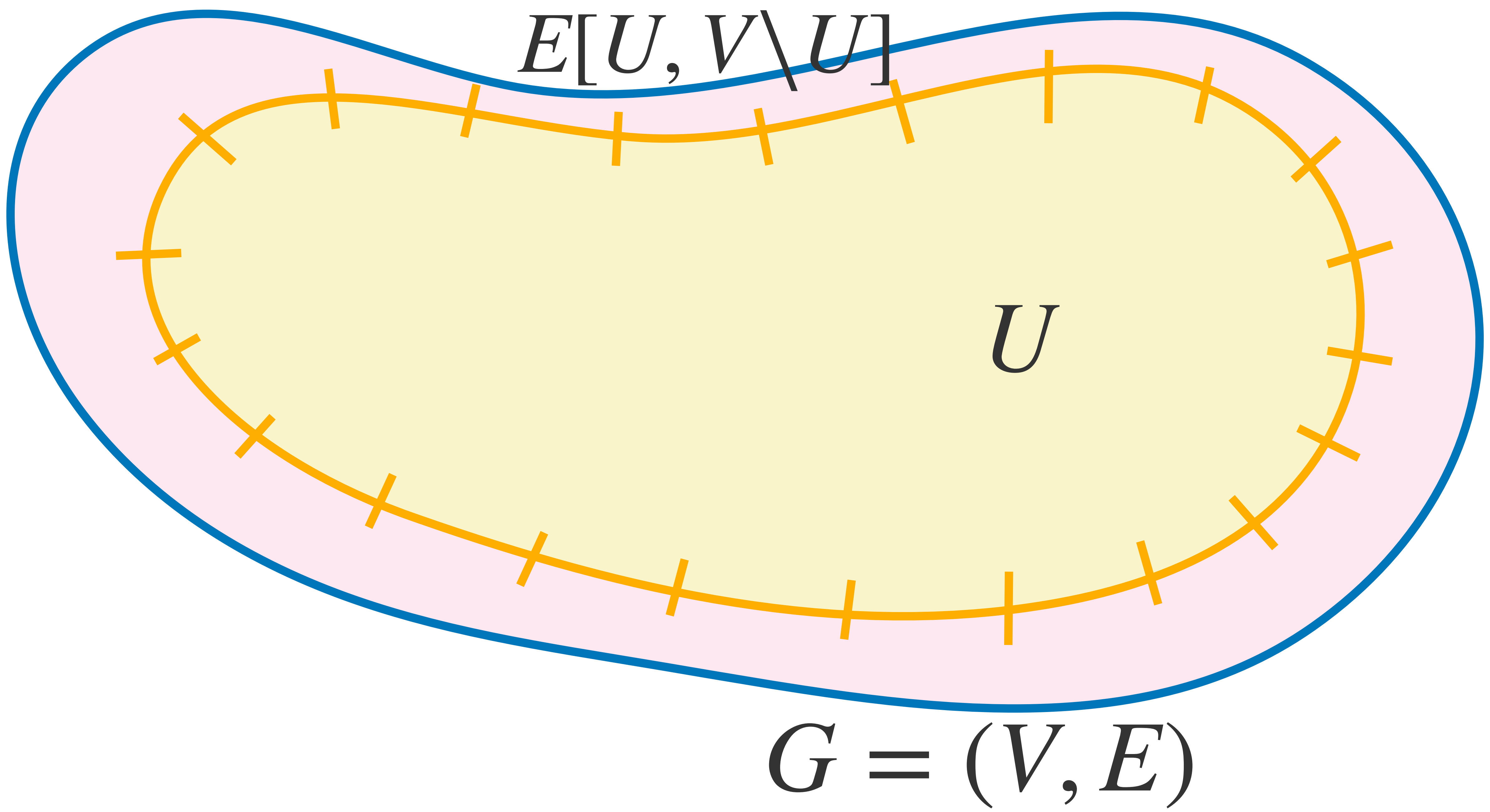

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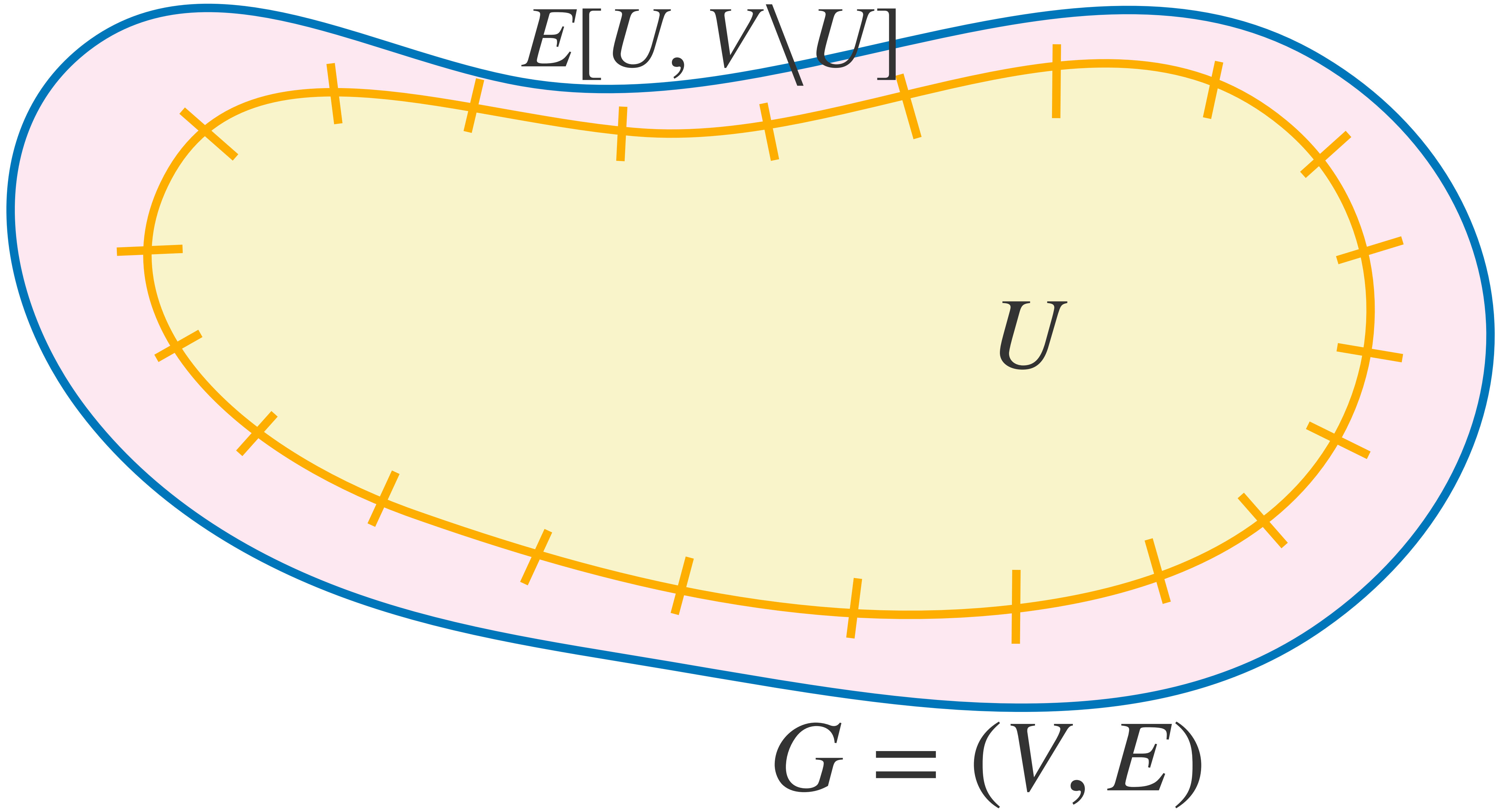

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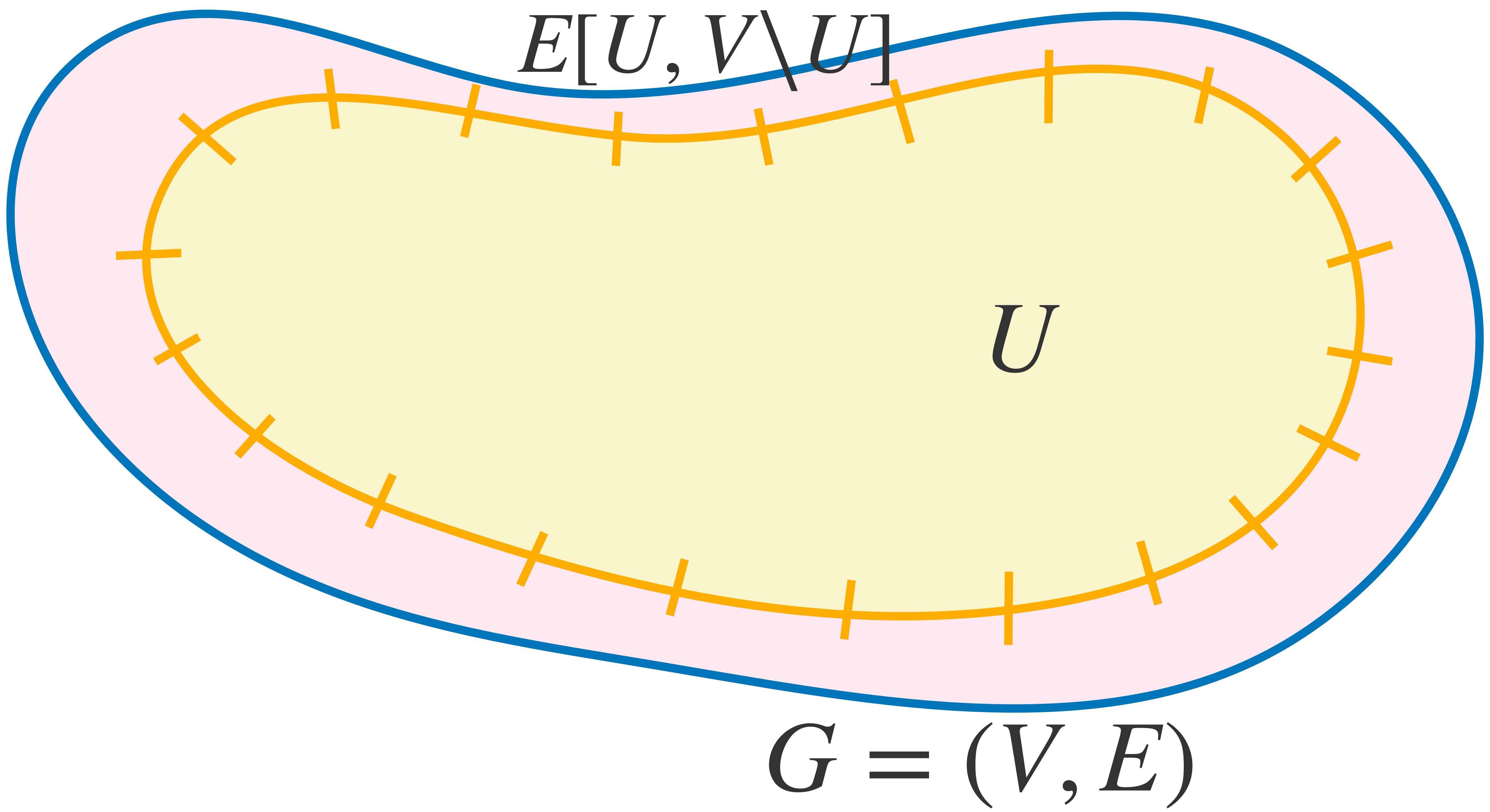

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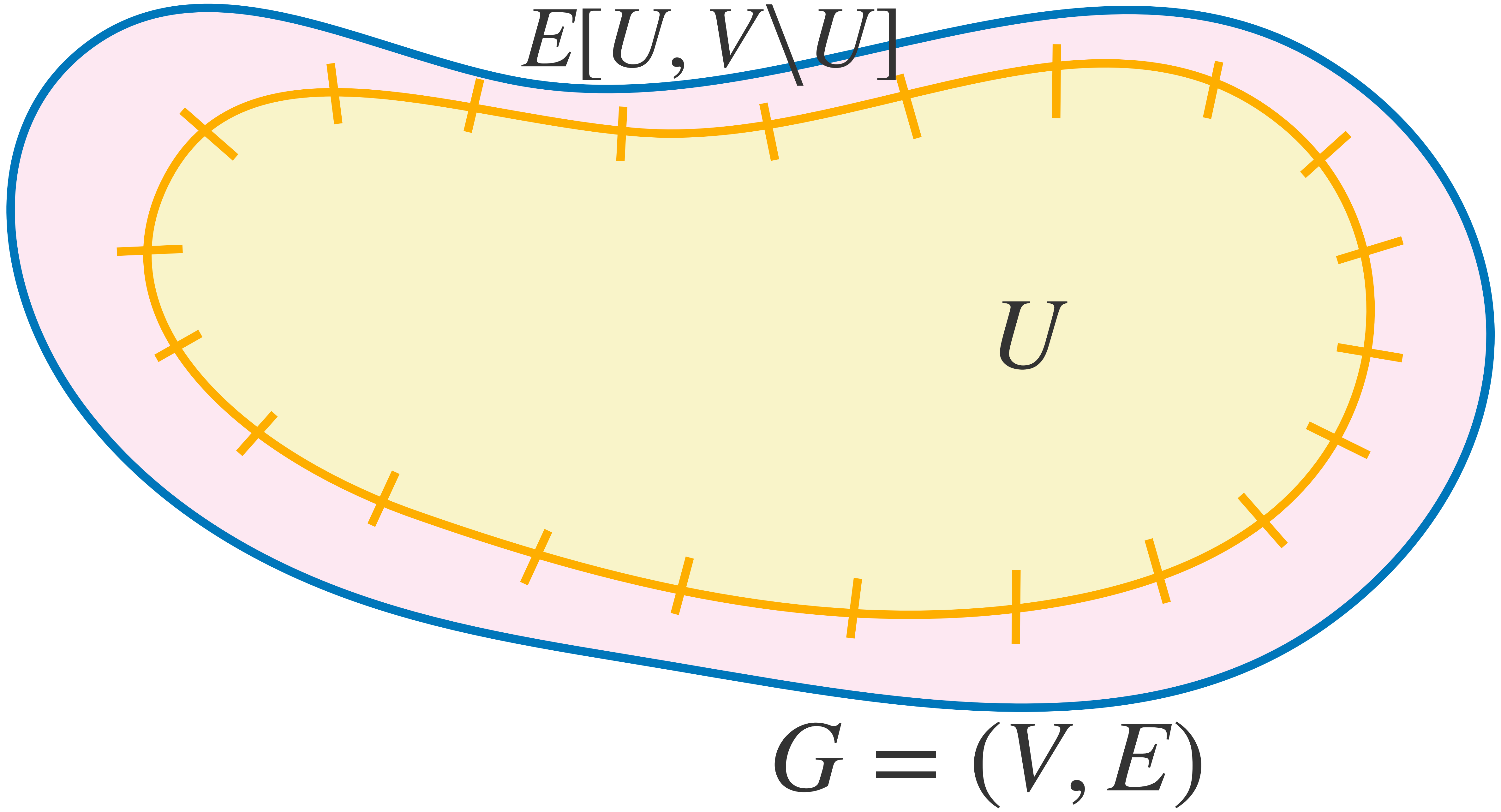

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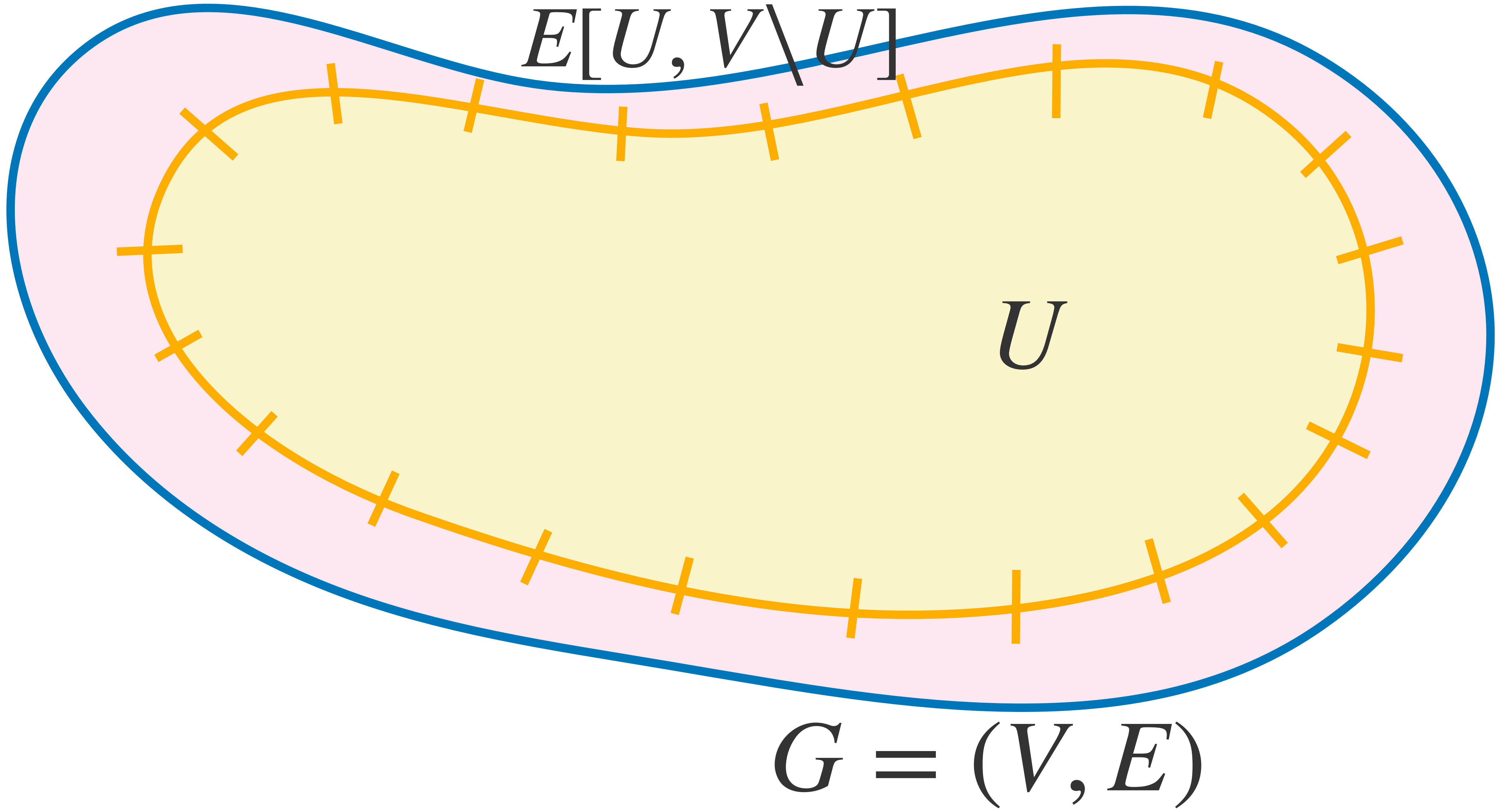
For U to be satisfiable we need

$\Sigma_{v \in U} \sigma(v) \equiv \Sigma_{v \in U} \Sigma_{v \in e} y_e \bmod 2$



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Given: $y \in \{0,1\}^n$ to the variables of $\text{Tseitin}_{G,\sigma}$

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Goal: find $v \in V$ such that $\bigoplus_{e \ni v} y_e \neq \sigma(v)$ — a falsified constraint

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Edges with one endpoint in U one in $V \setminus U$

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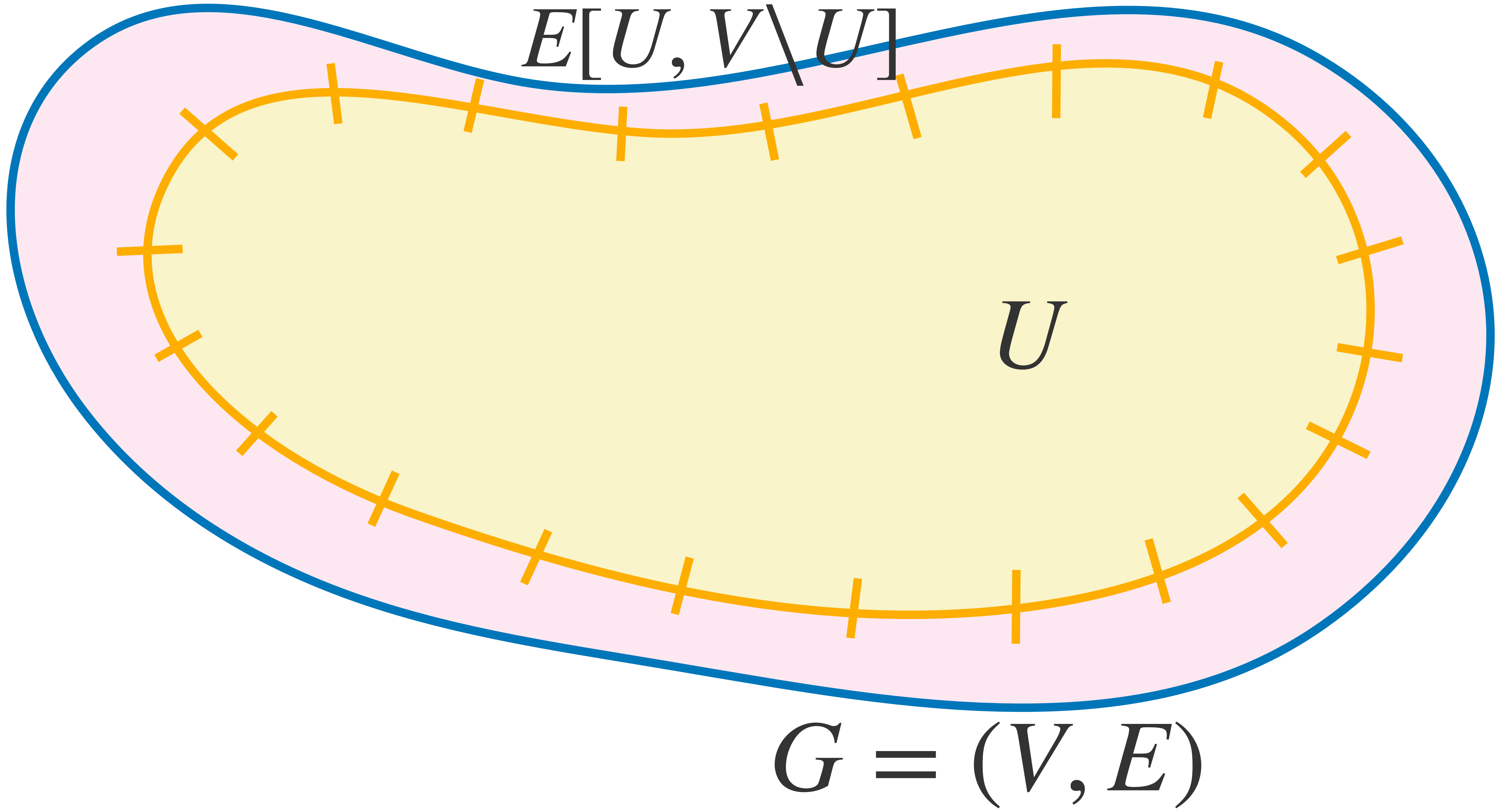

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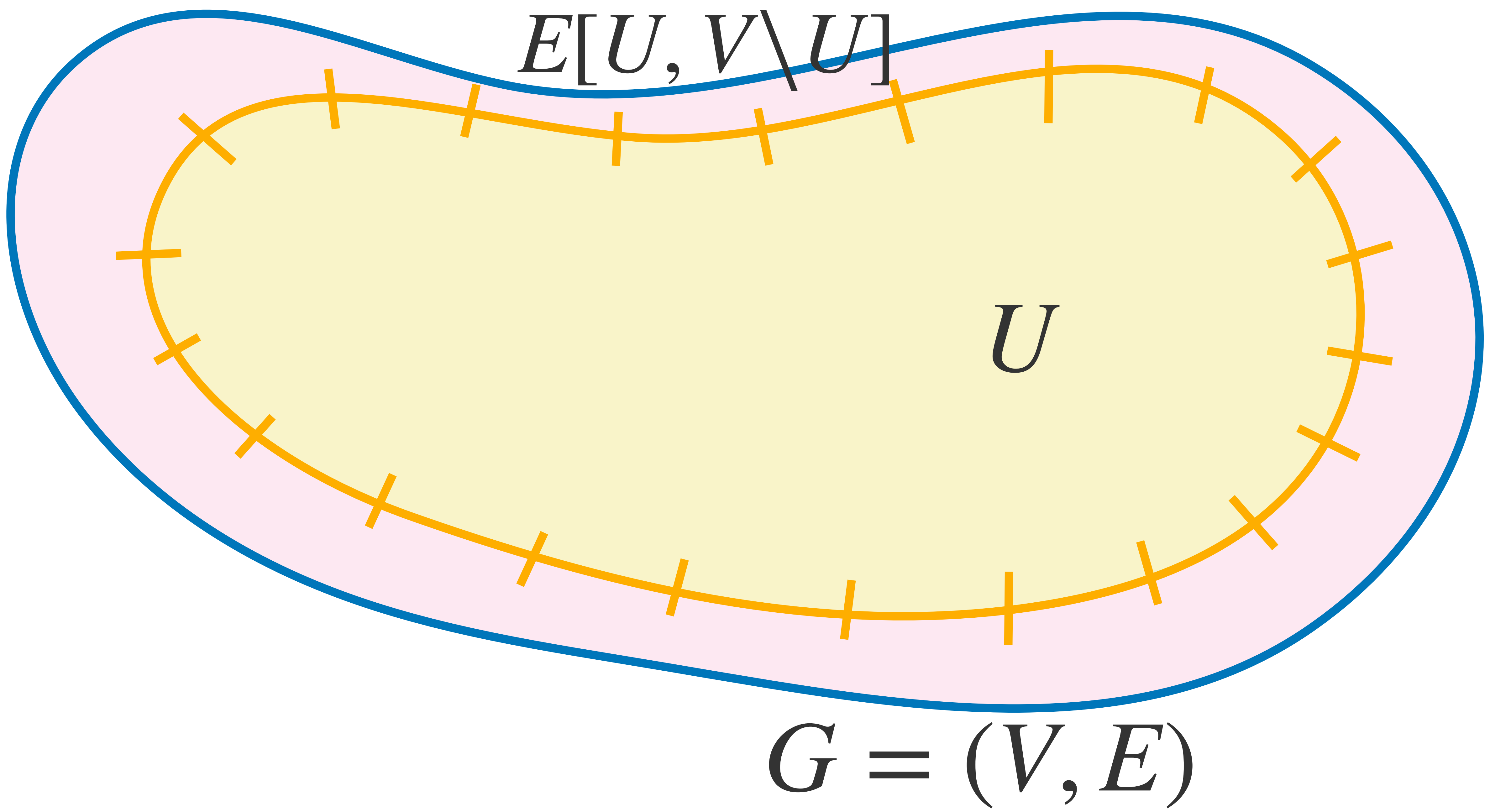

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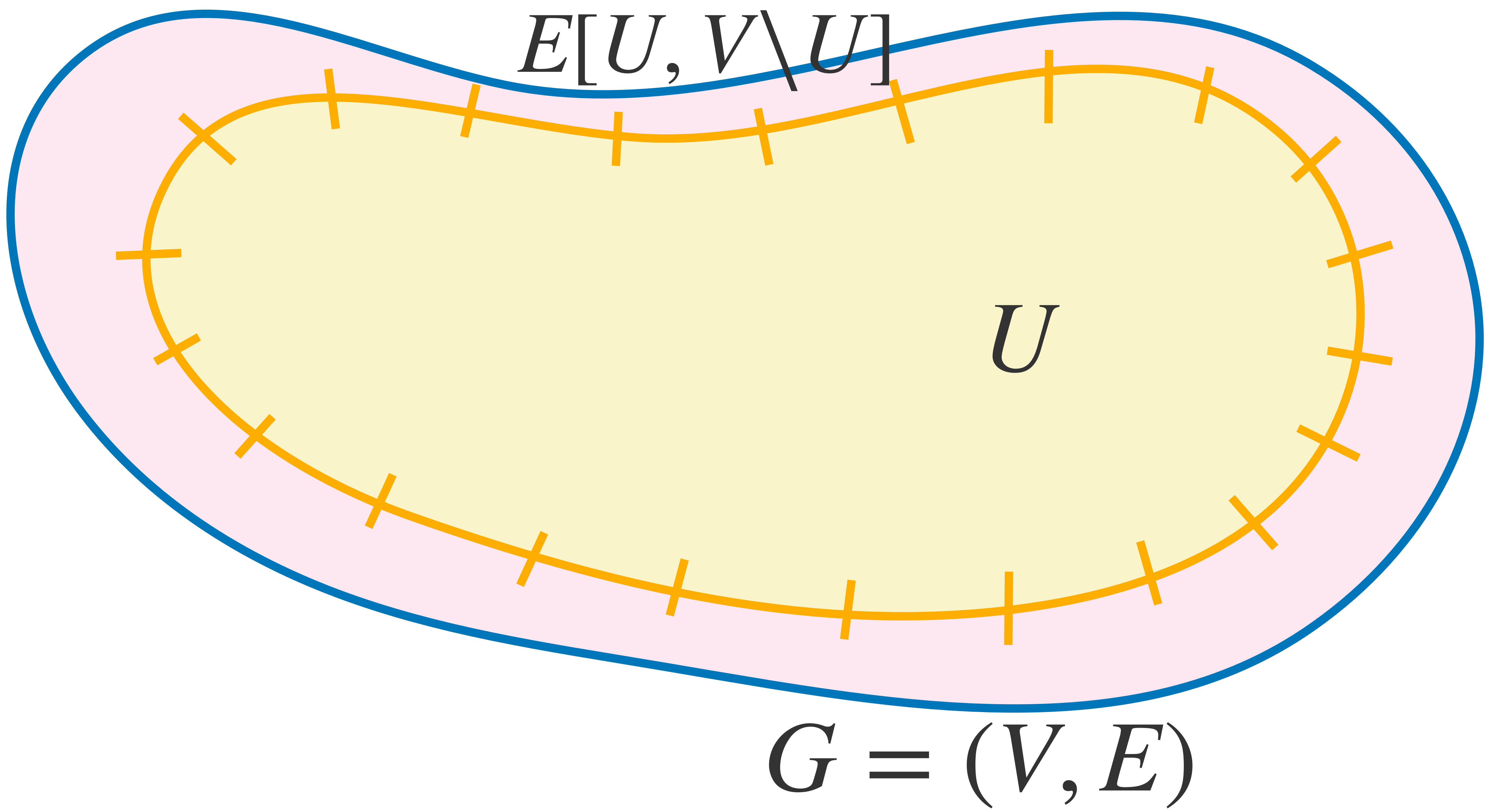

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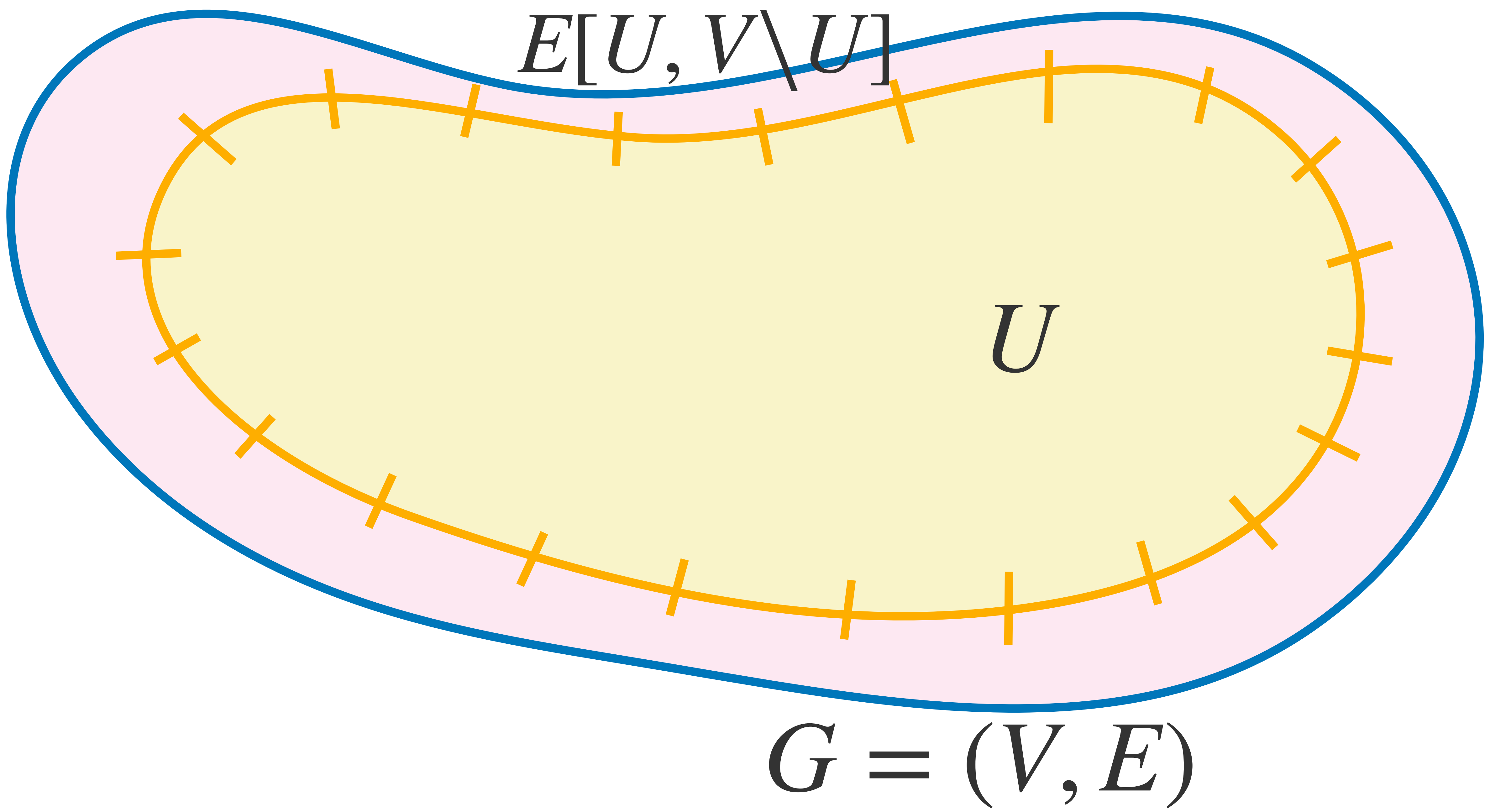

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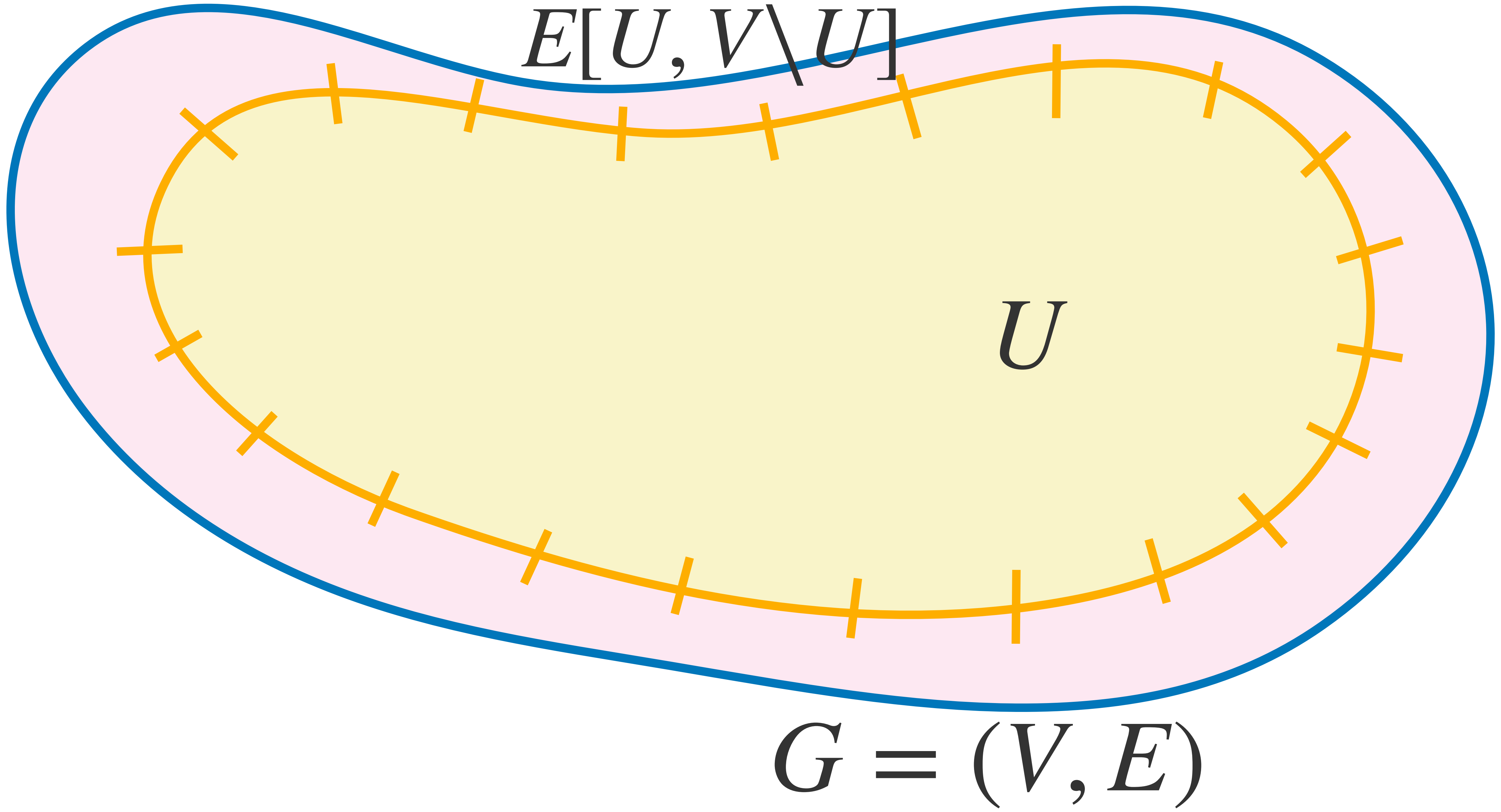

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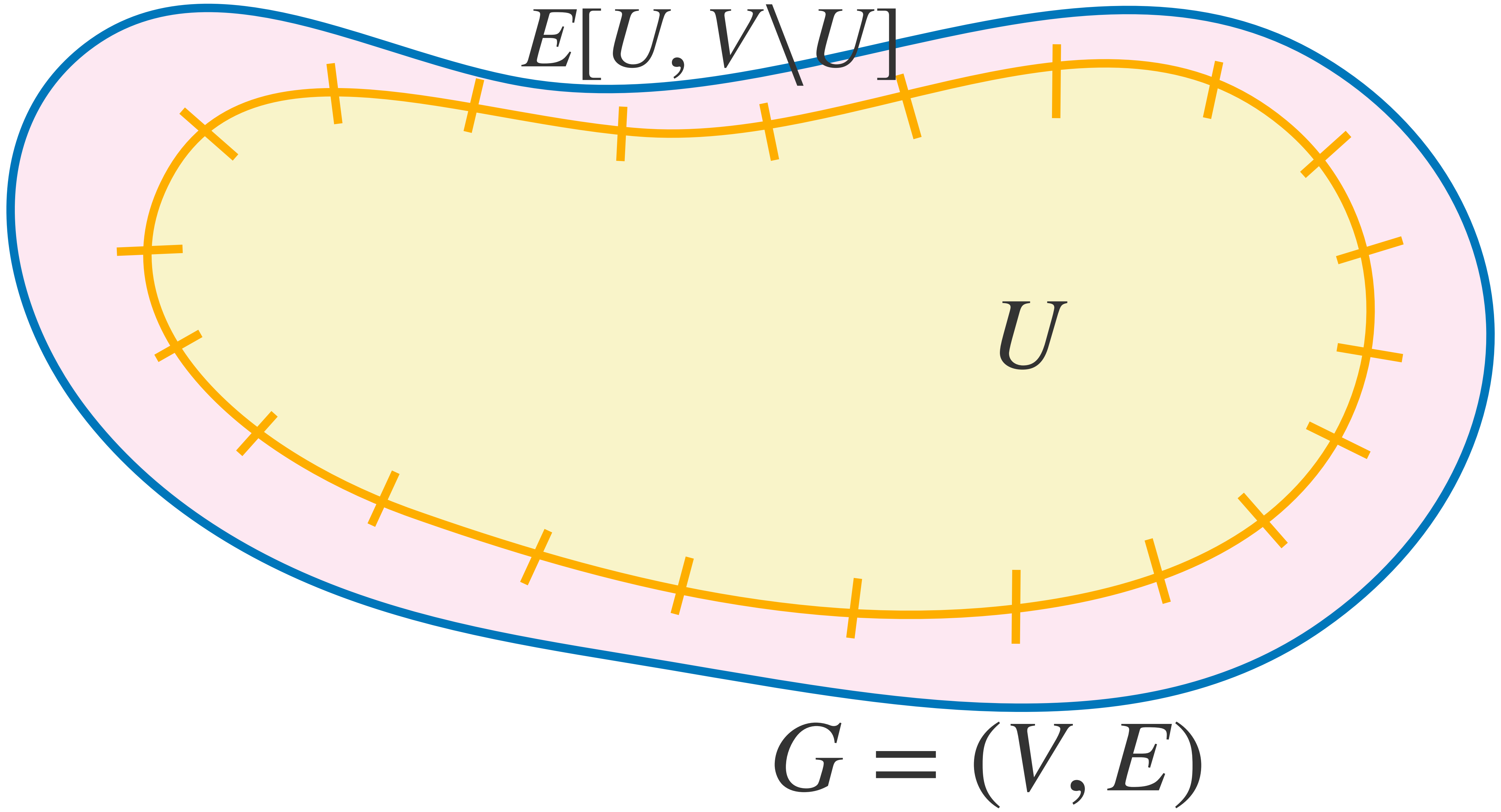

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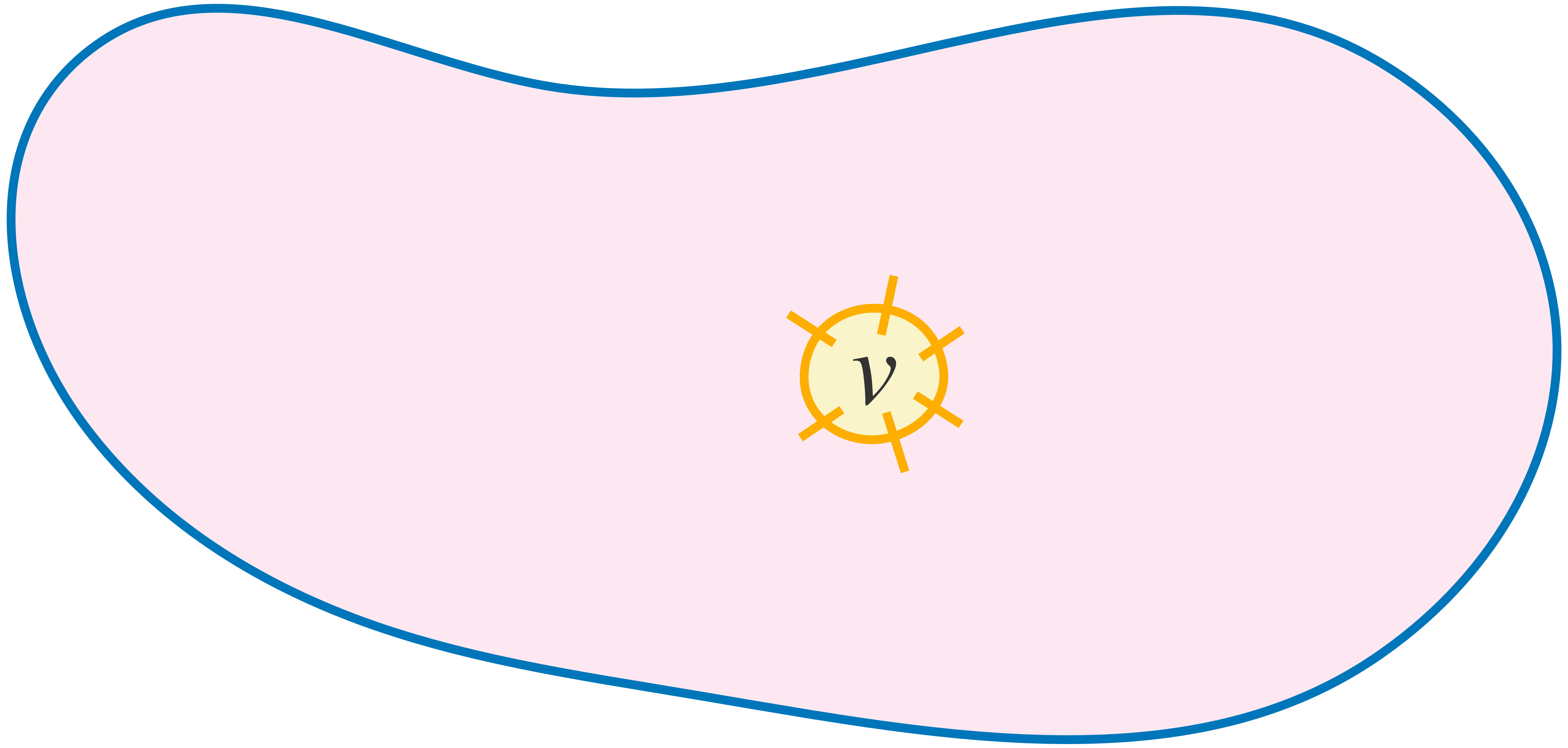
→ Once $U = \{v\}$



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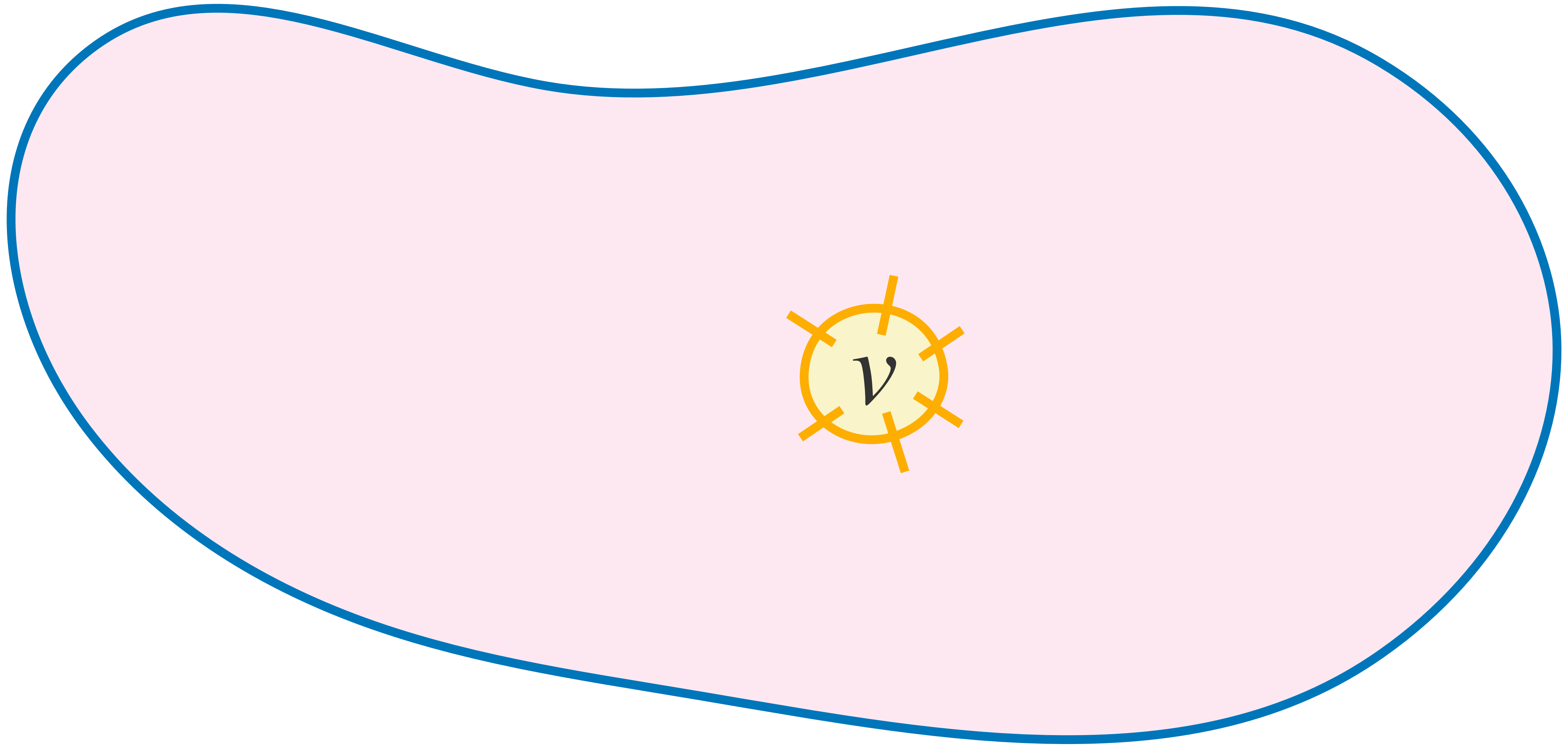
→ Once $U = \{v\}$ we have a vertex such that $\sigma(v) \neq (\kappa_v = \Sigma_{e: v \in e} y_e) \bmod 2$, a falsified constraint!



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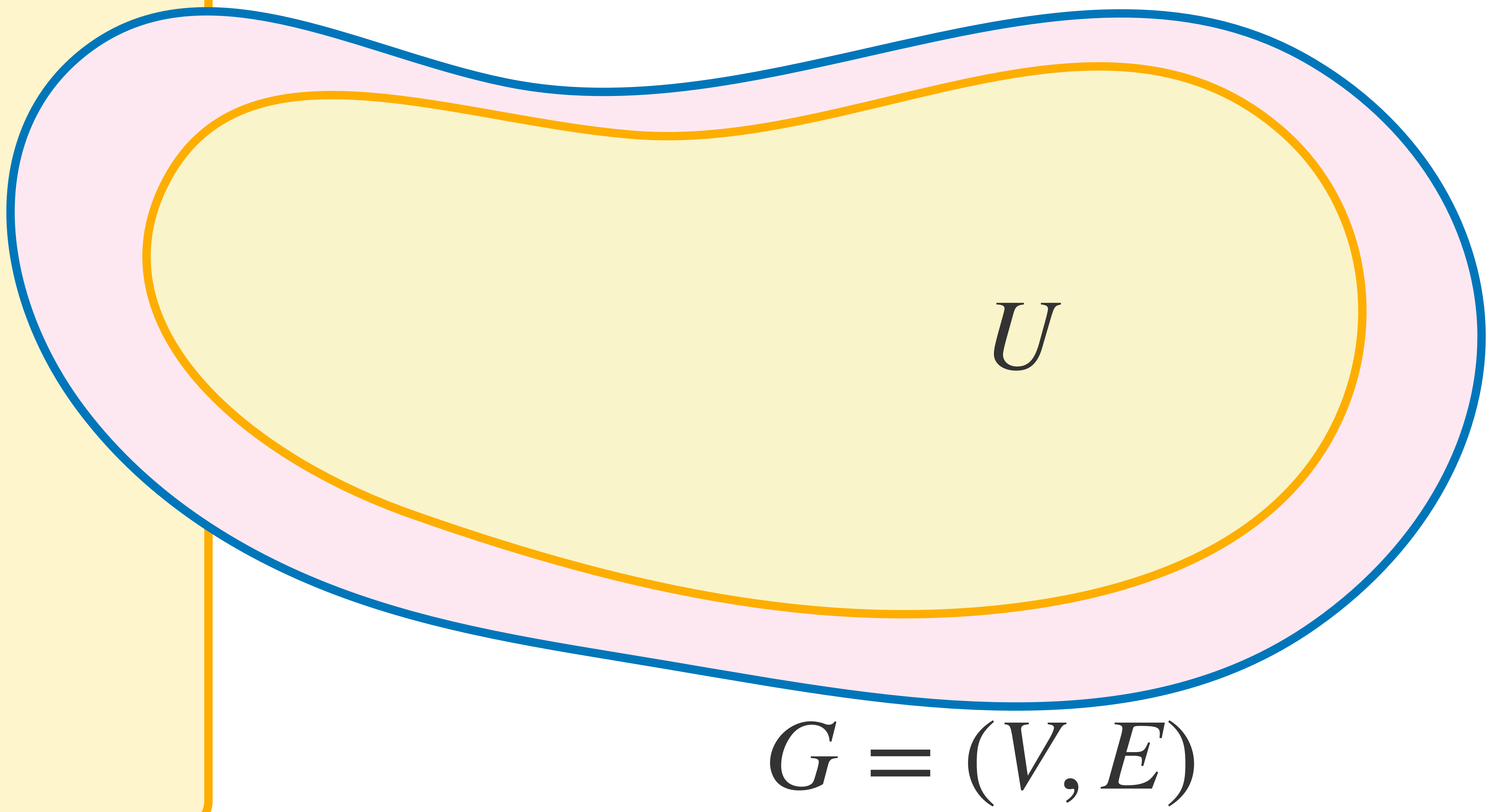

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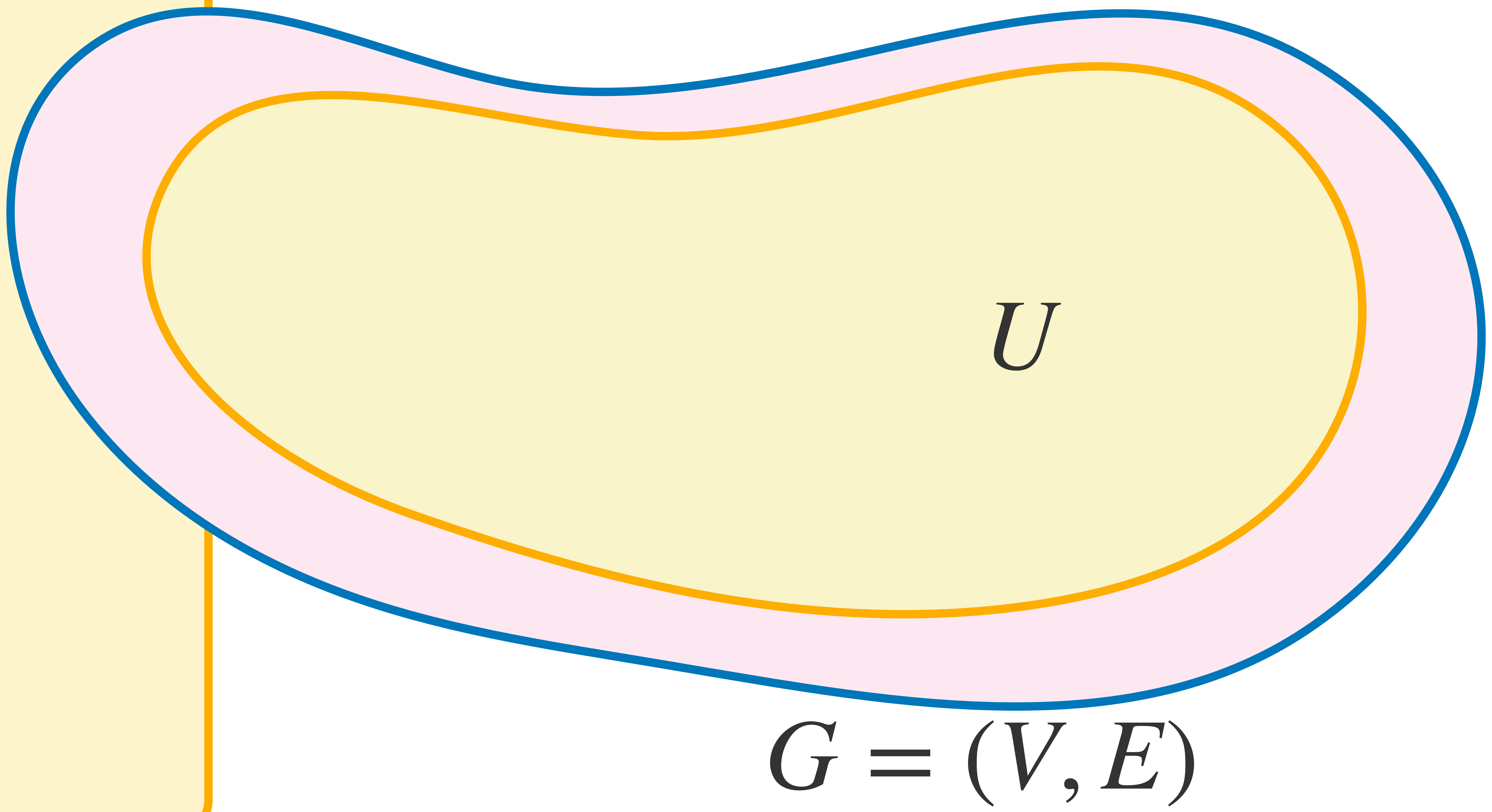

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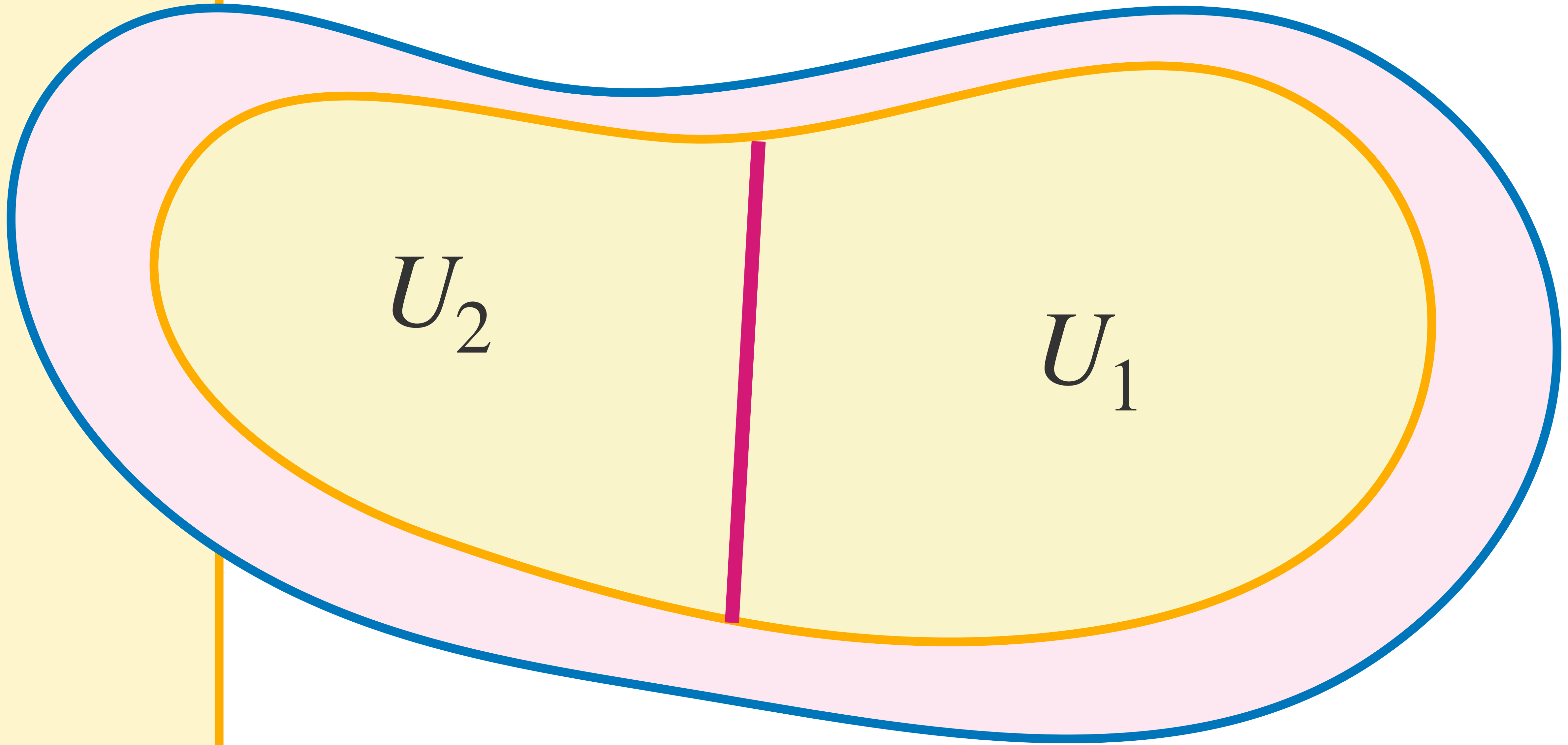

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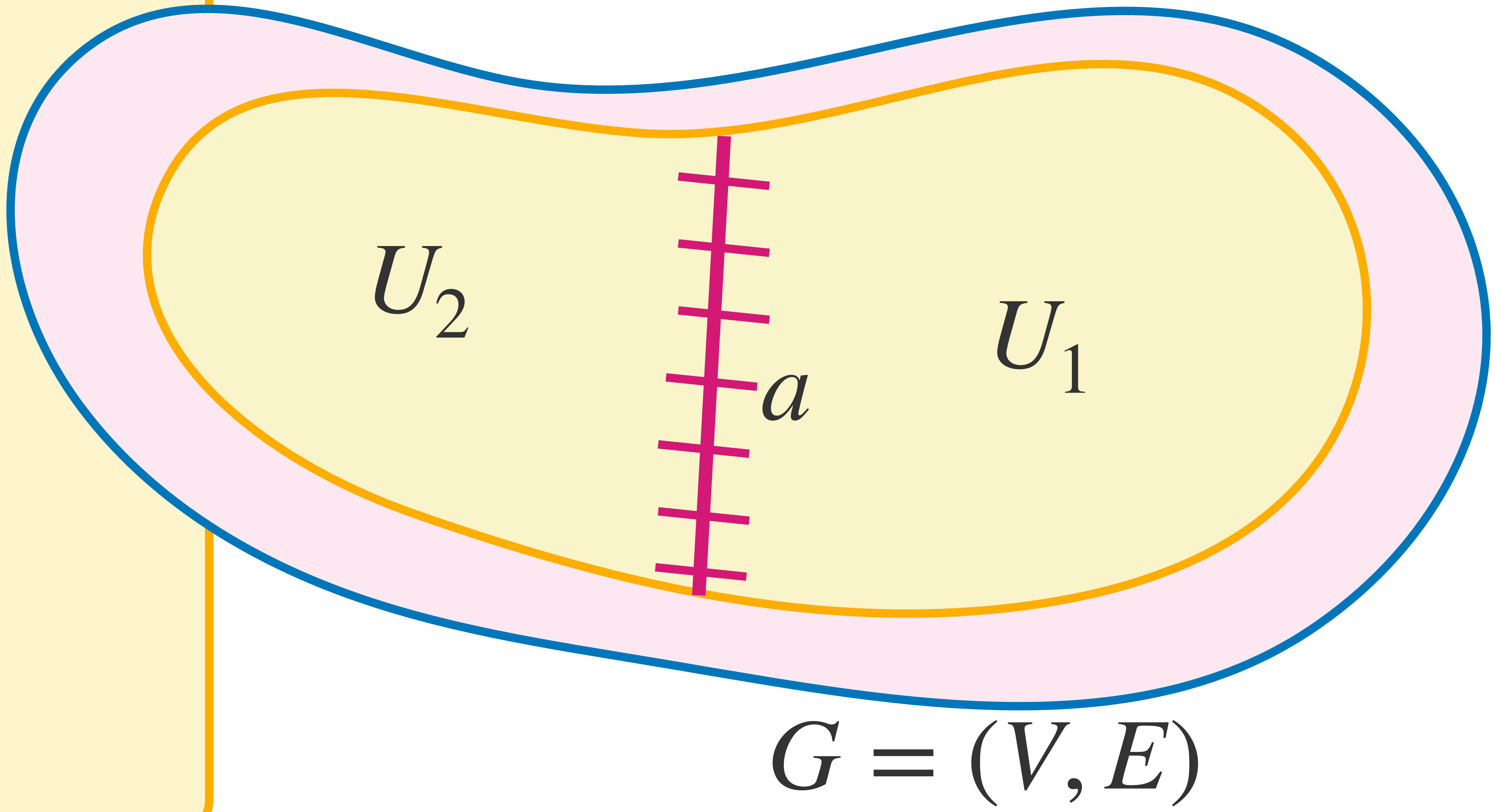

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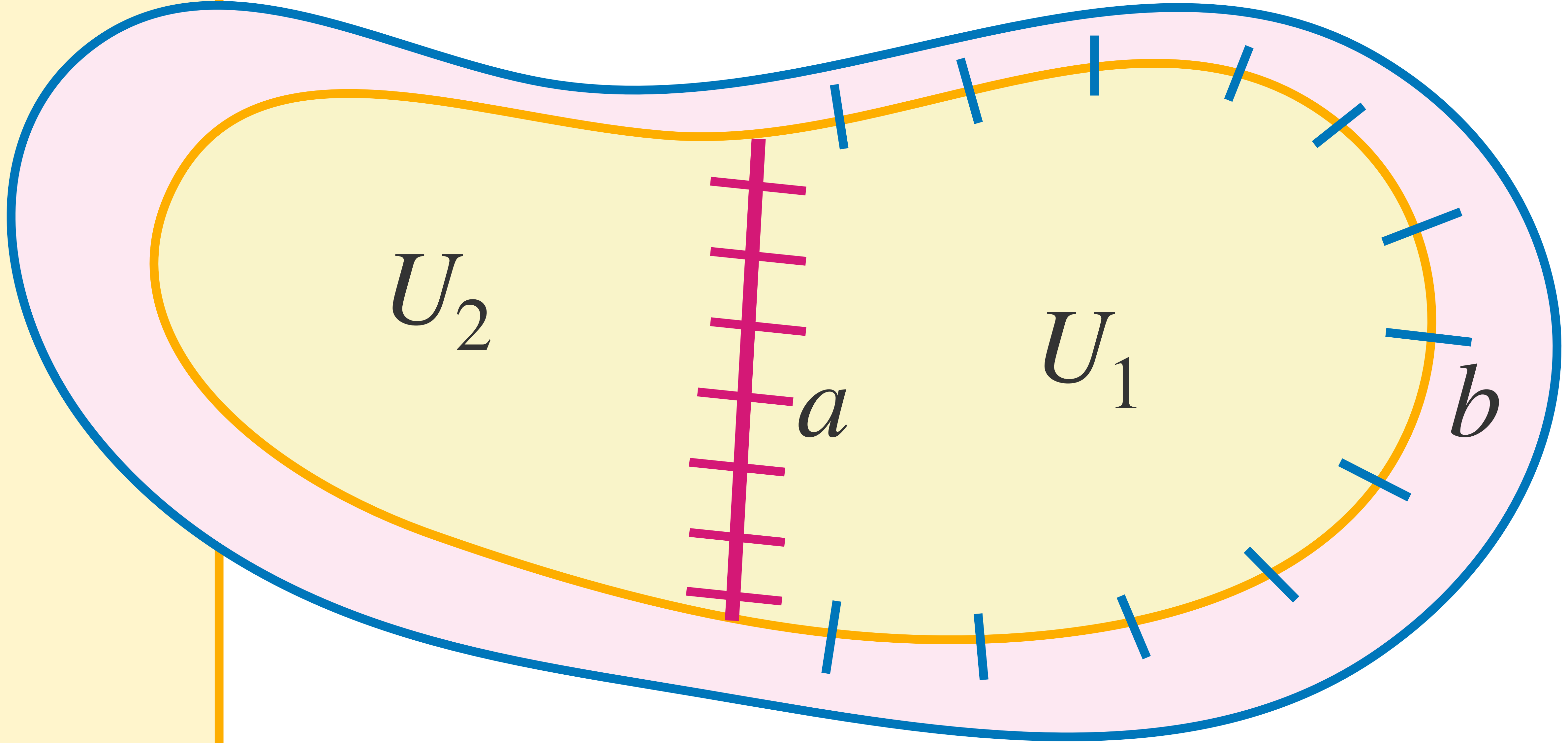

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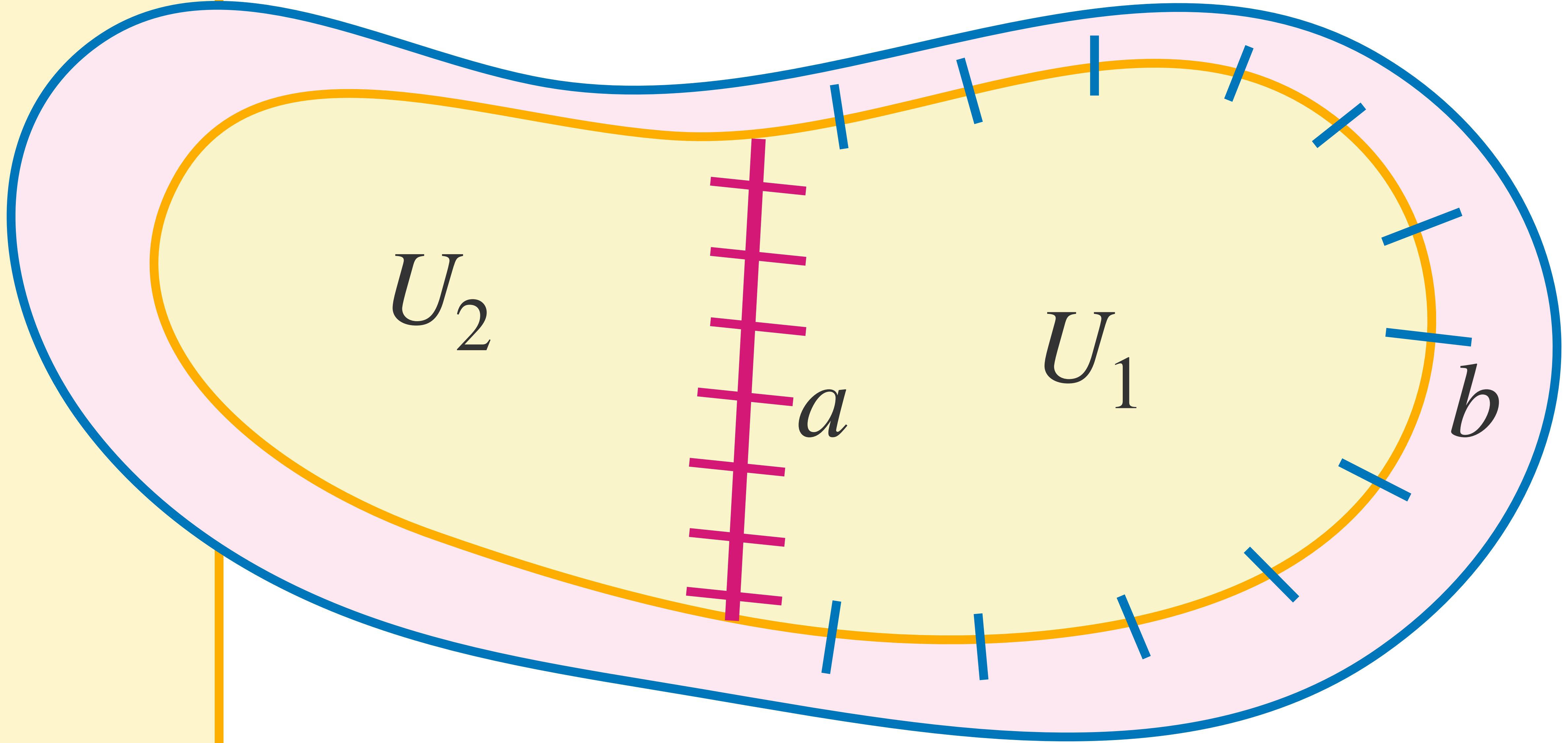
$\kappa_{U_1} = a + b$



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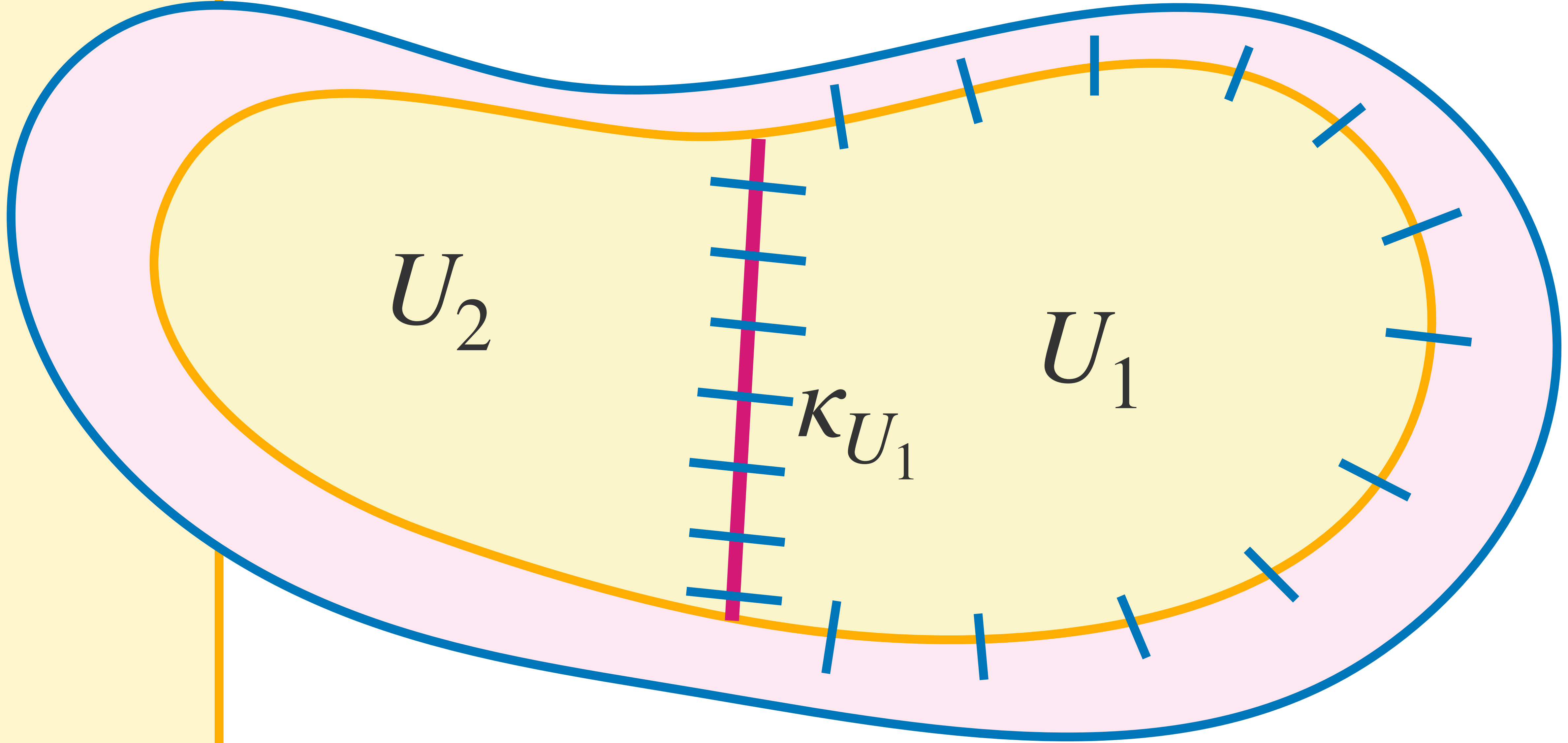

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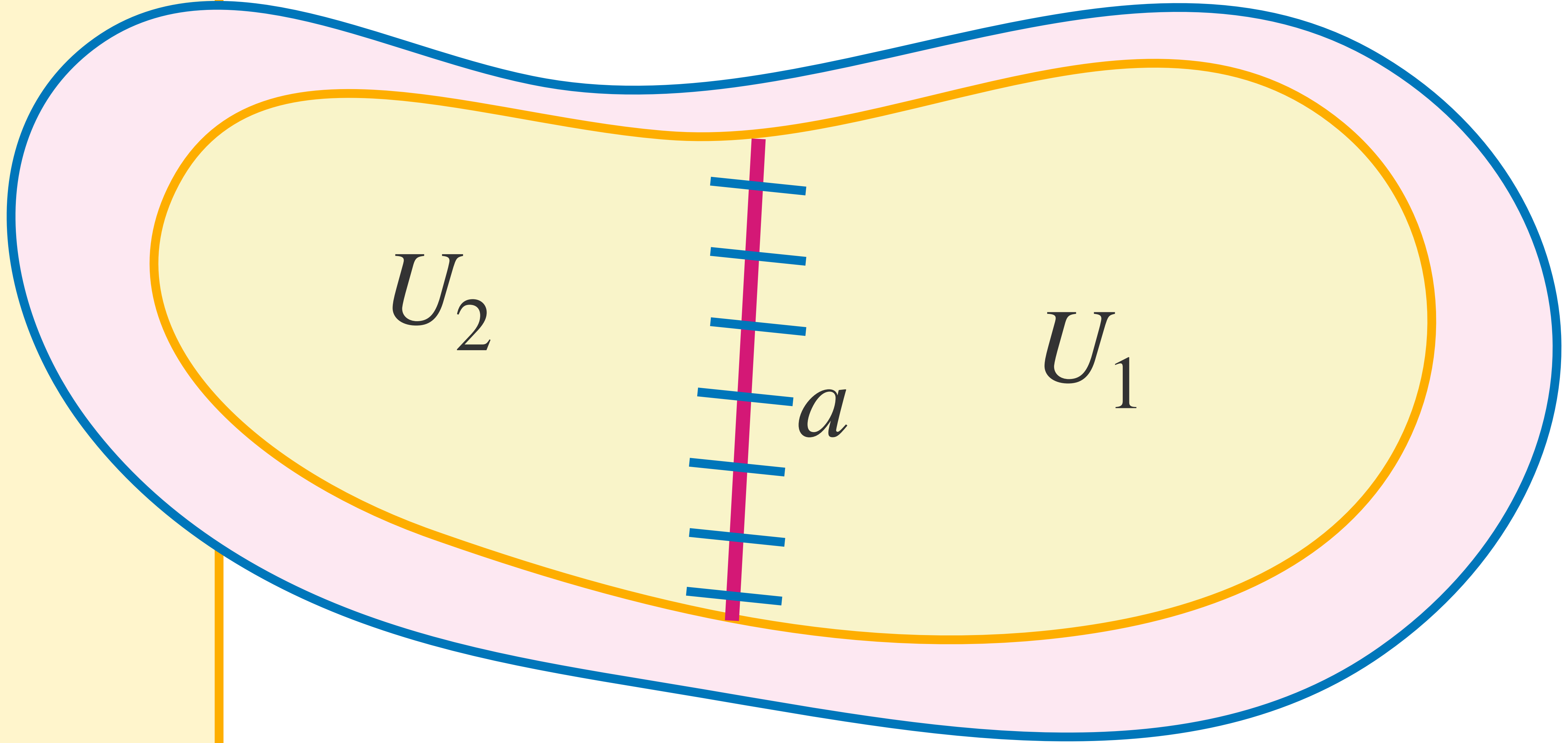

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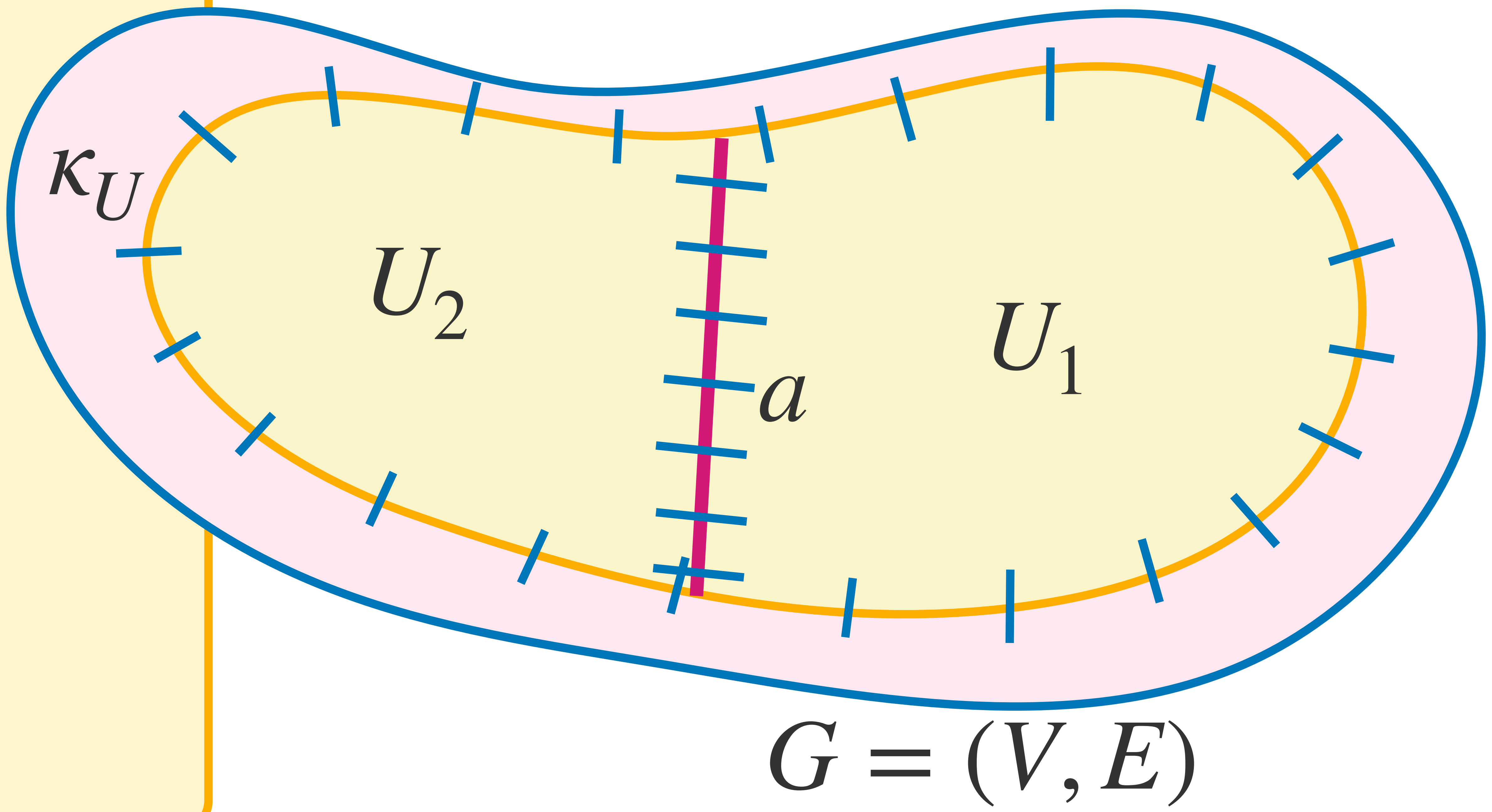

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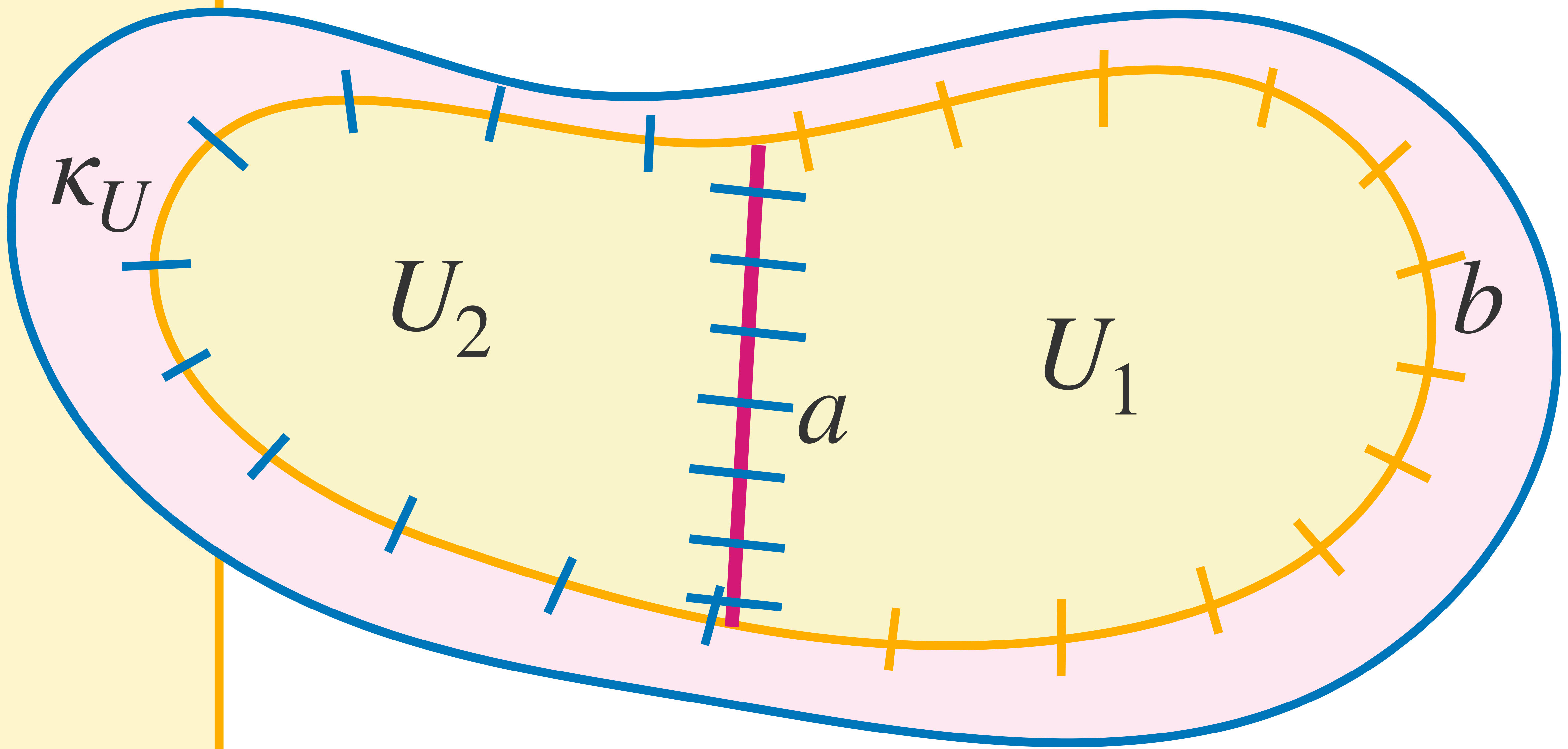

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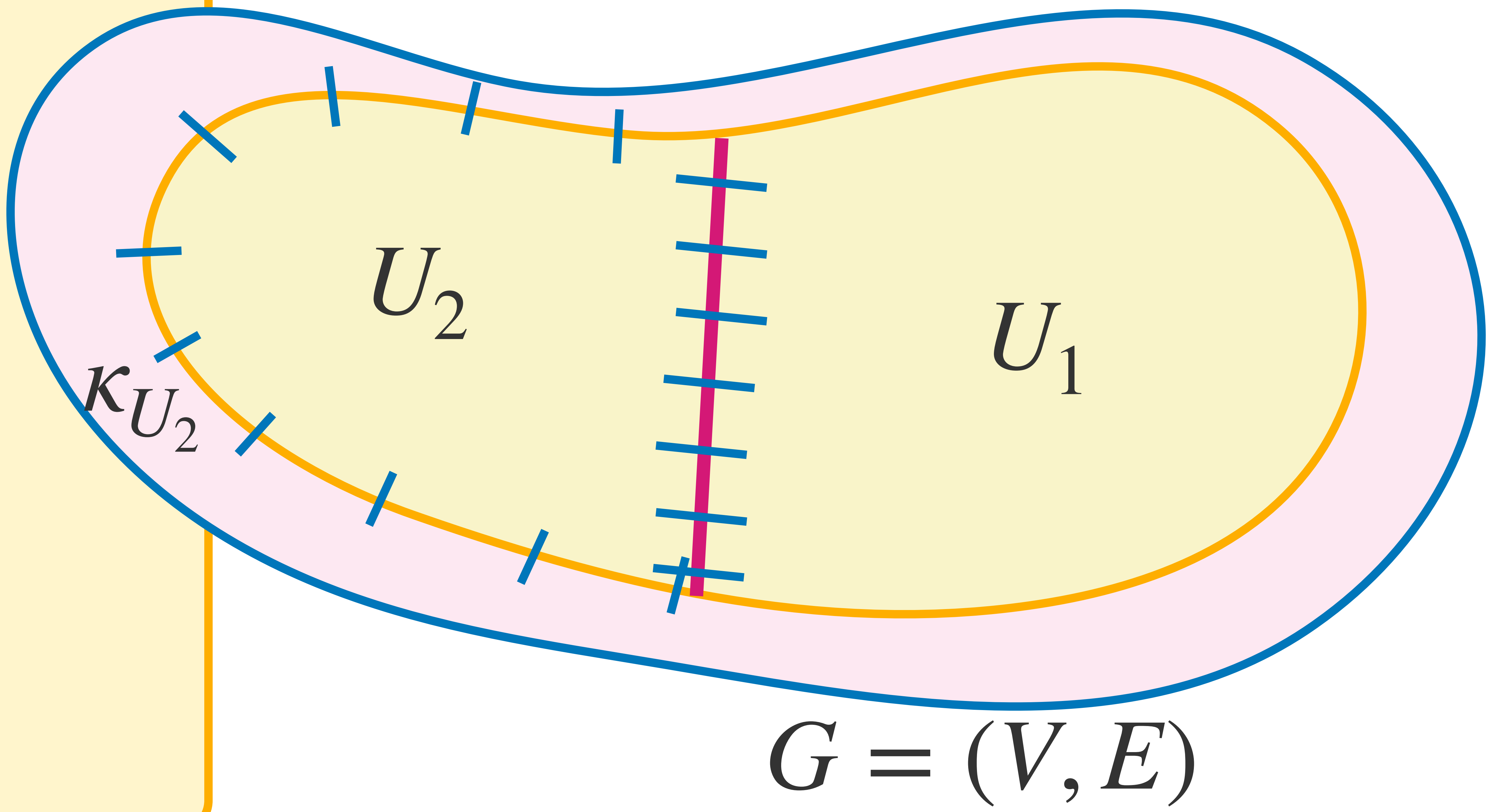

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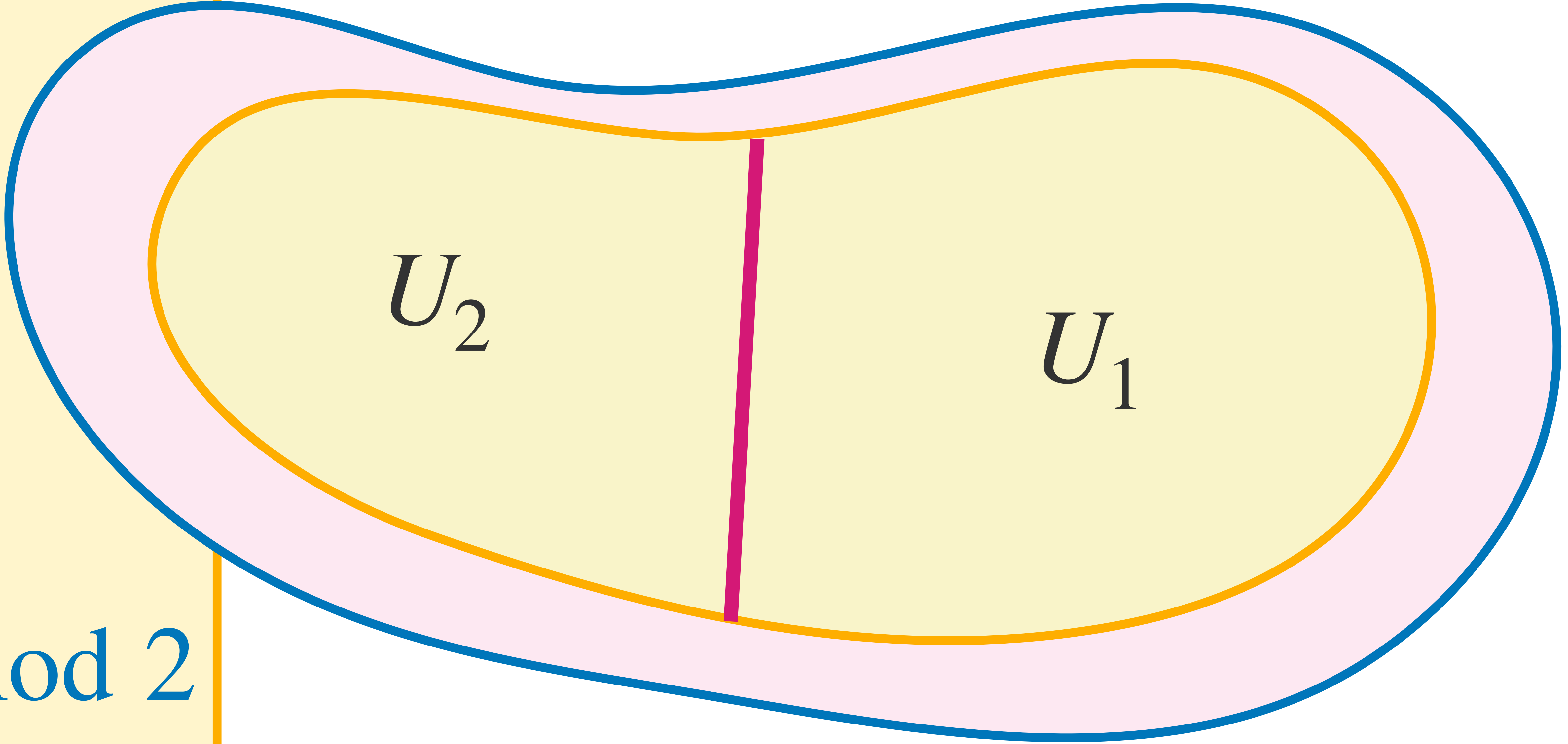

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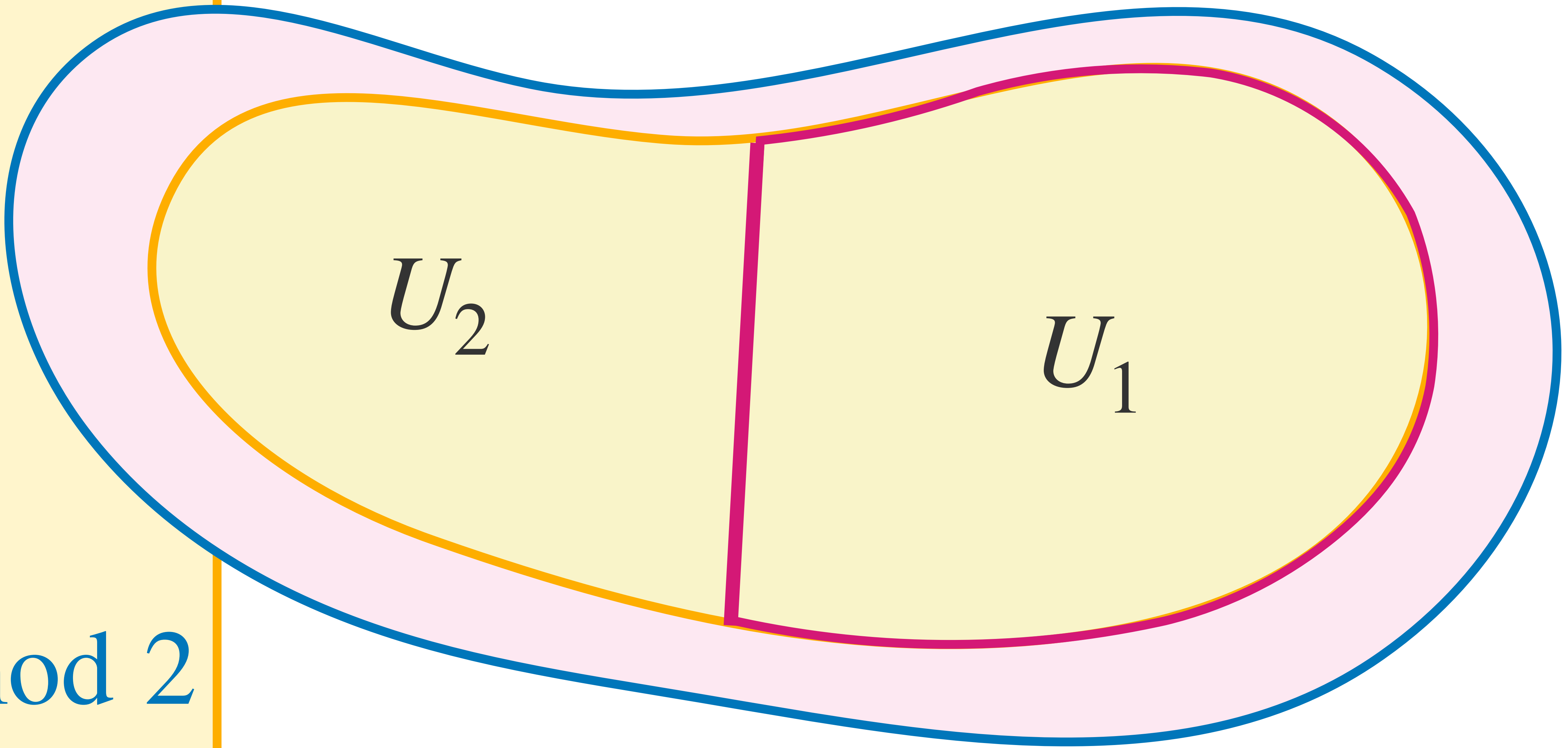
   $\rightarrow$ recurse on that one.



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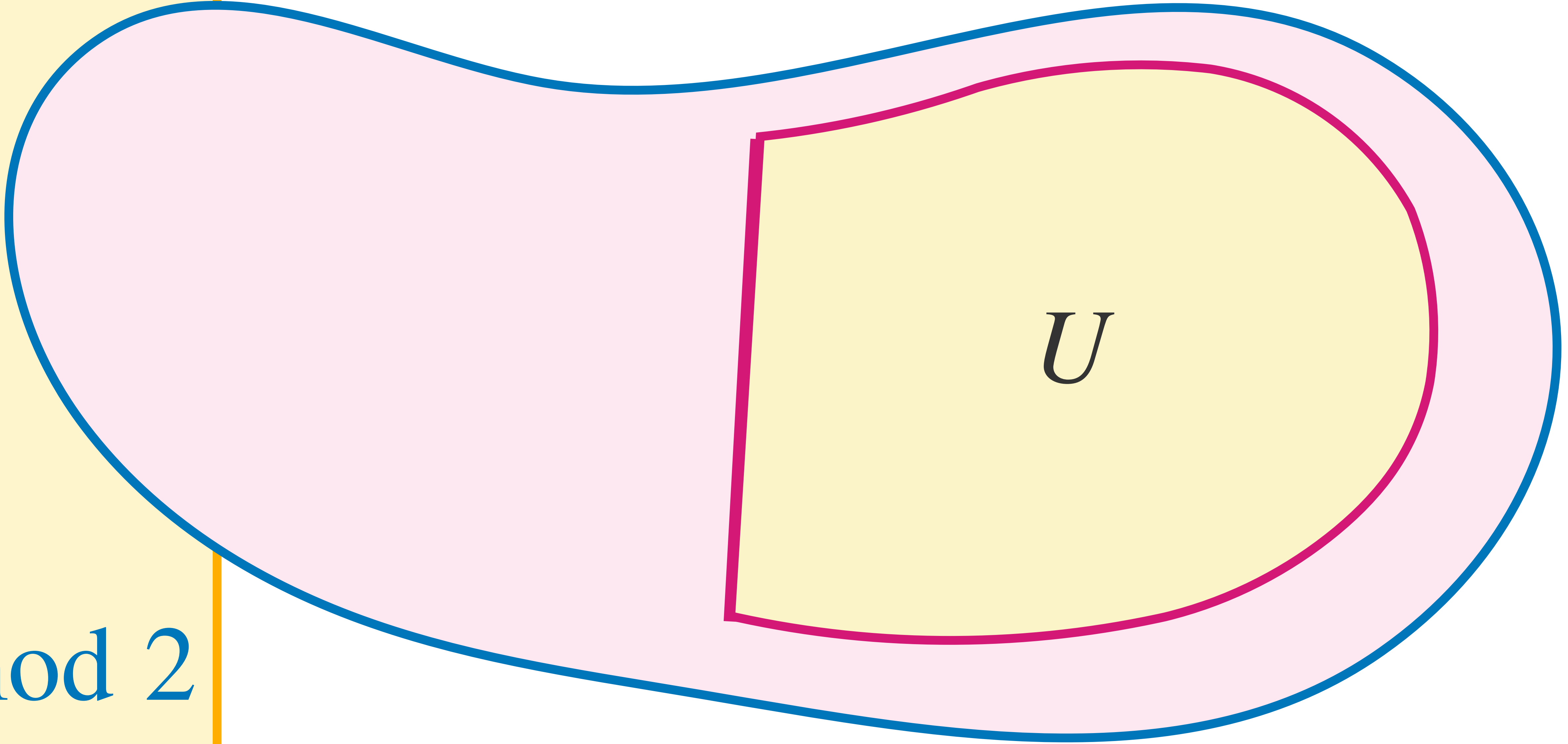

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Implementation in Stabbing Planes

1. Pick a balanced partition $U = U_1 \cup U_2$
2. Query:

$\boxed{a = \Sigma_{e \in [U_1, U_2]} y_e}$ $\boxed{b = \Sigma_{e \in [U_1, V \setminus U]} y_e}$

3. Either $\sigma(U_1) \neq \kappa_{U_1} \mod 2$ or $\sigma(U_2) \neq \kappa_{U_2} \mod 2$; recurse on that one.

Implementation in Stabbing Planes

To implement in SP we need to perform the queries a and b

1. Pick a balanced partition $U = U_1 \cup U_2$
2. Query:

$\boxed{a = \Sigma_{e \in [U_1, U_2]} y_e}$ $\boxed{b = \Sigma_{e \in [U_1, V \setminus U]} y_e}$

3. Either $\sigma(U_1) \neq \kappa_{U_1} \mod 2$ or $\sigma(U_2) \neq \kappa_{U_2} \mod 2$; recurse on that one.

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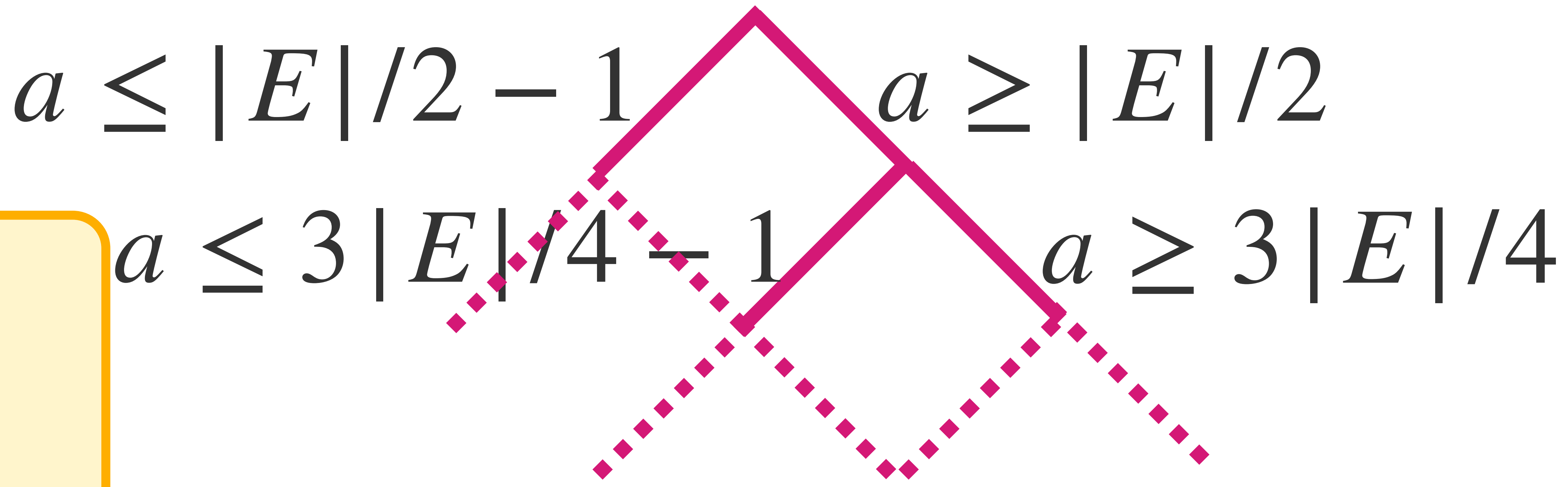

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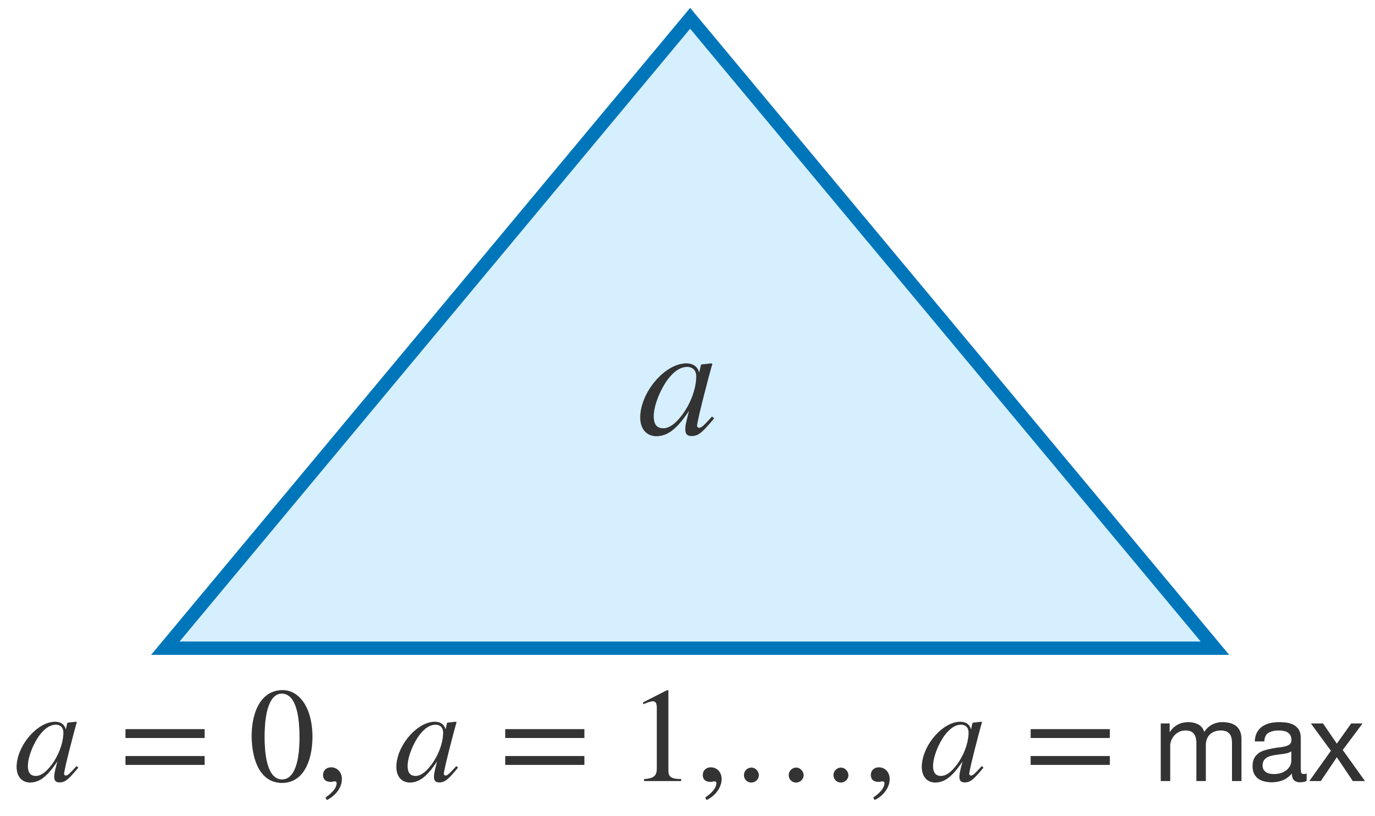

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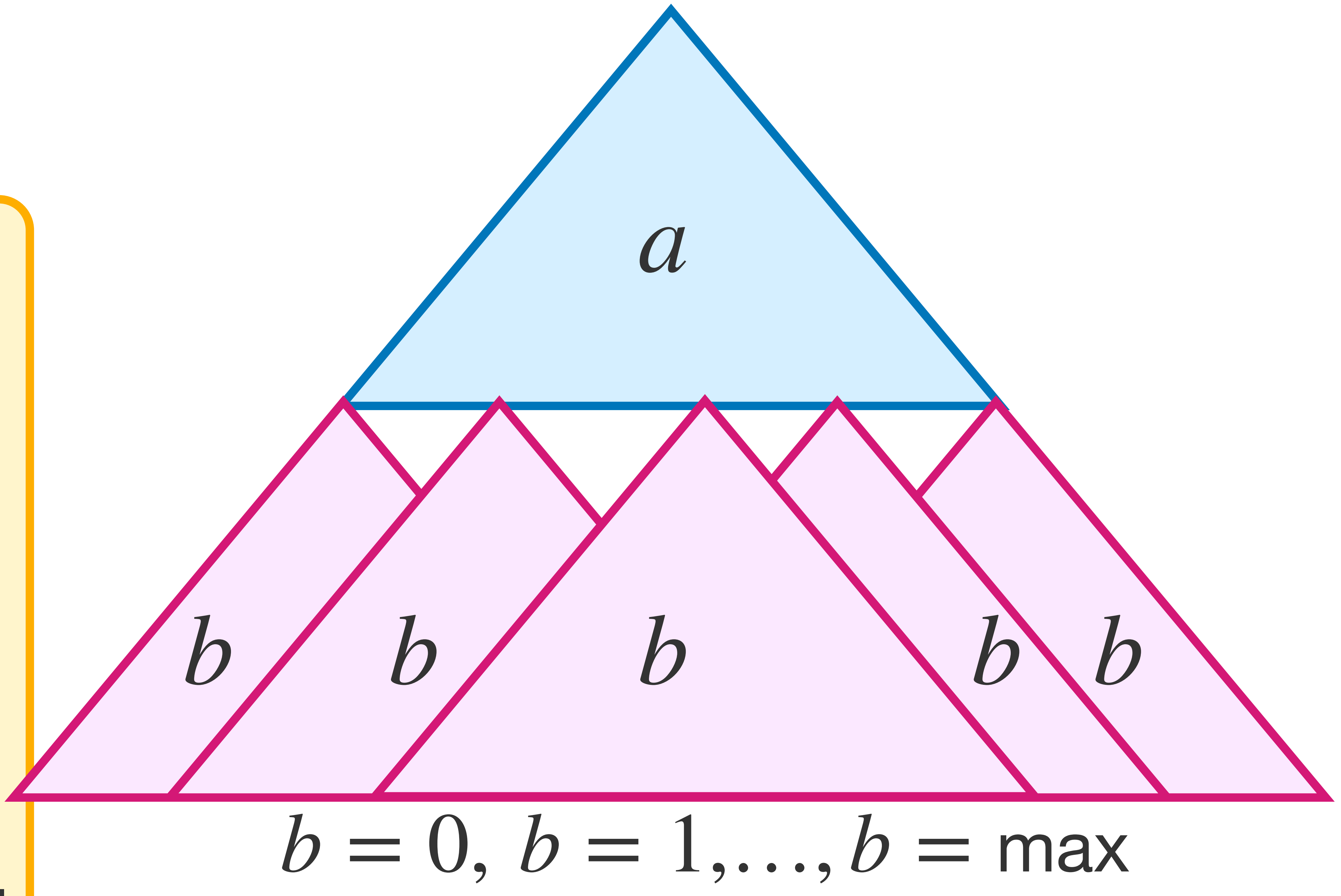

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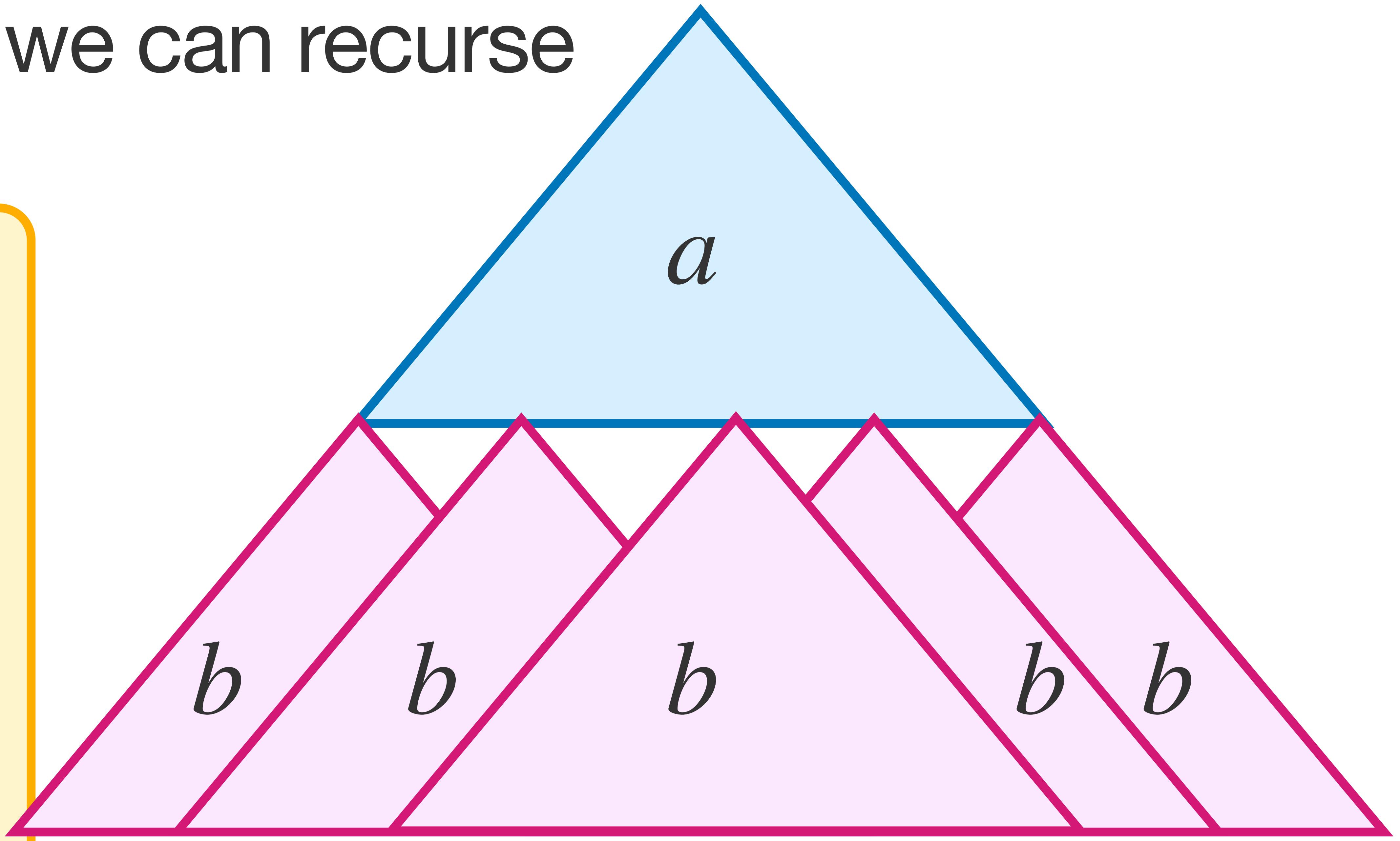

$b = 0,\ b = 1, \ldots, b =$ max

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Complexity:

- $\log|V|$ rounds

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- $\log|V|$ rounds
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→ total **depth:** $2\log|E|\log|V| = O(\log^2 n)$

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Complexity:

SP proofs are binary trees

depth $O(\log^2 n) \rightarrow$ size $2^{O(\log^2 n)}$

Cutting Planes Proves Tseitin!

Thm: There are quasipolynomial size Cutting Planes proofs of Tseitin

High Level:

1. Exhibit a quasipolynomial size Stabbing Planes proof of Tseitin
2. Translate that proof into Cutting Planes

In fact, almost every SP proof can be translated into CP!

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Thm [FGI+21]

Any Stabbing Planes proof with coefficients at most $2^{\mathrm{polylog} n}$ (SP*) can be translated into Cutting Planes with a quasi-polynomial blow-up in the size.

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Idea:

1. Turn the proof SP* into a facelike SP proof — one that branches on the faces of the current polytope

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Idea:

1. Turn the proof SP* into a facelike SP proof — one that branches on the faces of the current polytope (causes a quasipolynomial blow-up)
2. Show that facelike SP proofs are equivalent to Cutting Planes proofs