A Gentle Introduction to Modern SAT Solving — Part 2

Noah Fleming
University of California, San Diego

Last Time — Resolution

Resolution: A method for proving that a CNF formula is **unsatisfiable**

Derive new clauses from old using:

→ **Resolution rule:**

$$\frac{C_1 \vee x, \quad C_2 \vee \neg x}{C_1 \vee C_2}$$

Goal: derive empty clause Λ

Resolution rule is **sound**

$\Longrightarrow$ Derivation of Λ certifies unsatisfiability

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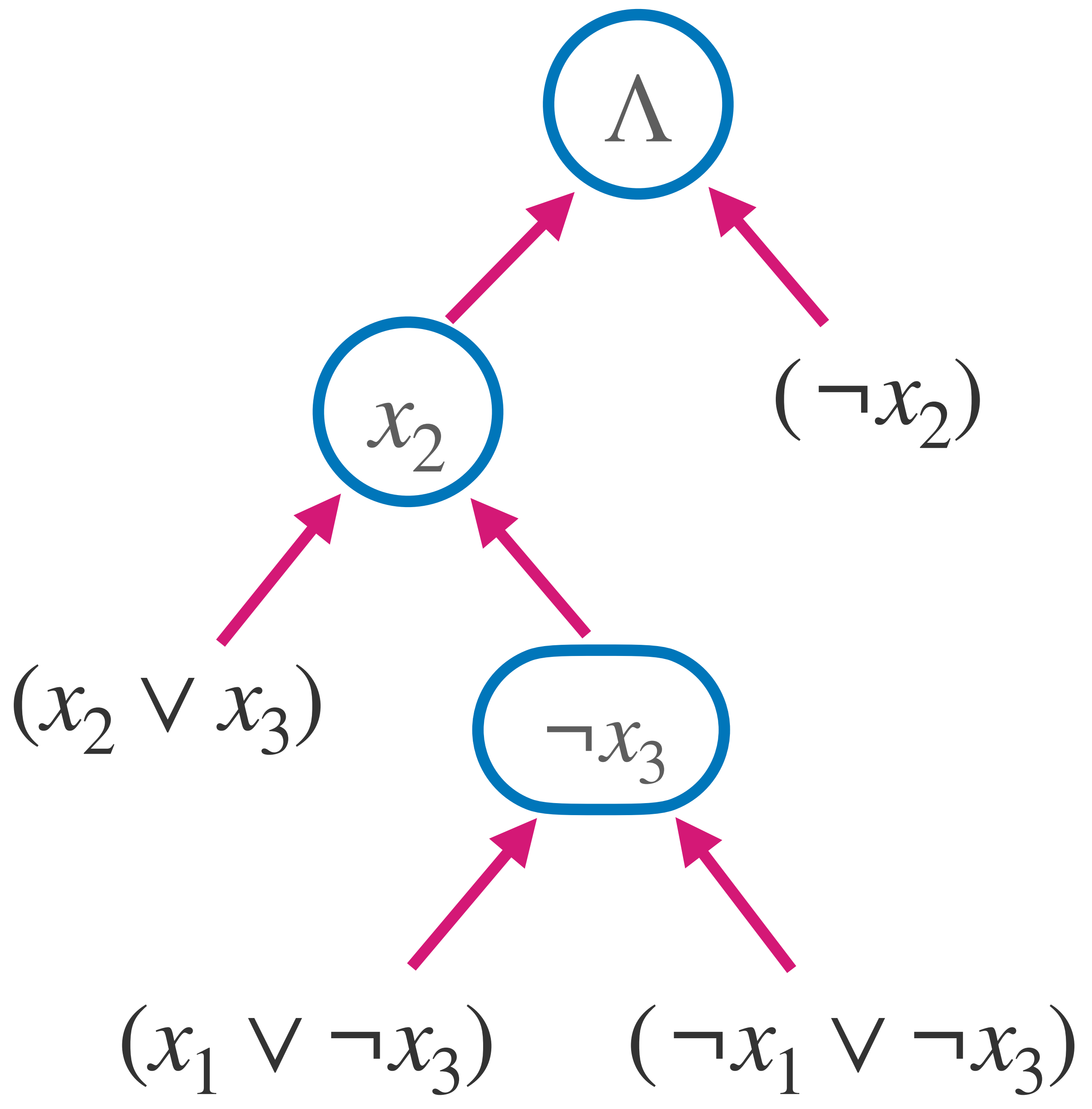

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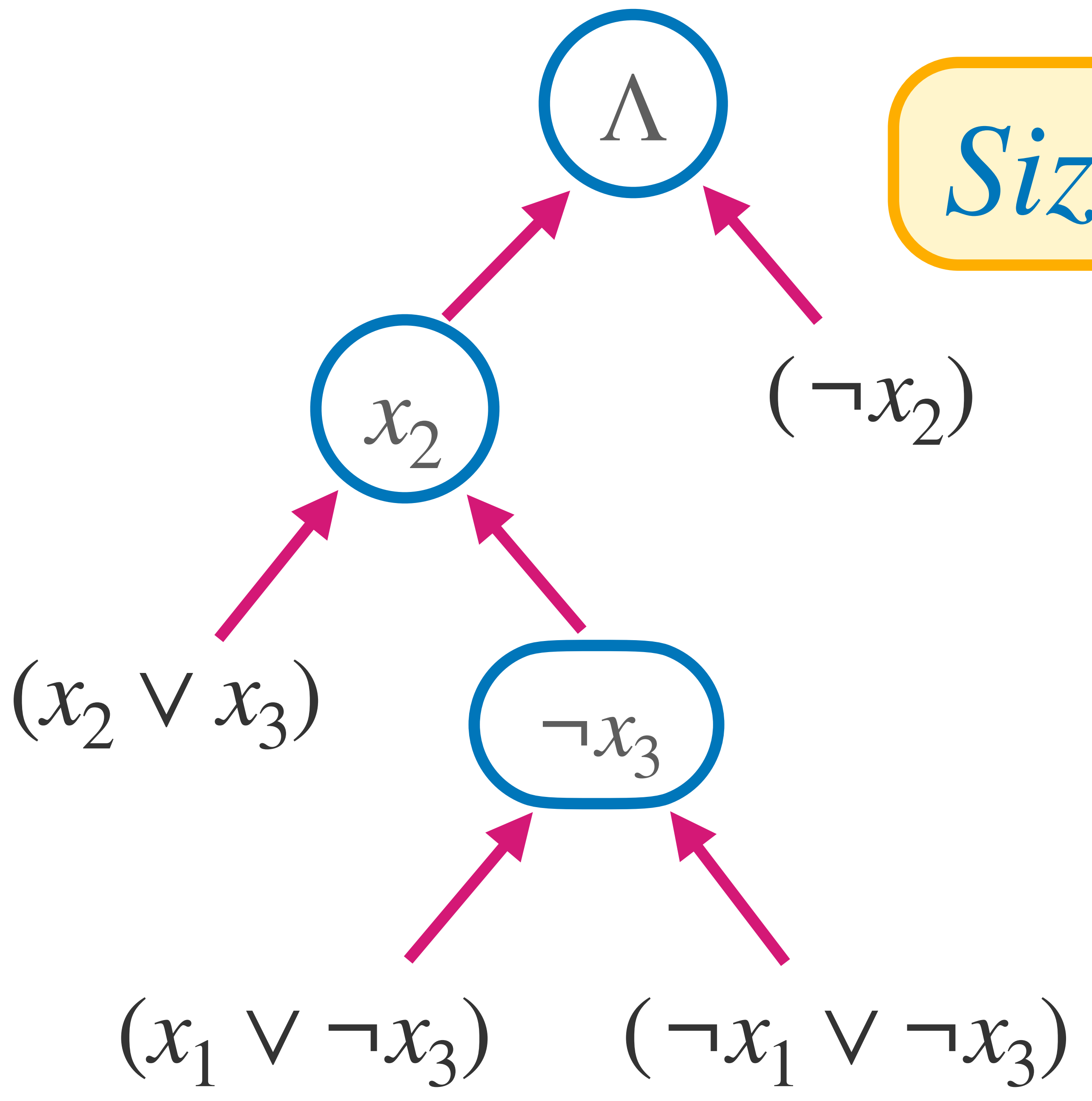

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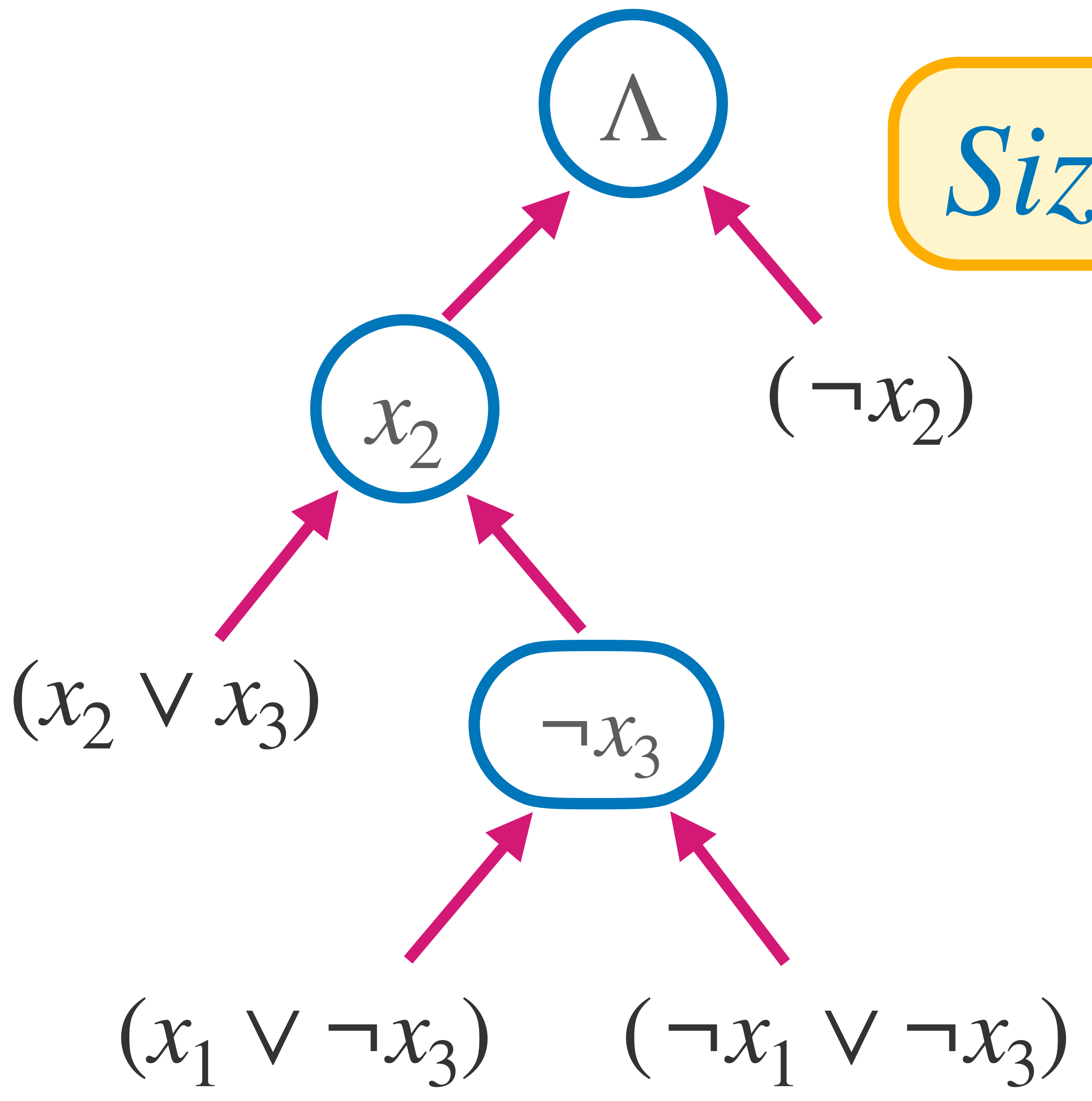

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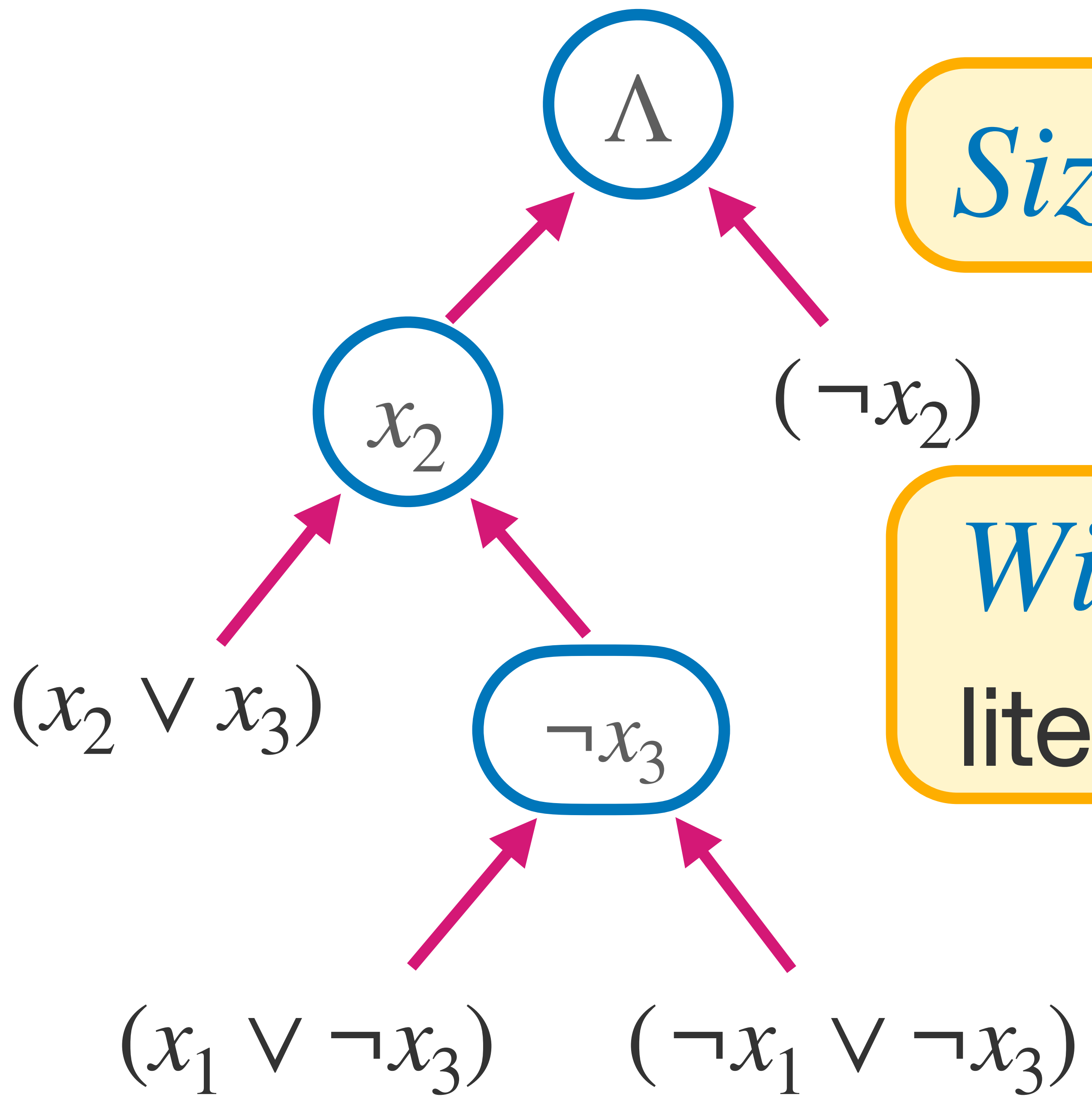

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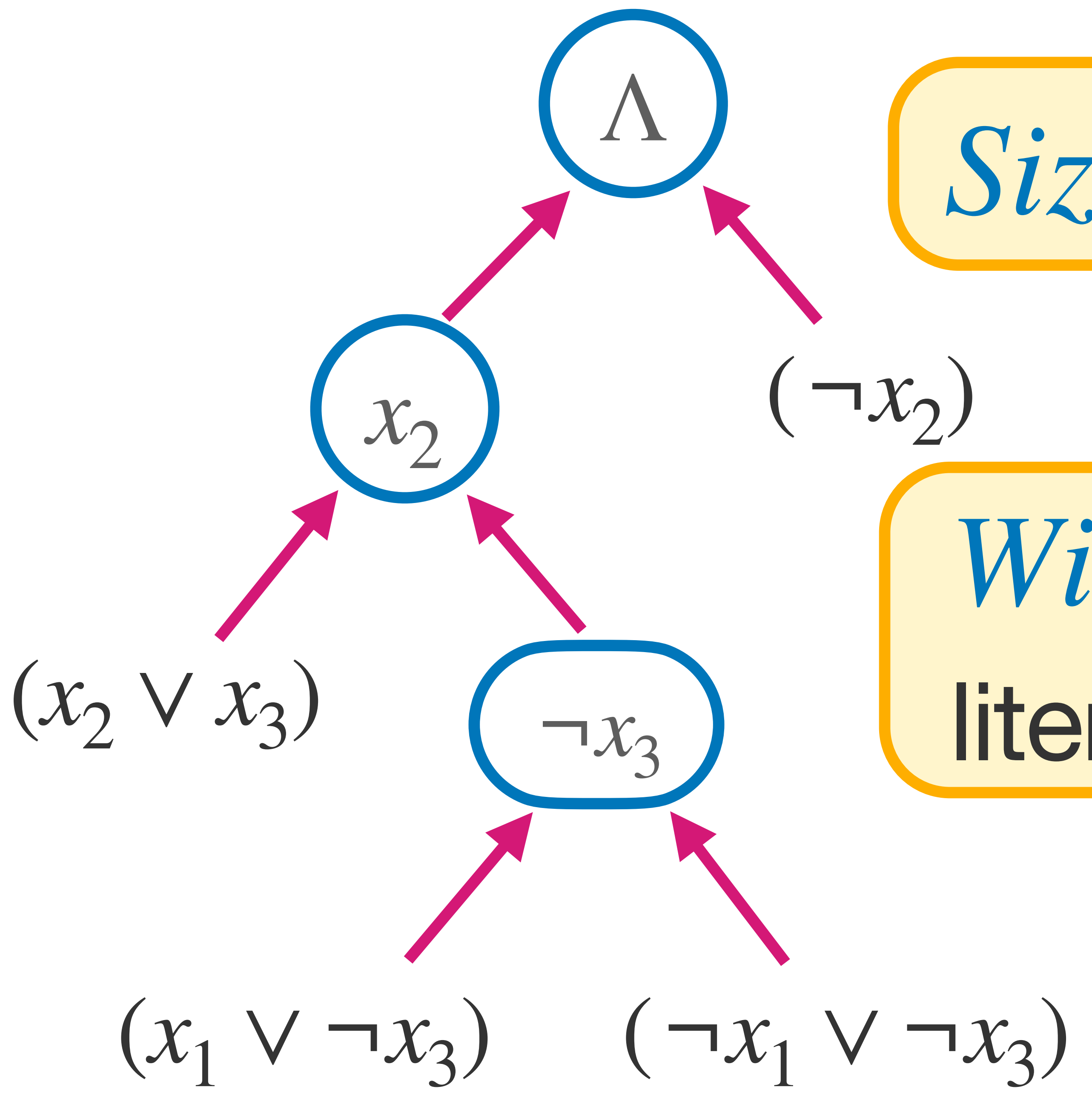

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Last Time

- Introduced the DPLL algorithm

  → Lower bounds on the runtime of DPLL follow from lower bounds on tree Resolution proofs

- Introduced the CDCL algorithm by extending DPLL with

  → Unit Propagation

  → Clause Learning

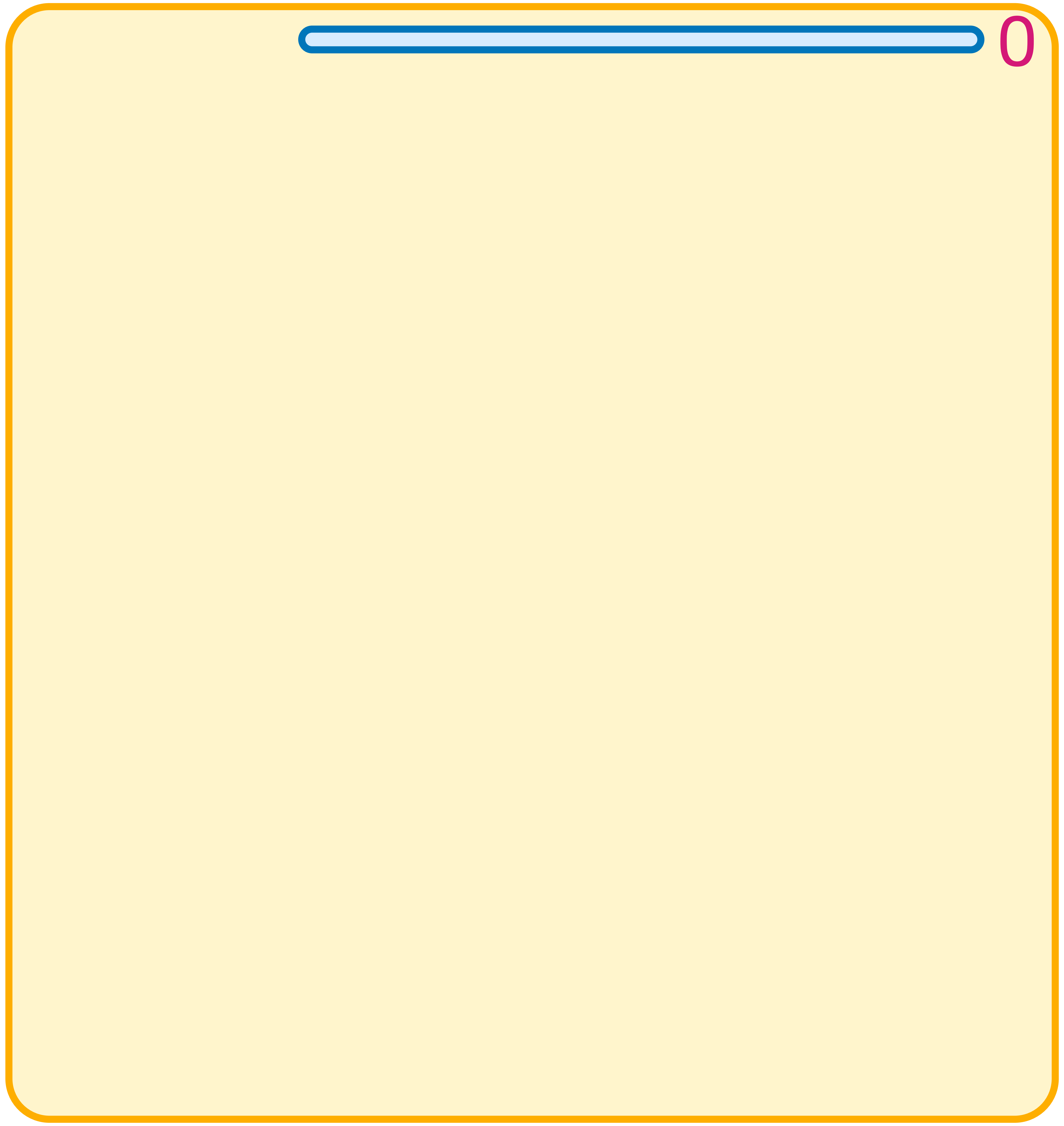
  → Restarts

Last Time — Unit Prop

$(x \vee y) \wedge (z \vee w) \wedge (h \vee \bar{z} \vee \bar{y}) \wedge (\bar{i} \vee \bar{z}) \wedge (i \vee \bar{z} \vee \bar{y})$

Unit clause: a clause containing a **single** literal ℓ

Unit Propagation: if F contains a unit clause (under the current assignment), set $\ell = 1$



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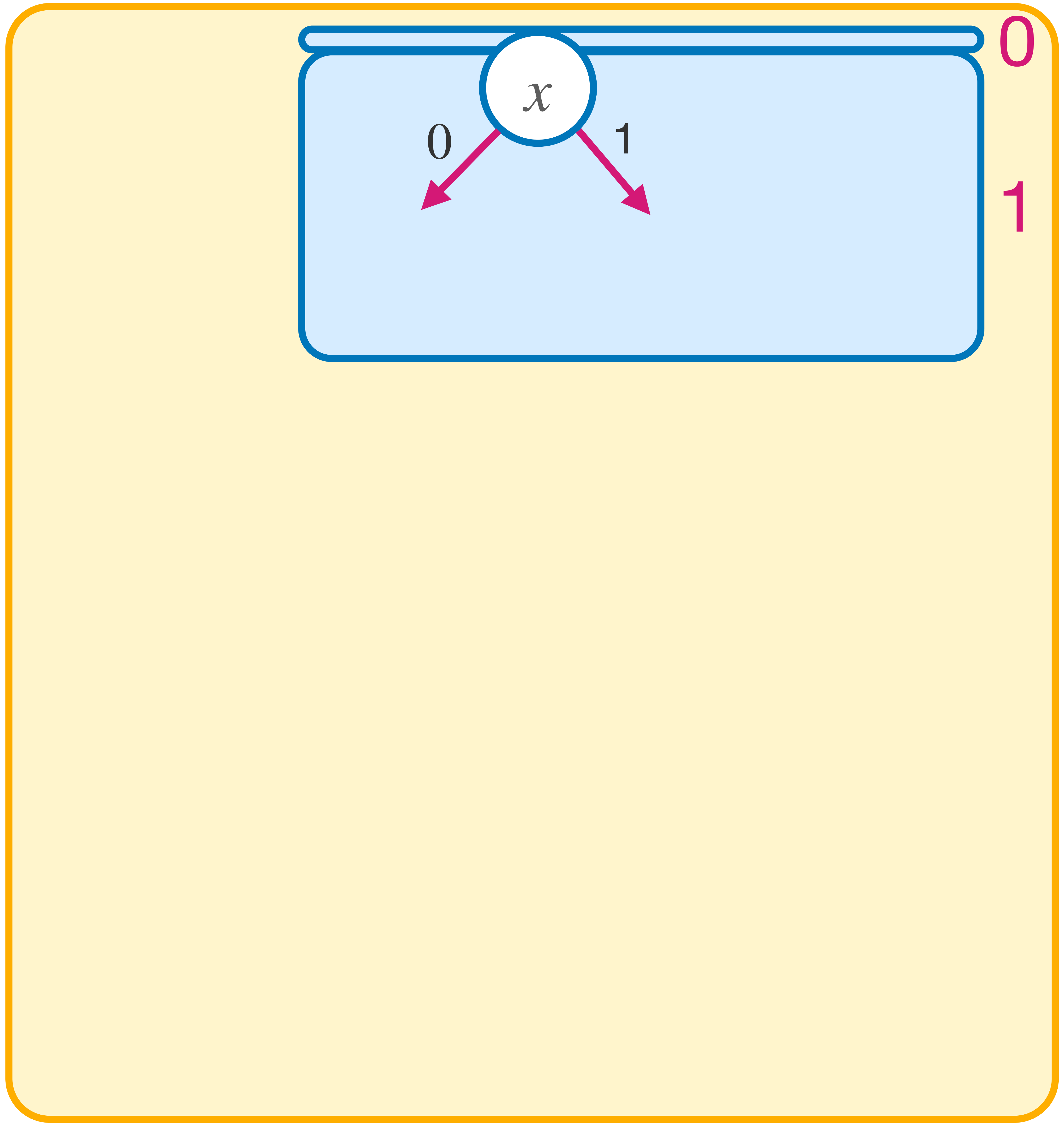

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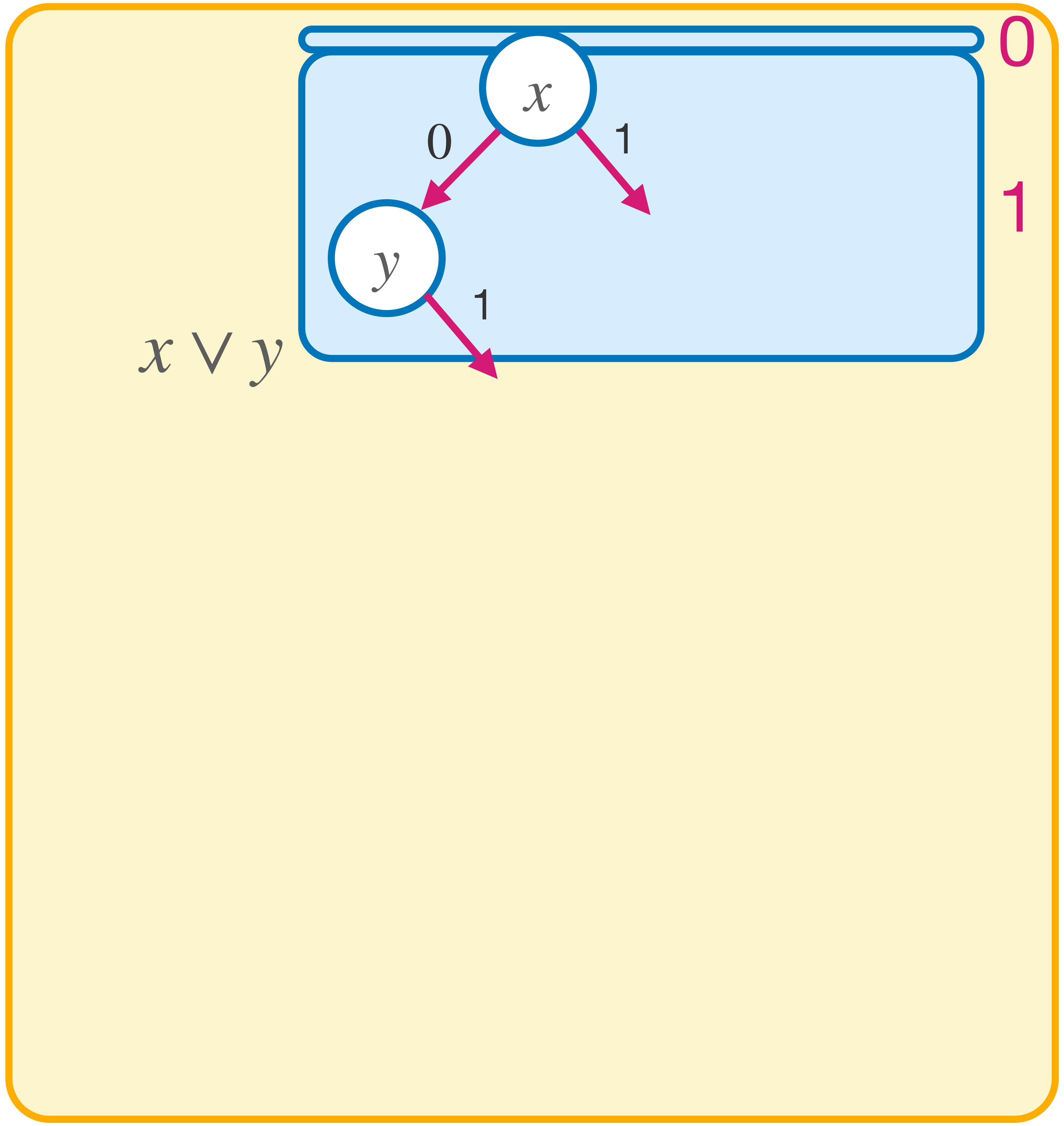

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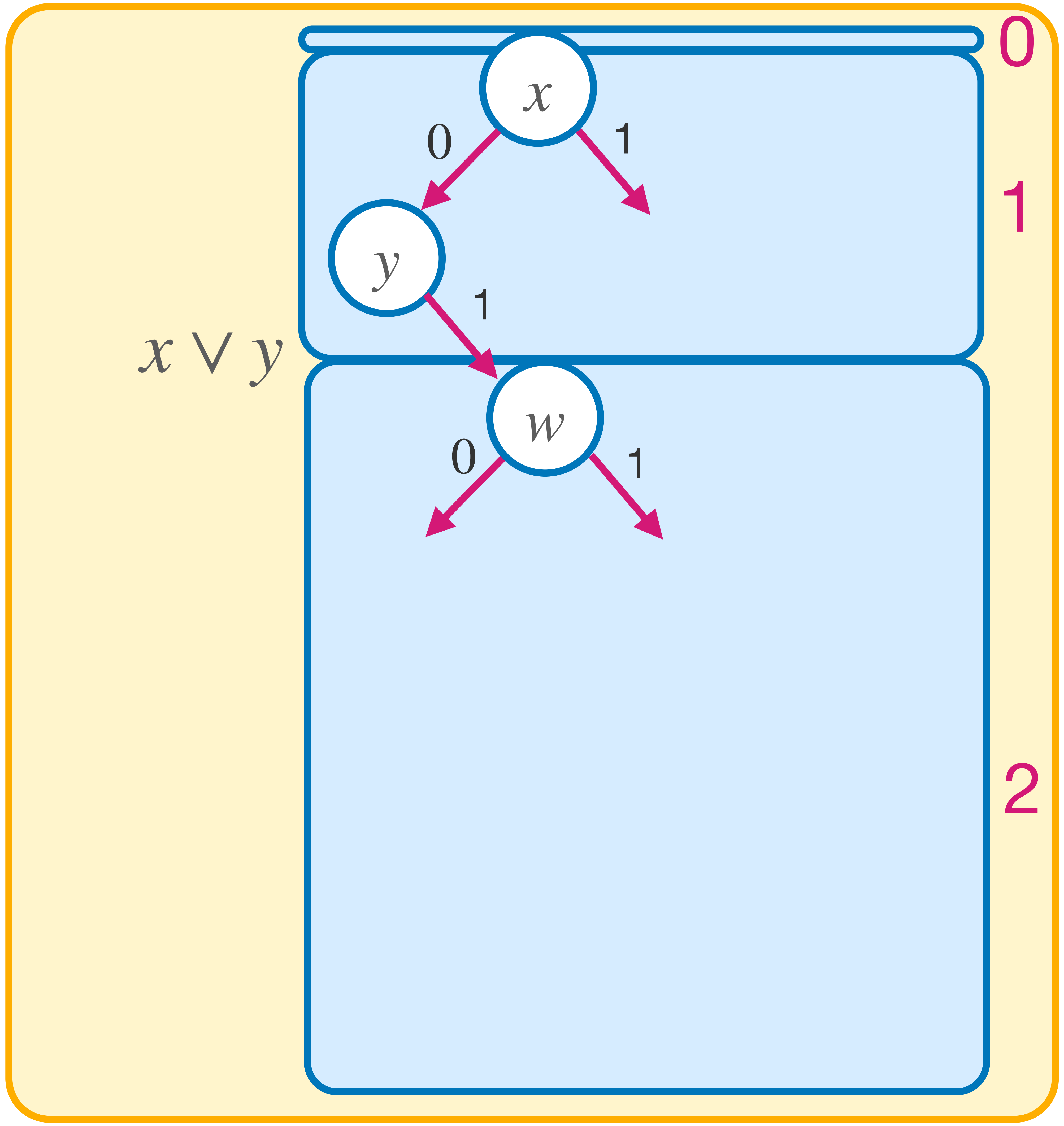

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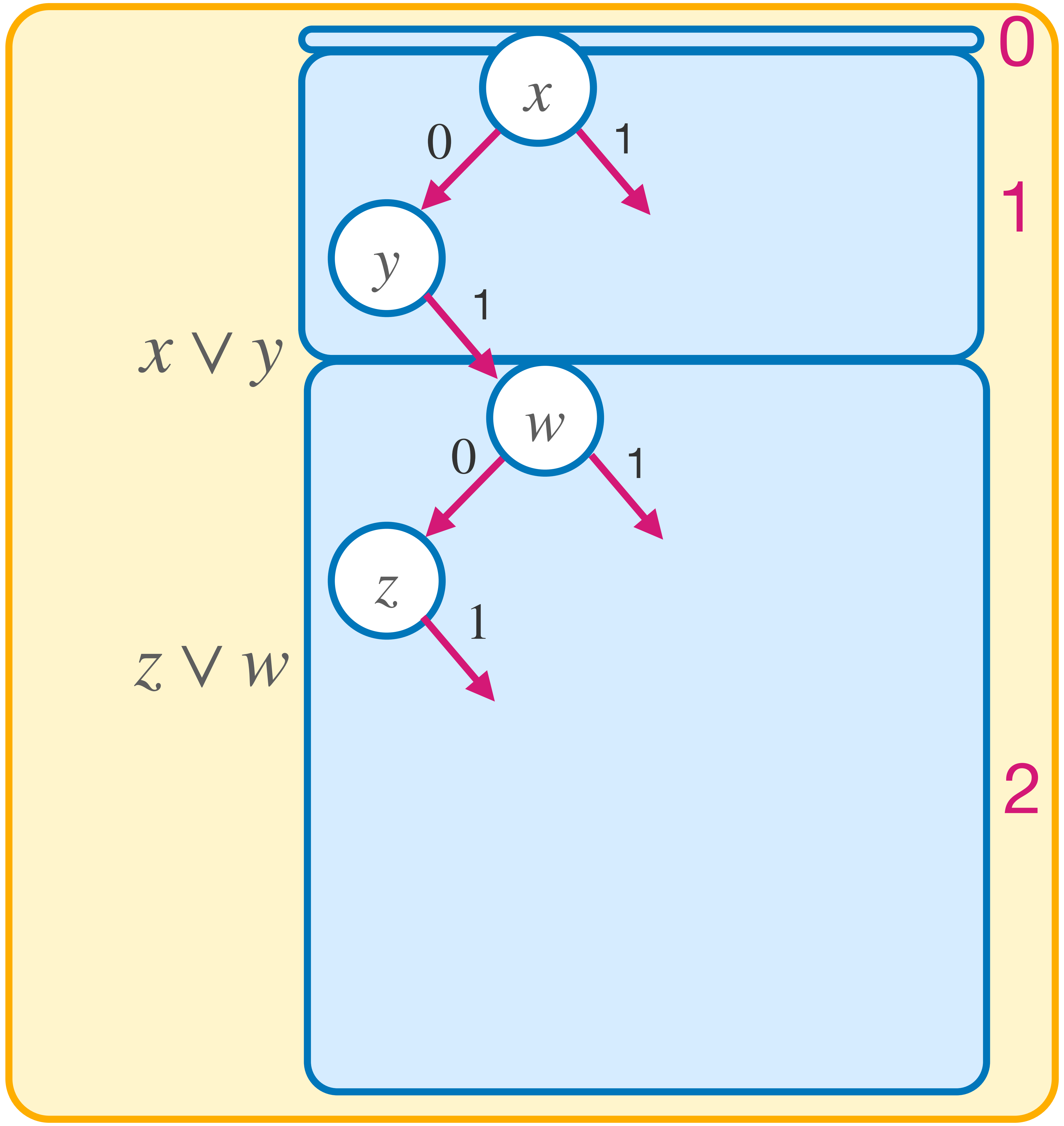

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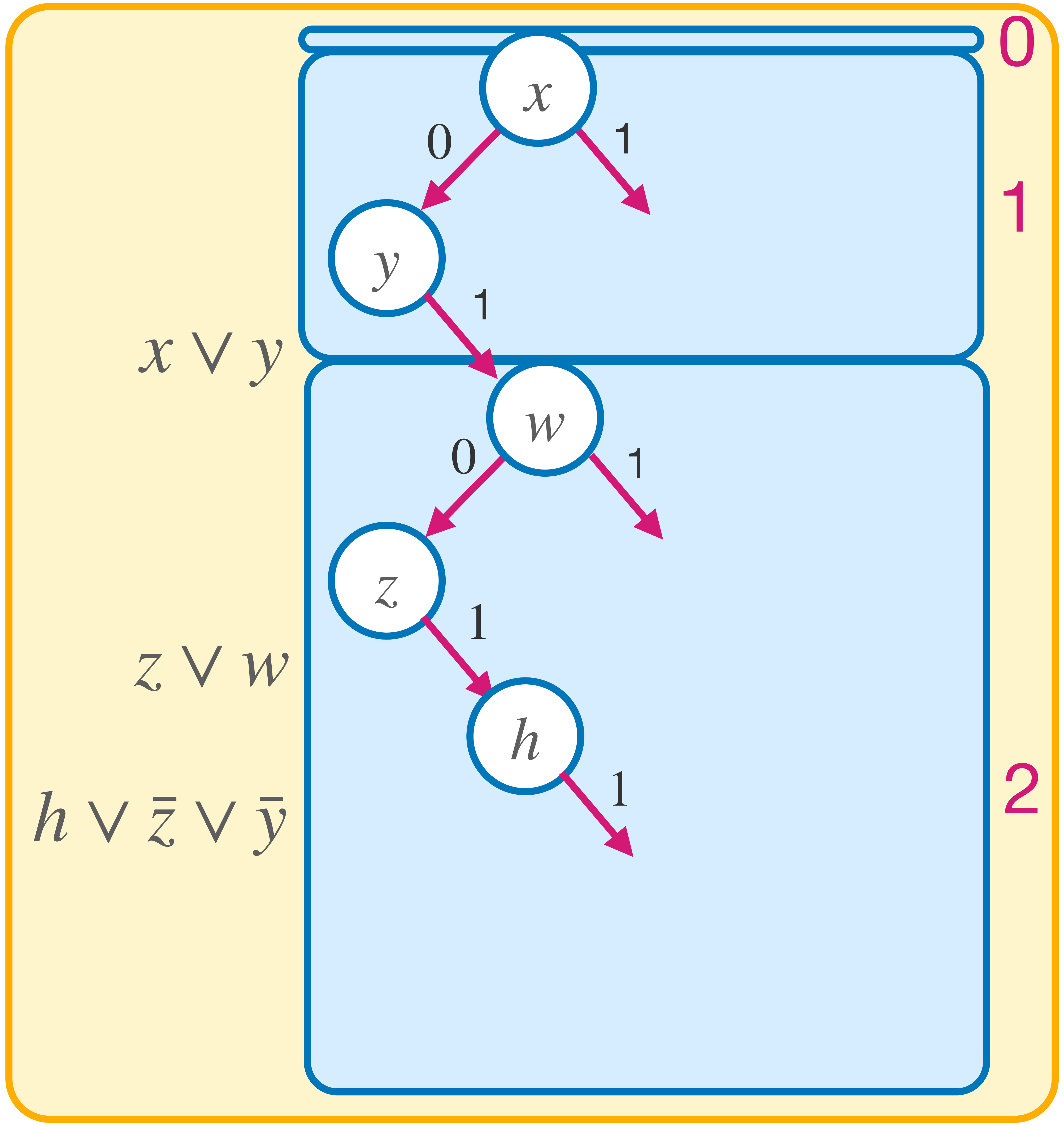

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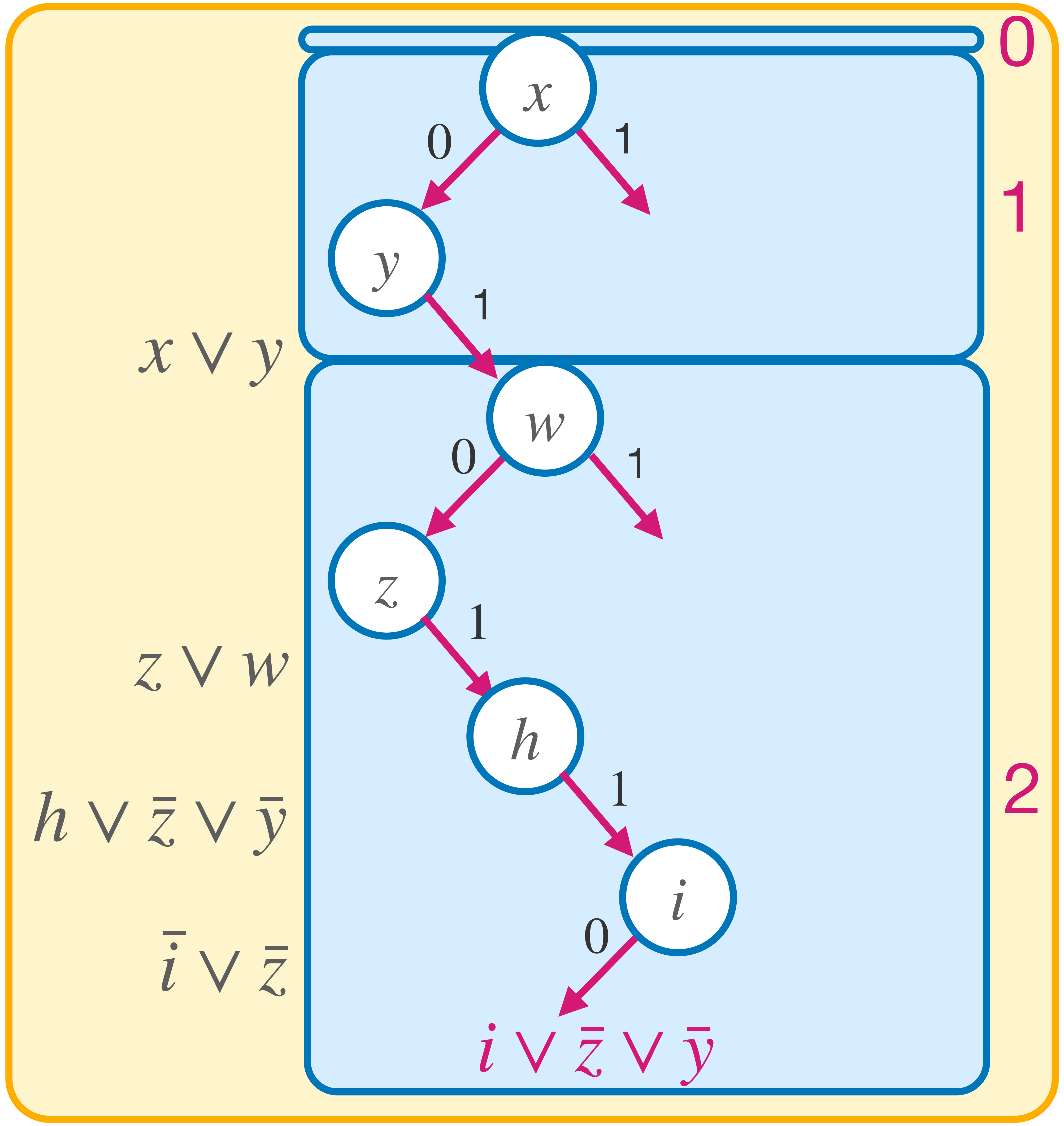

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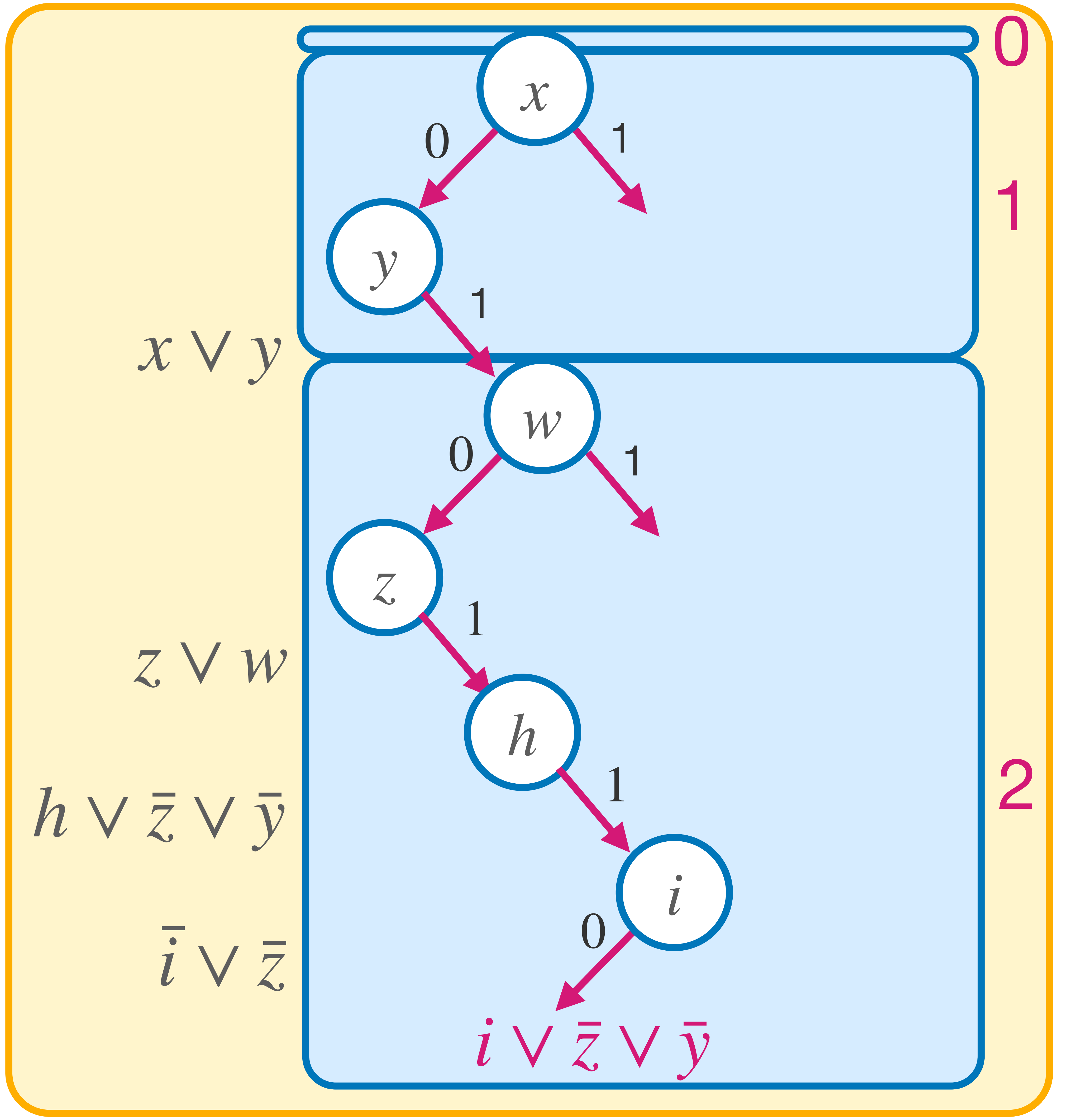
When a conflict, use Resolution to learn a clause



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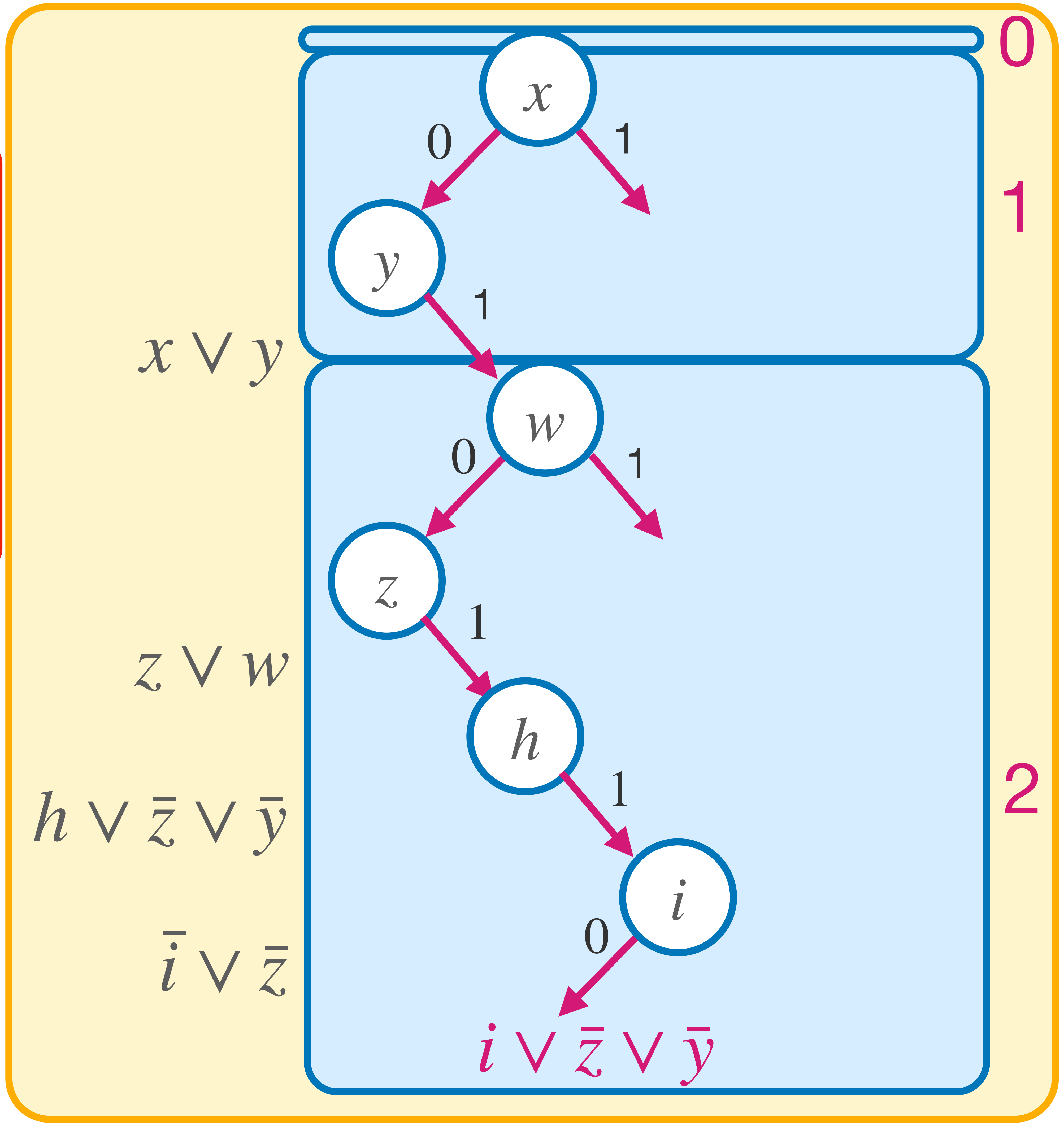
1-UIP Clause

Obtained by resolving the conflict clause along the path until there is only one literal in the clause at the largest decision level



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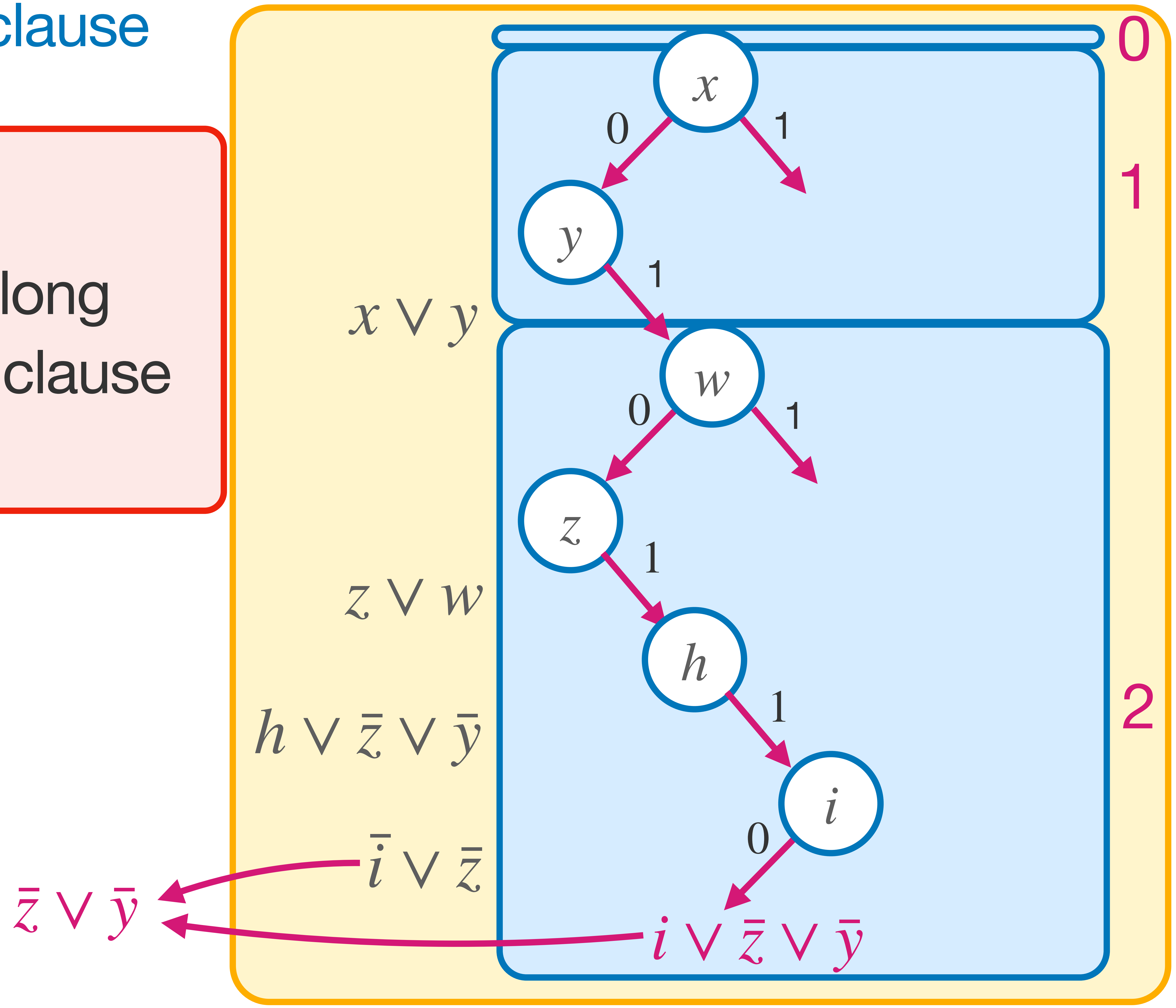

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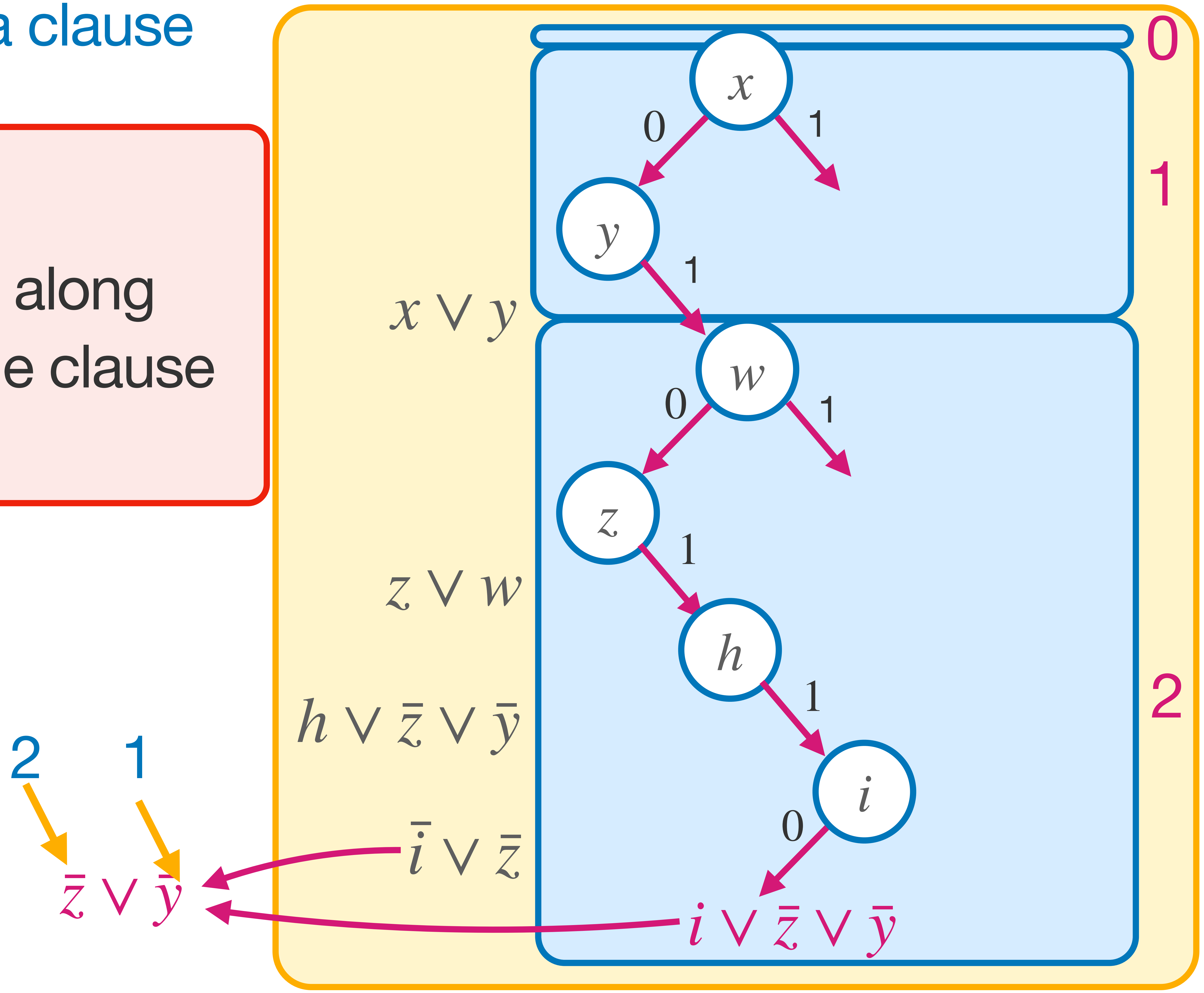

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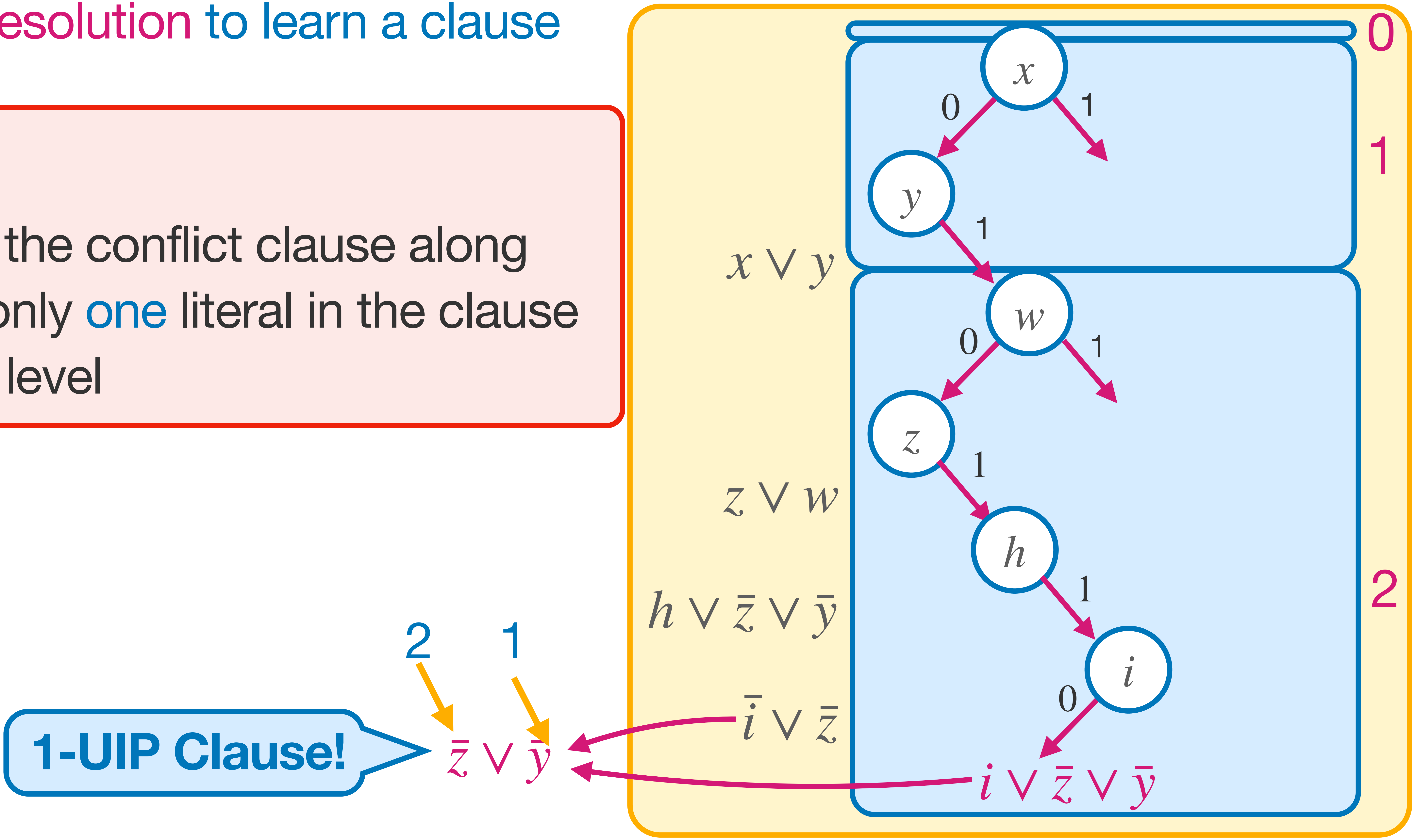

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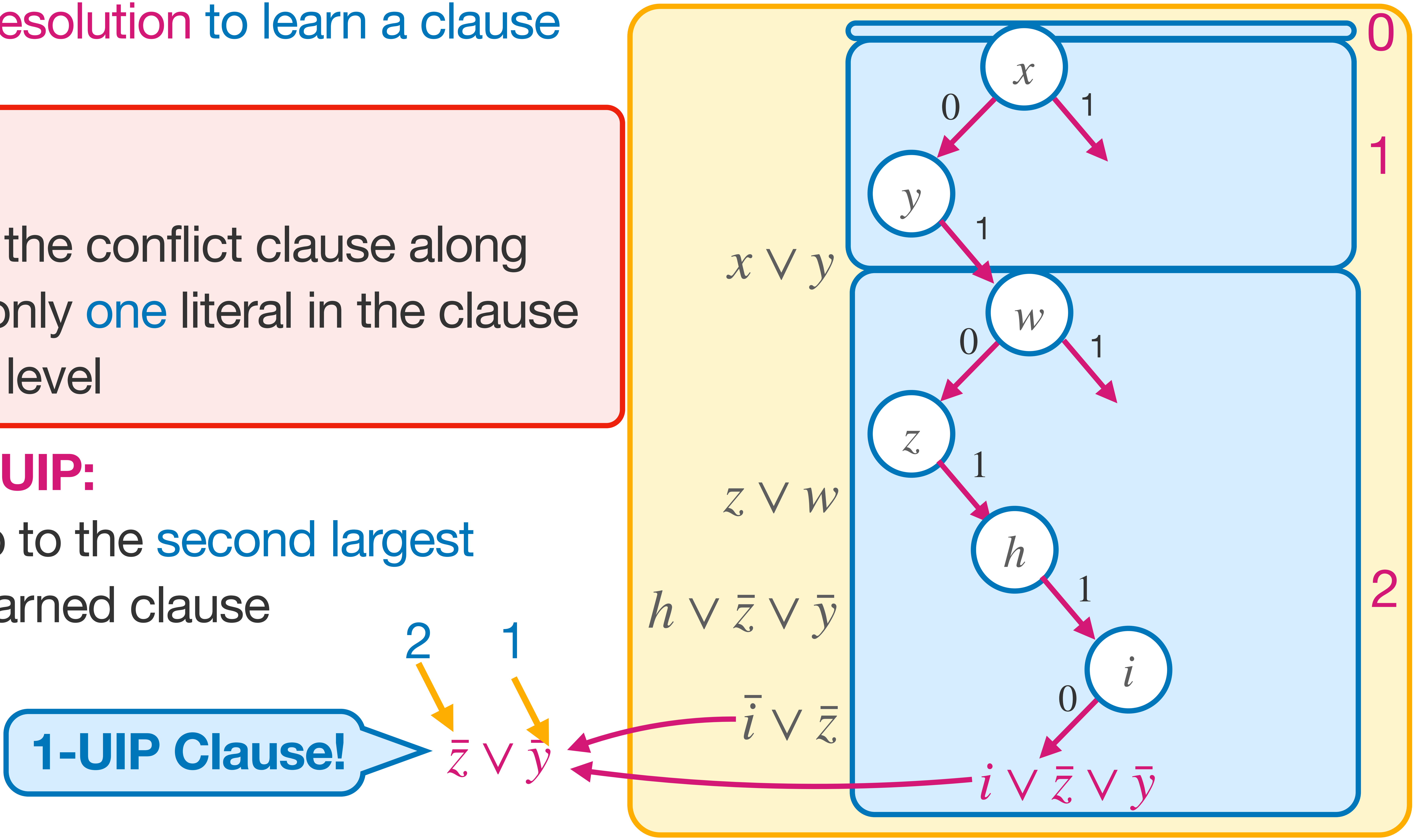
Backtracking with 1-UIP:

Remove everything up to the second largest decision level in the learned clause



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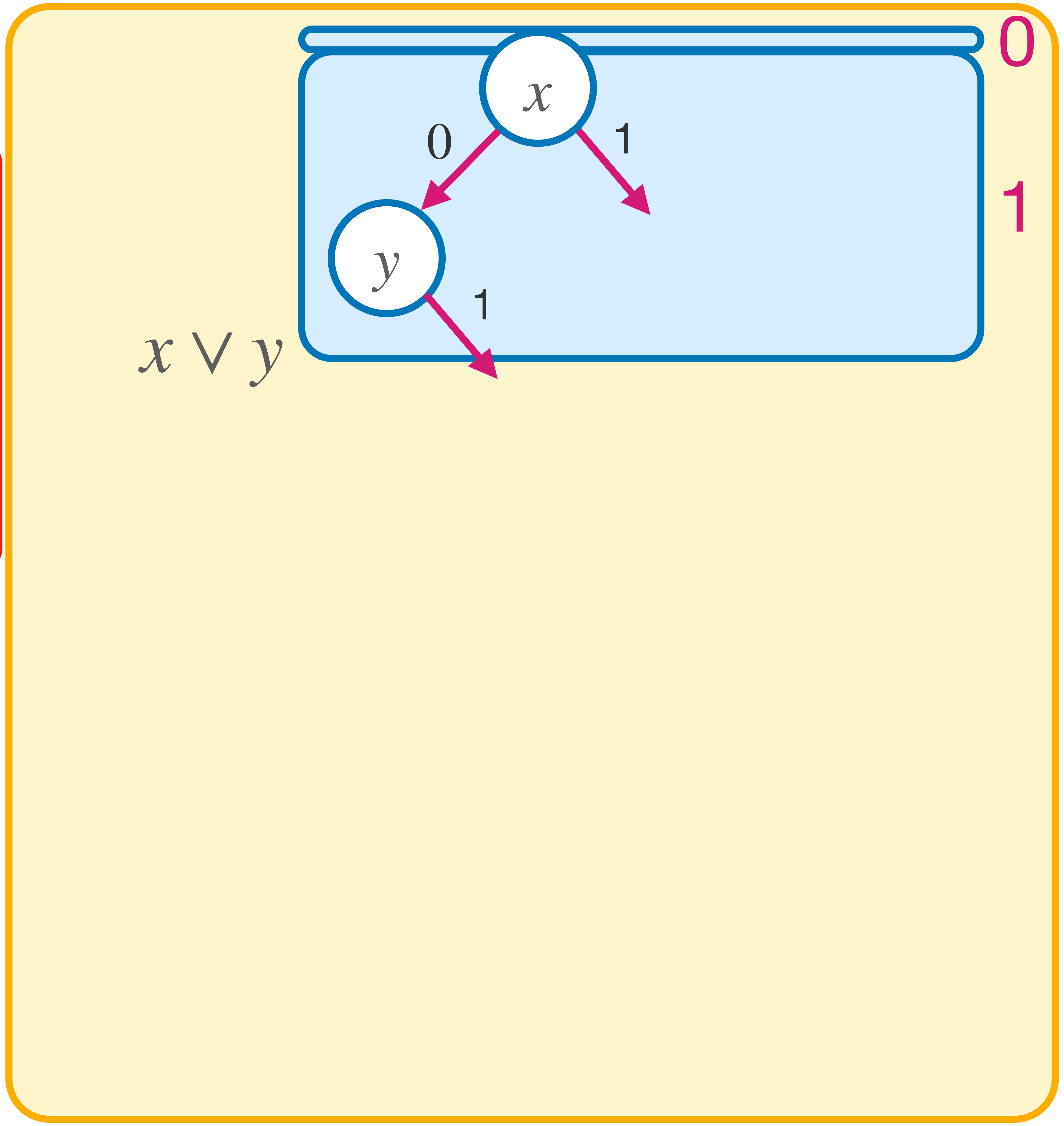

Analyzing CDCL

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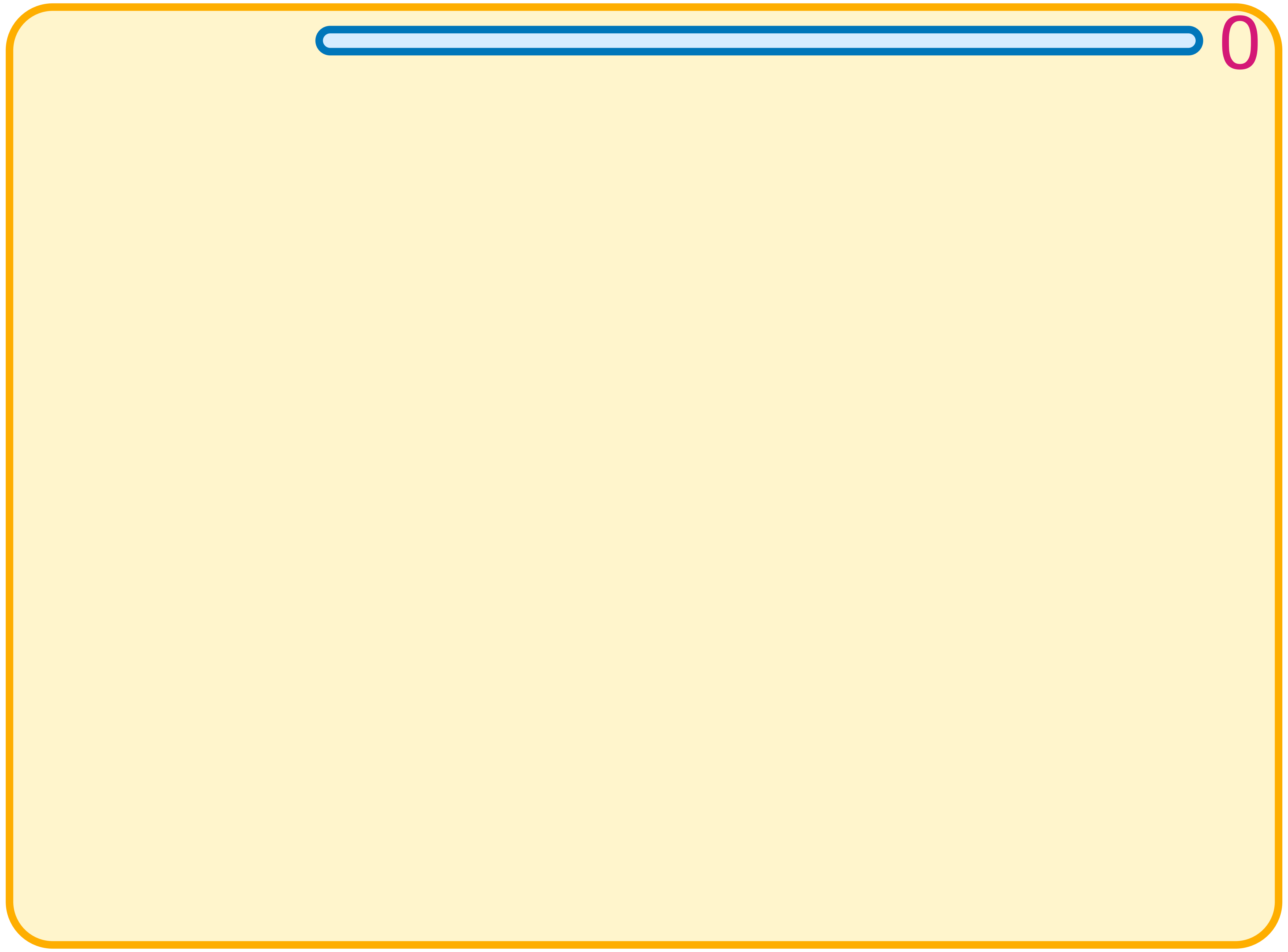

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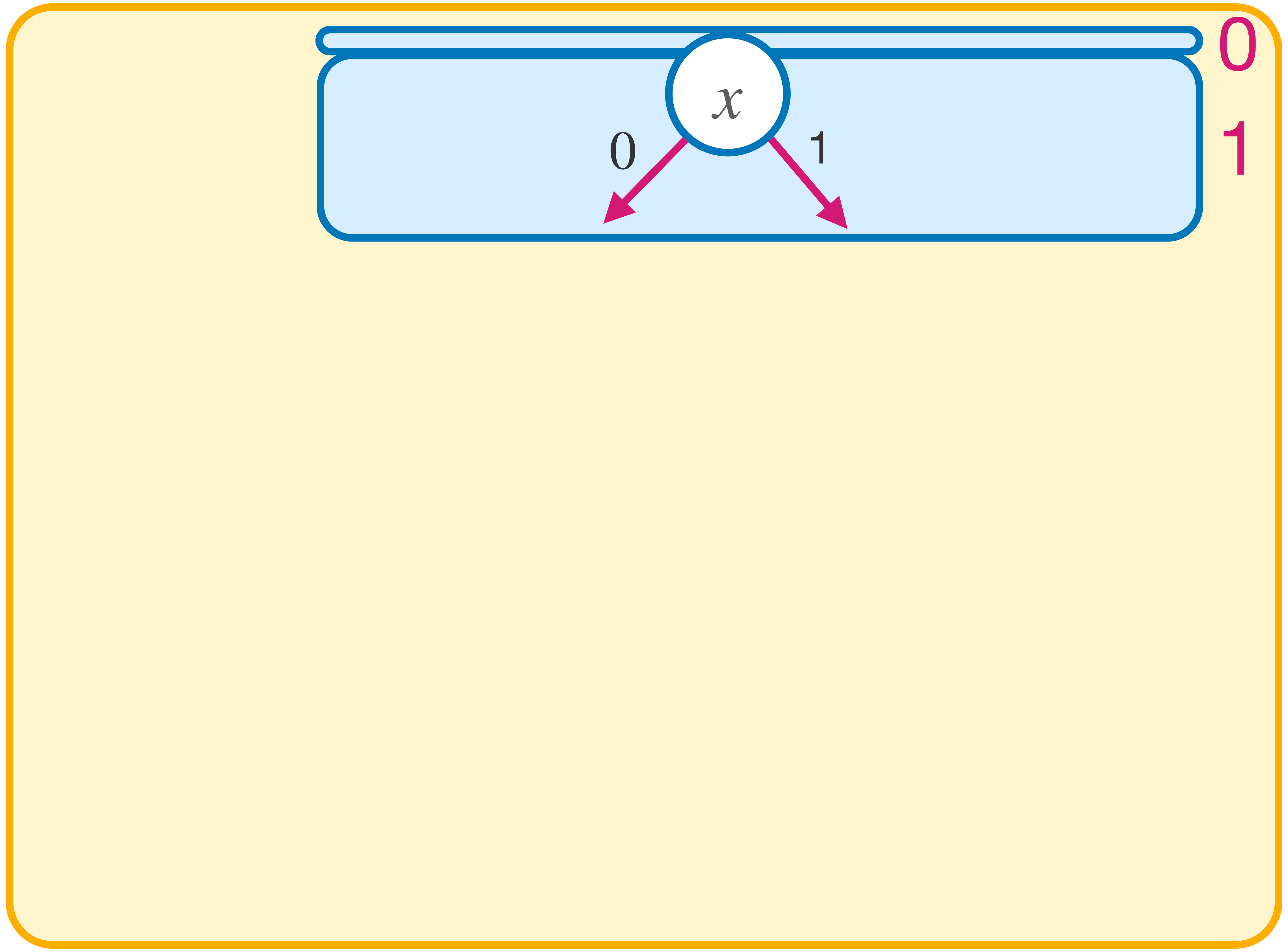

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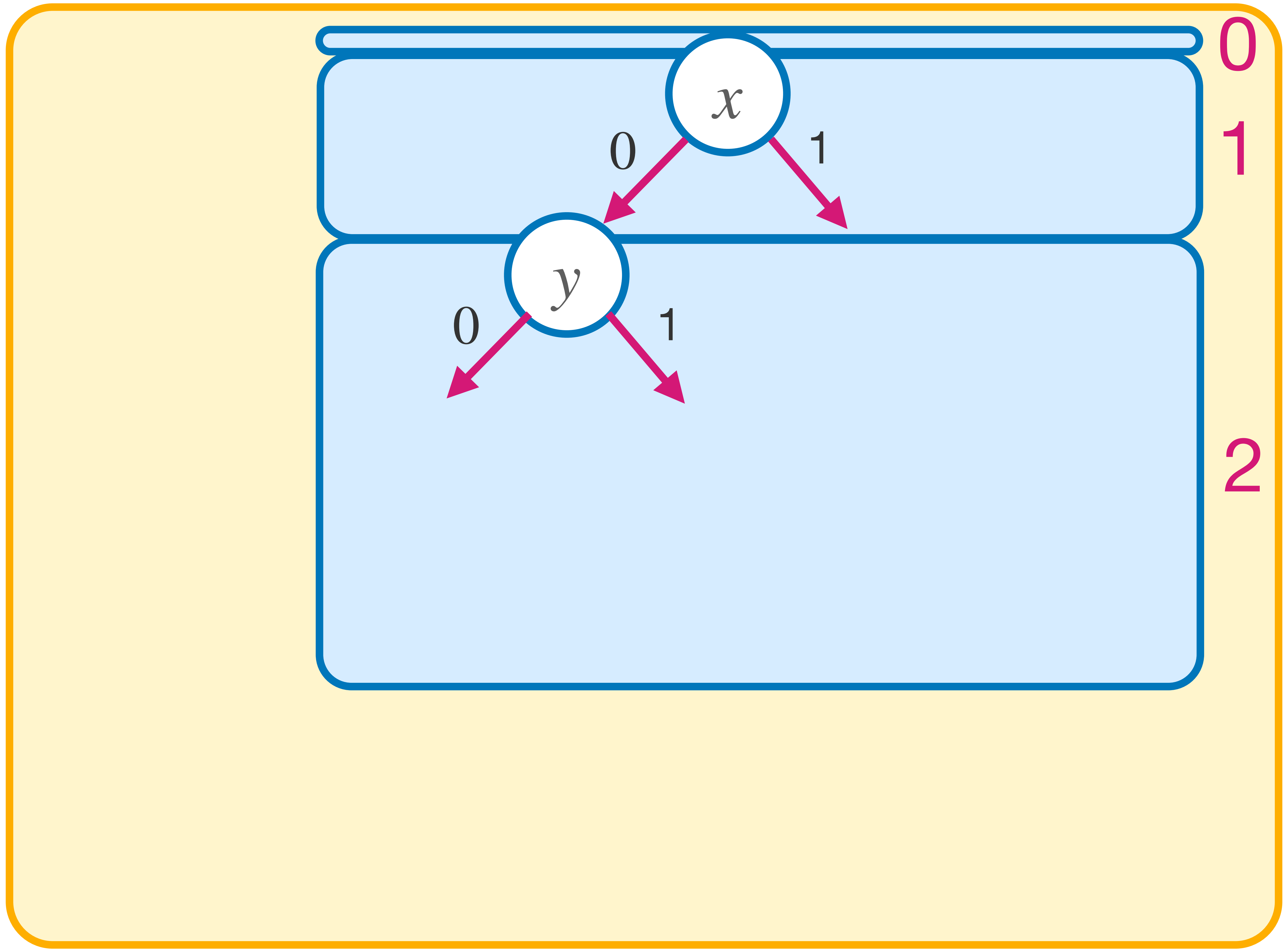

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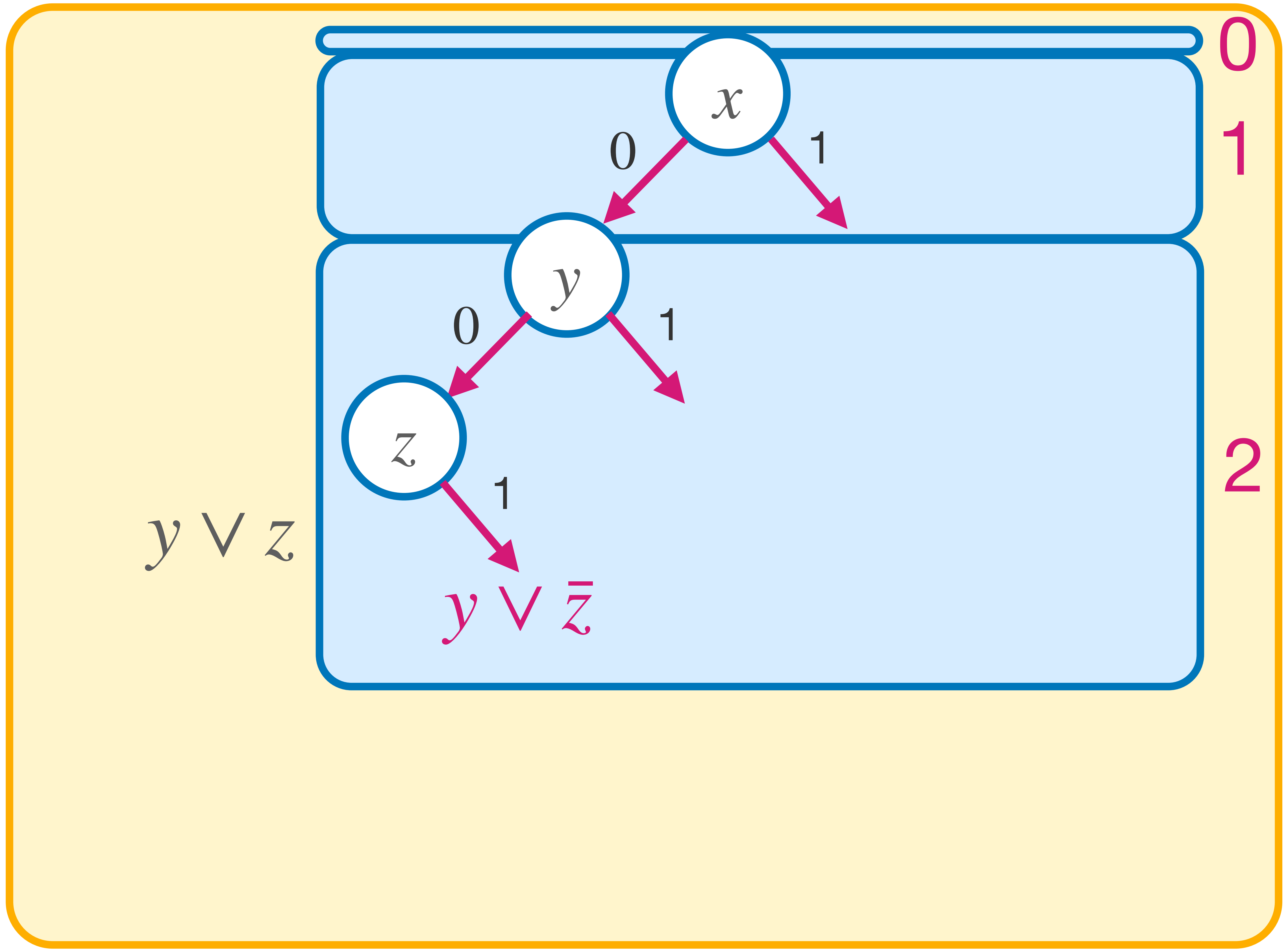

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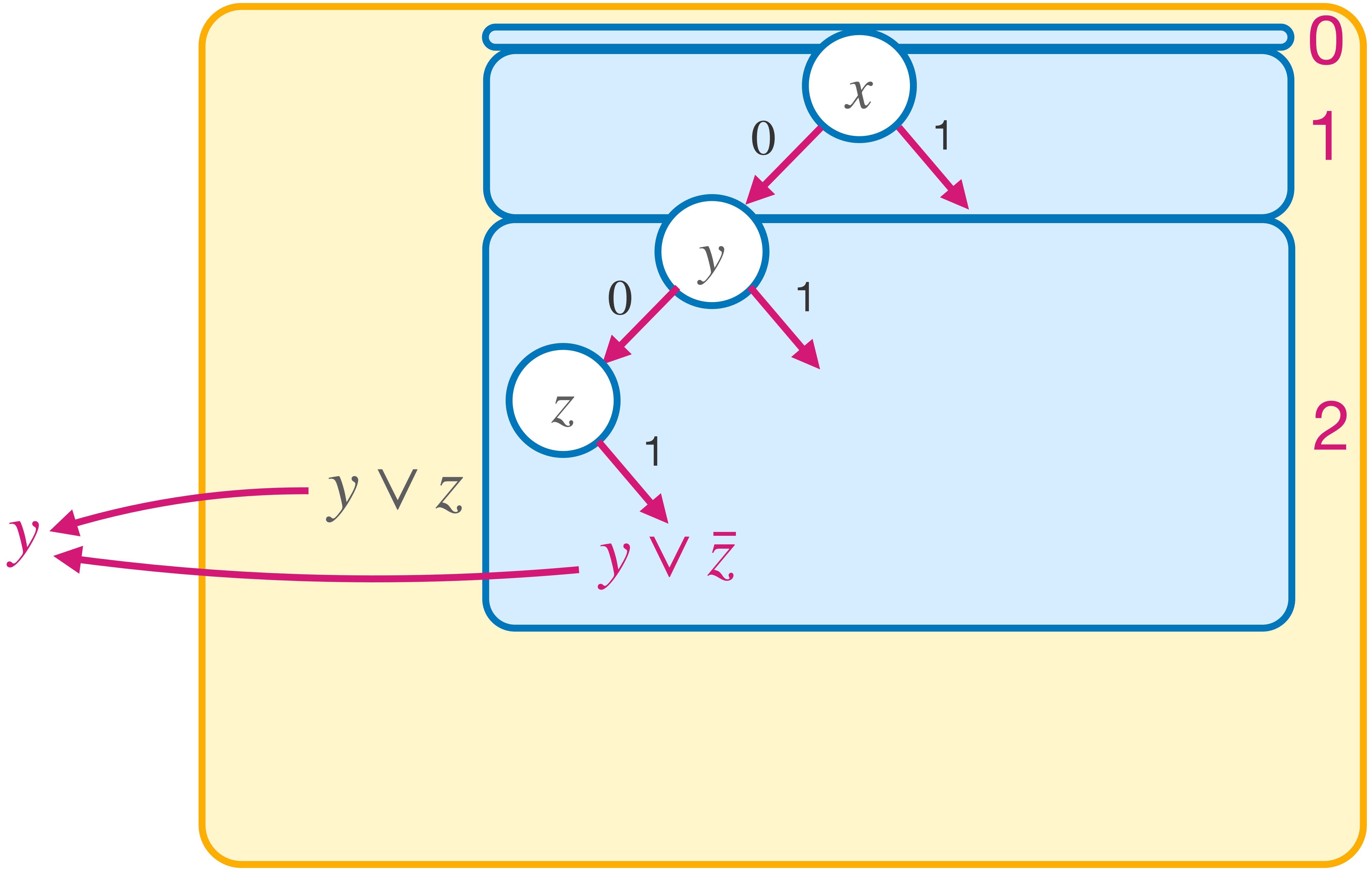

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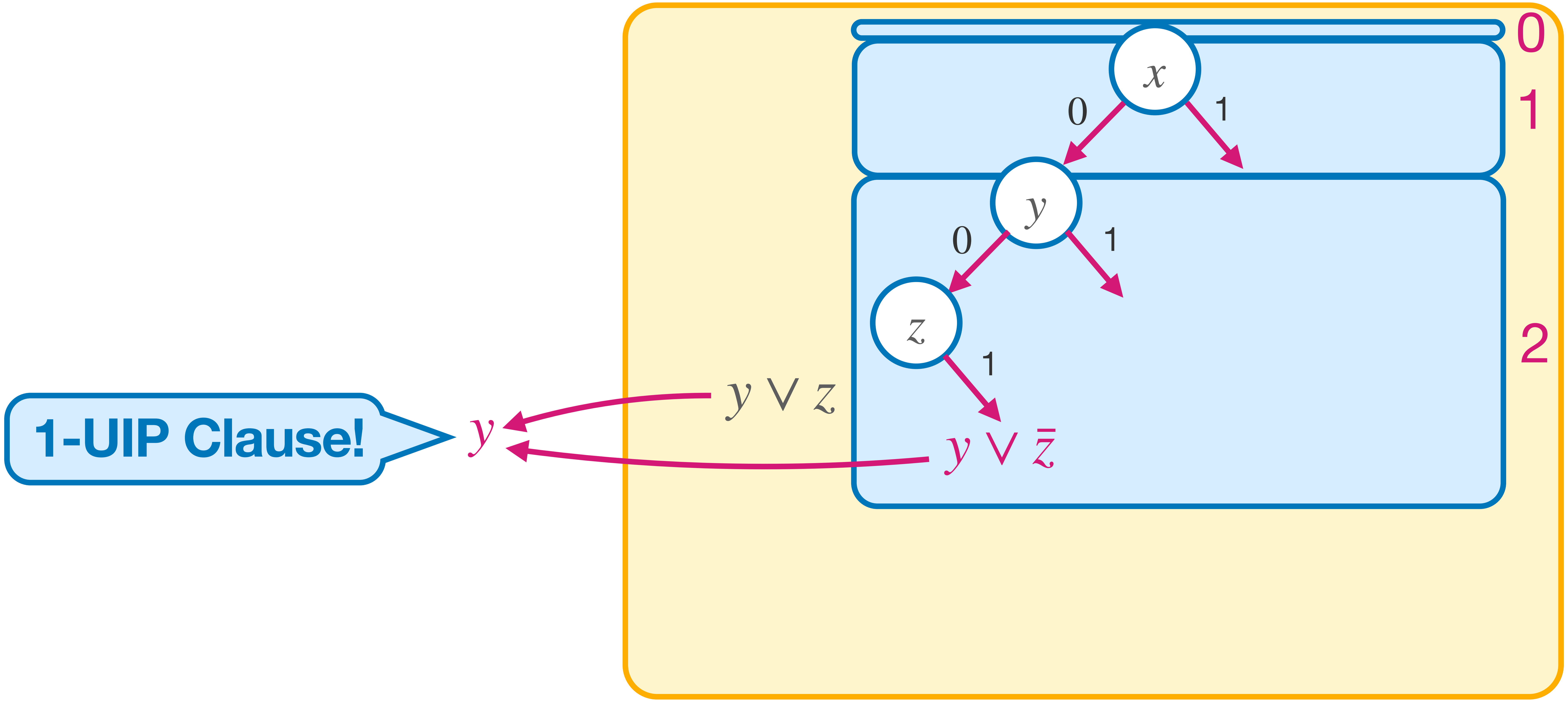

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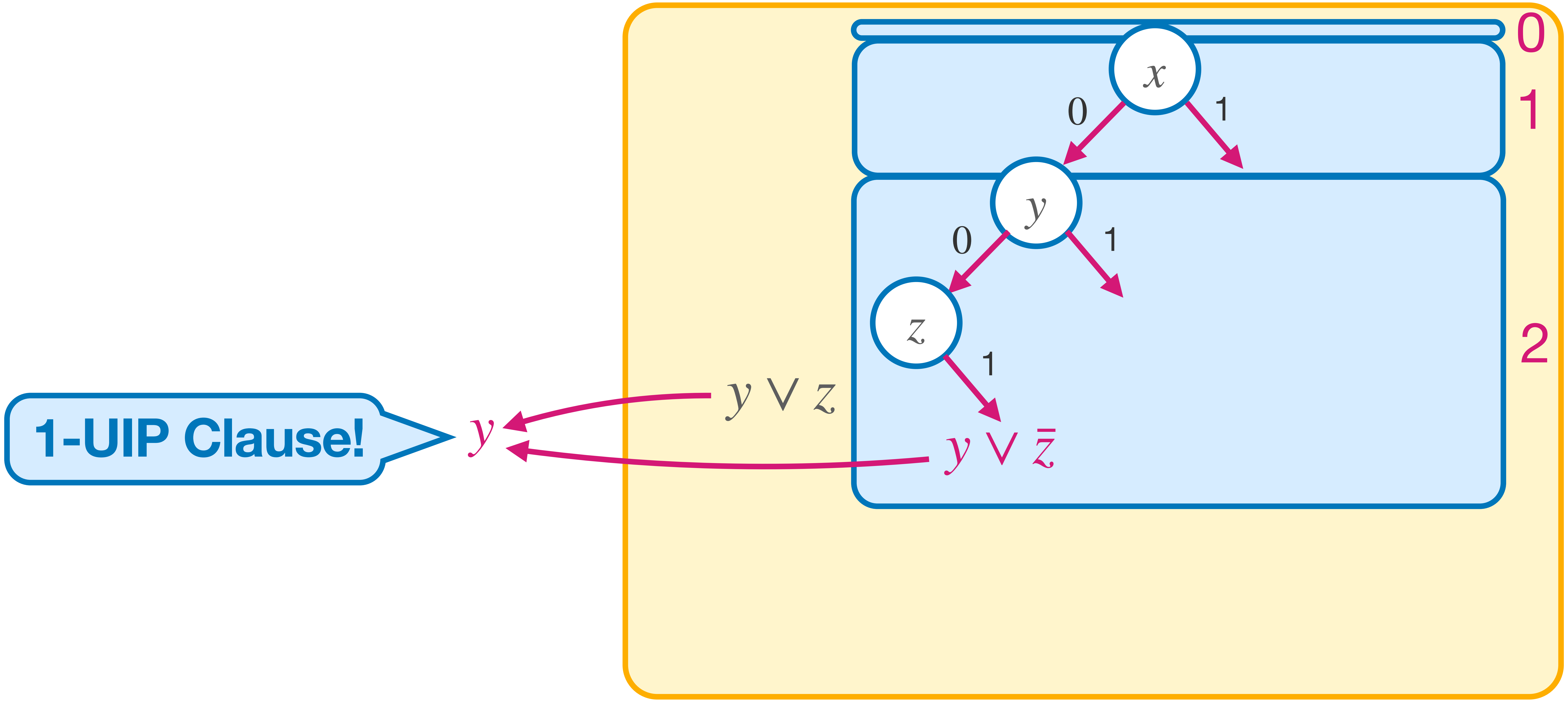

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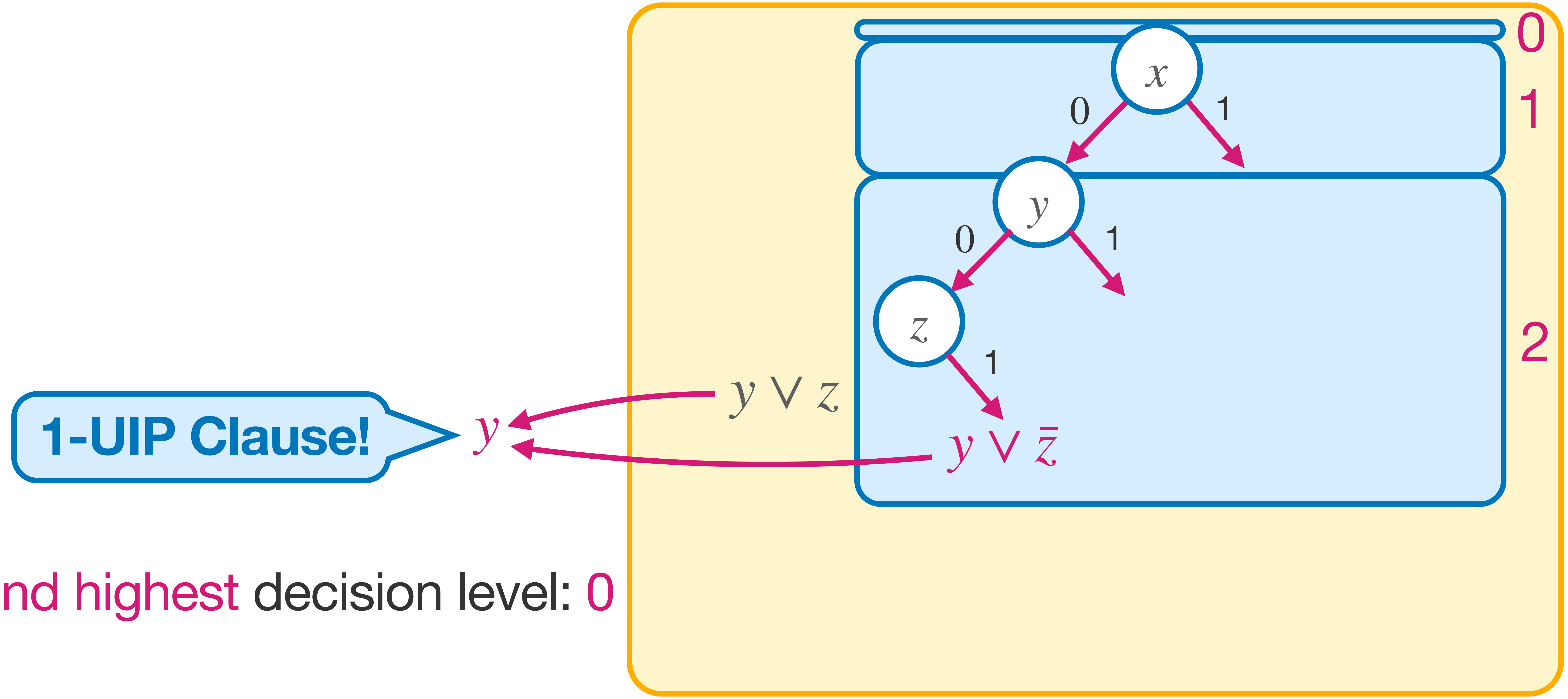

Backjump to second highest decision level: 0

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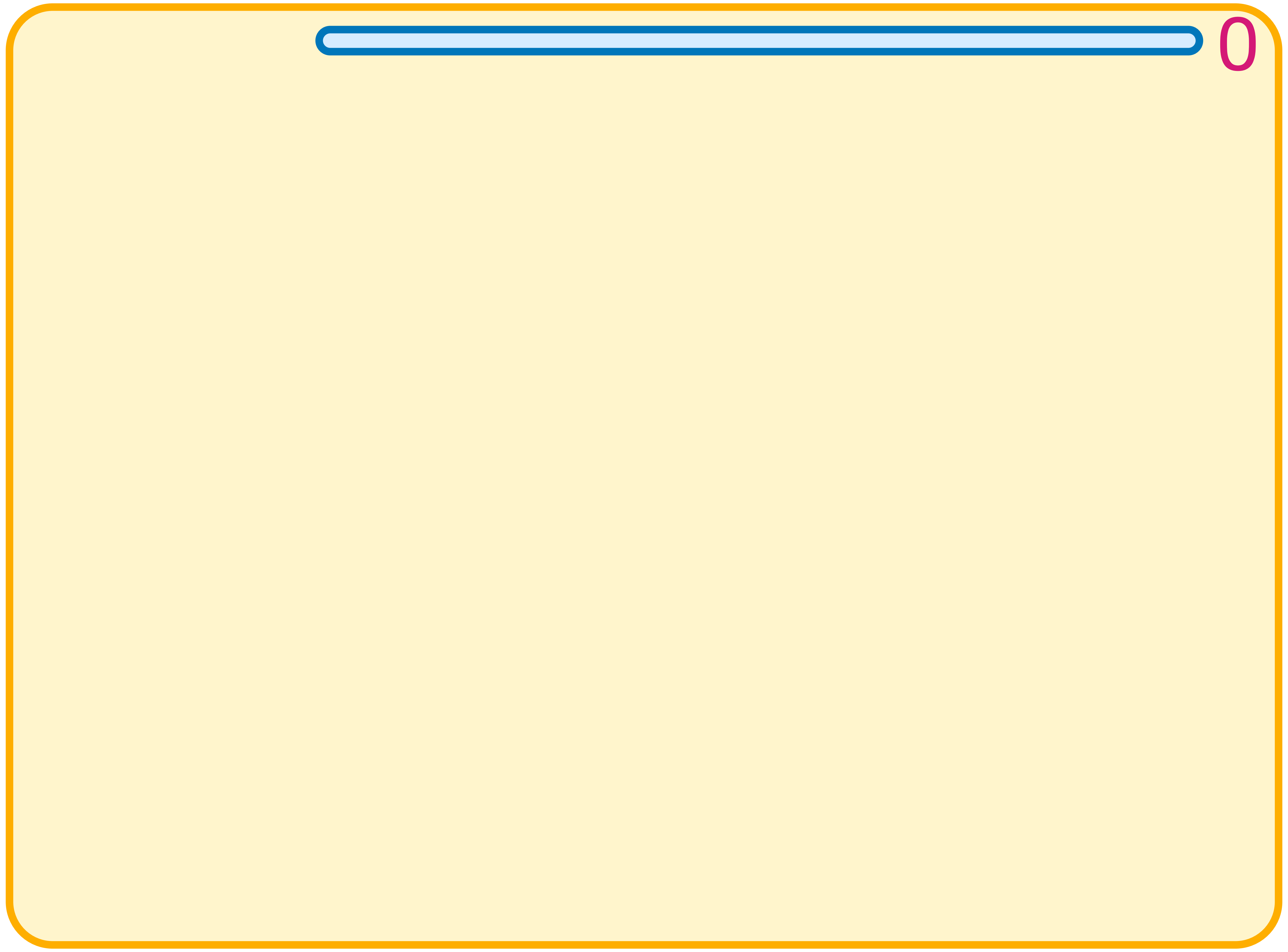

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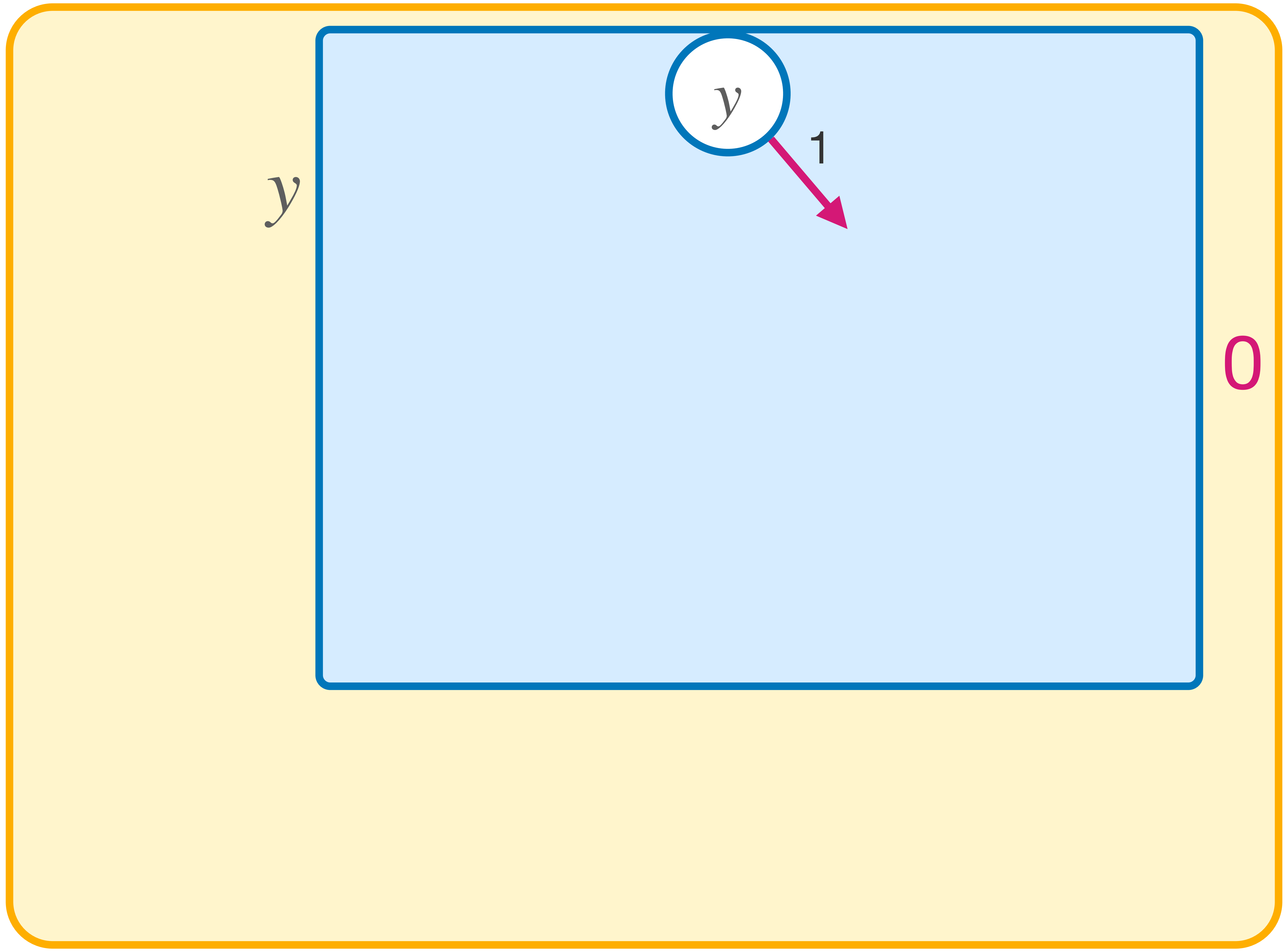

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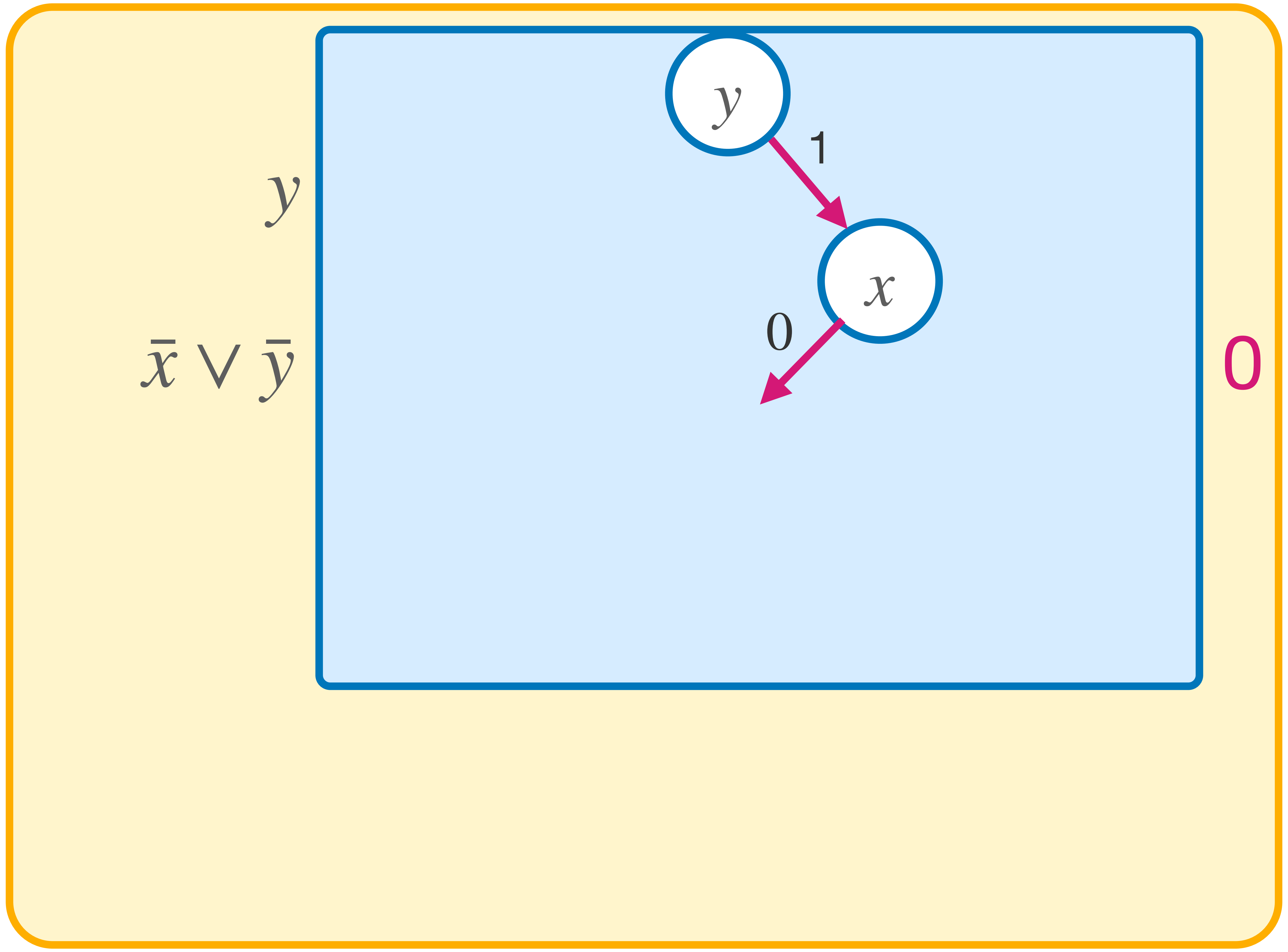

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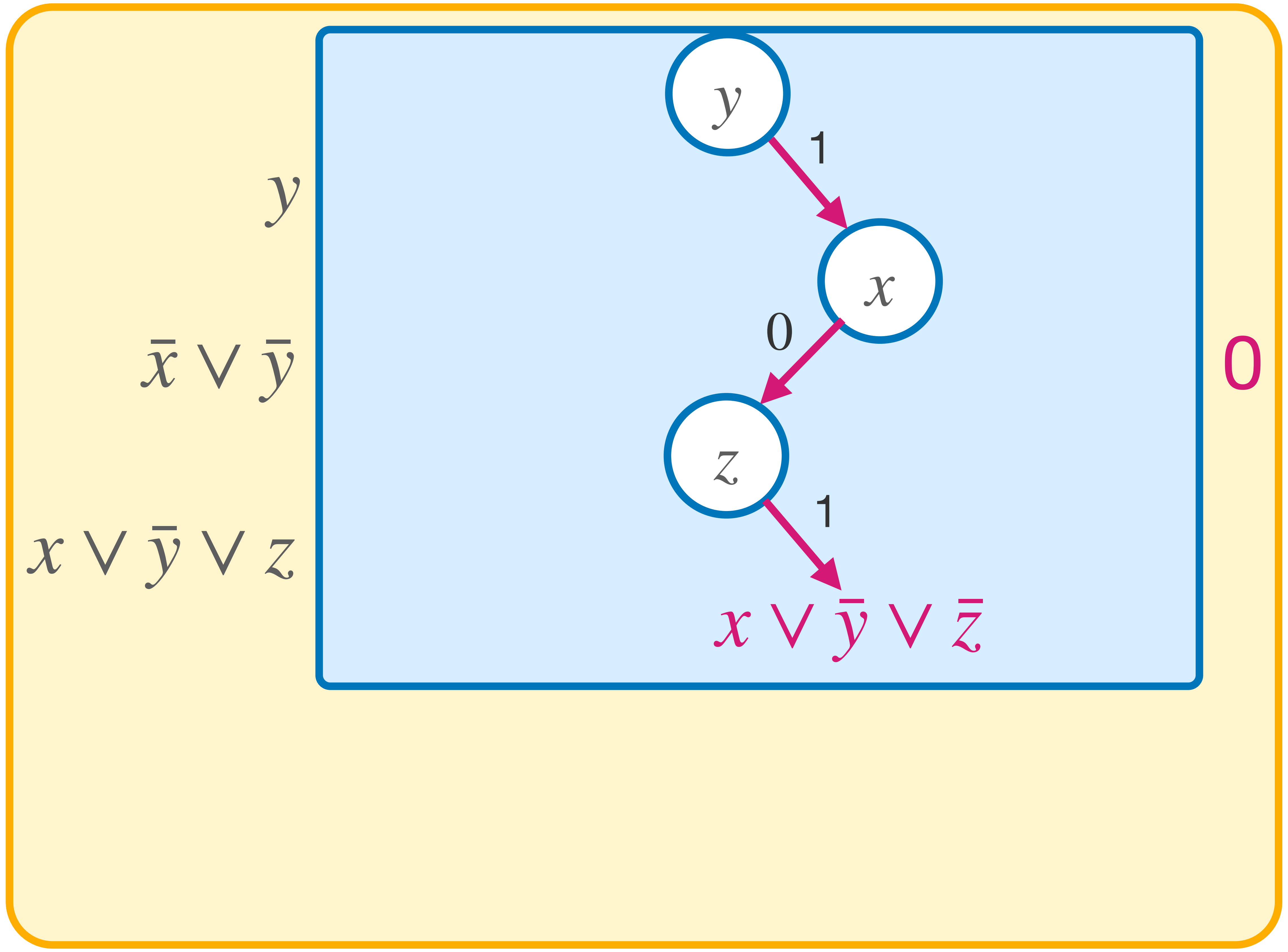

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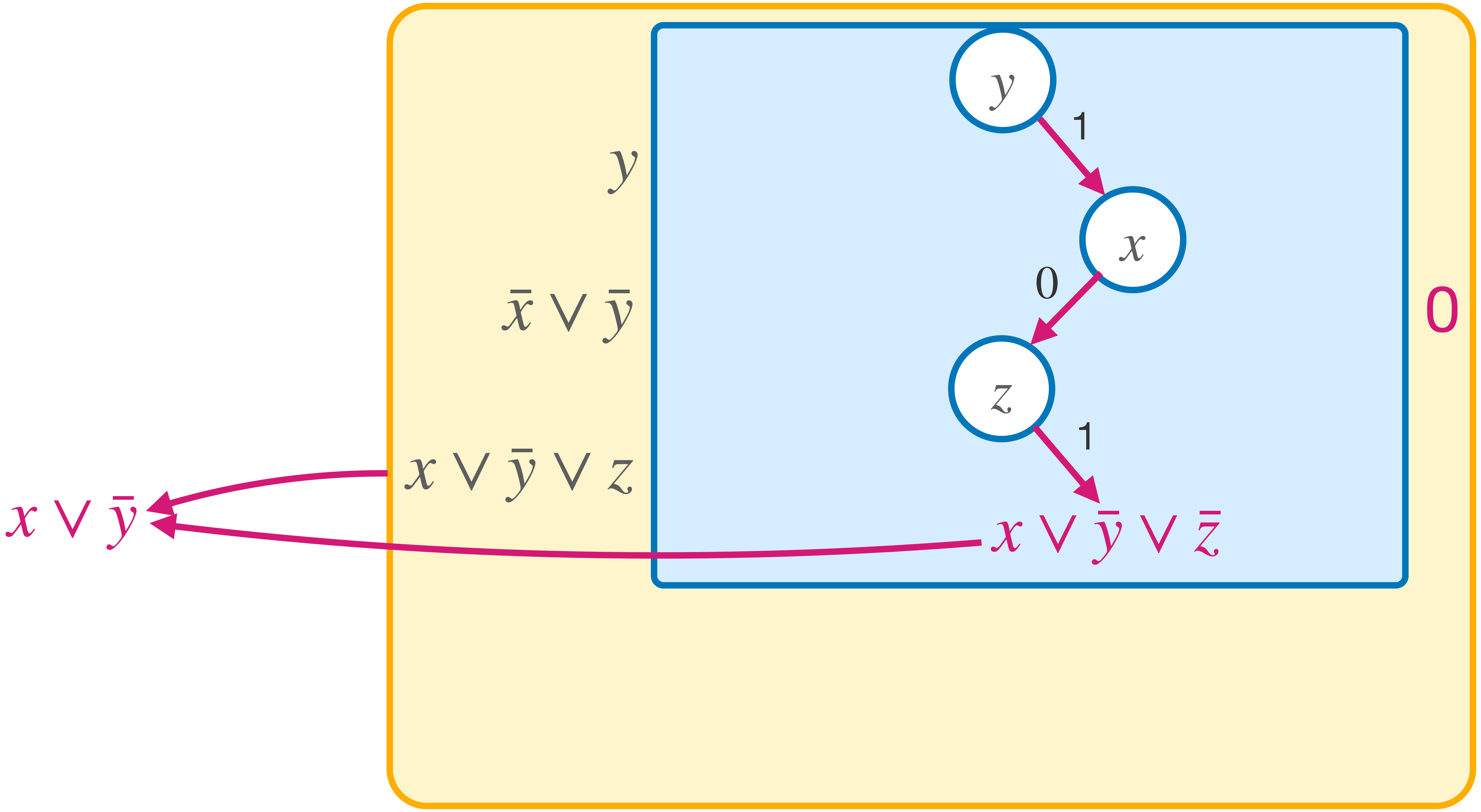

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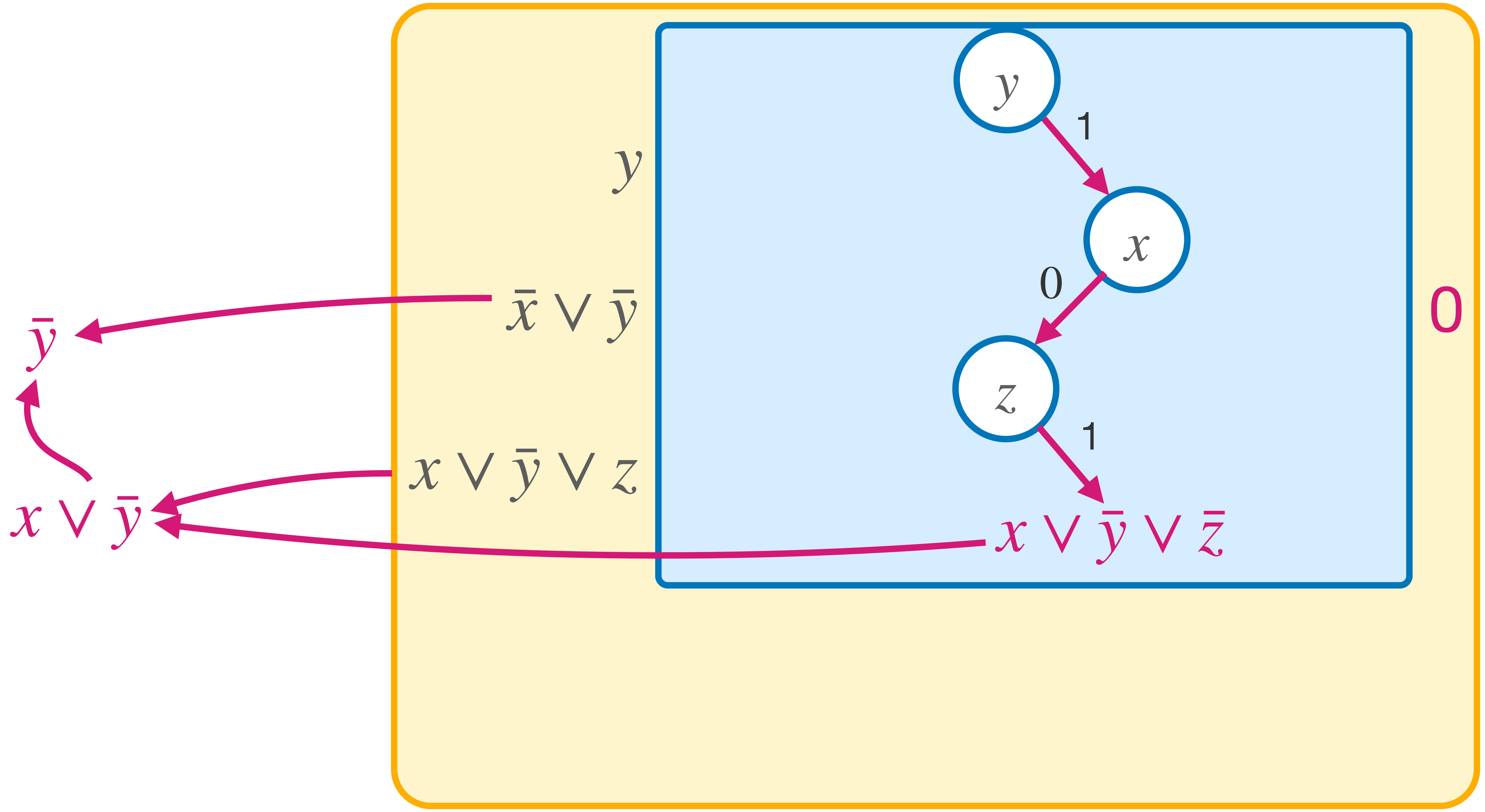

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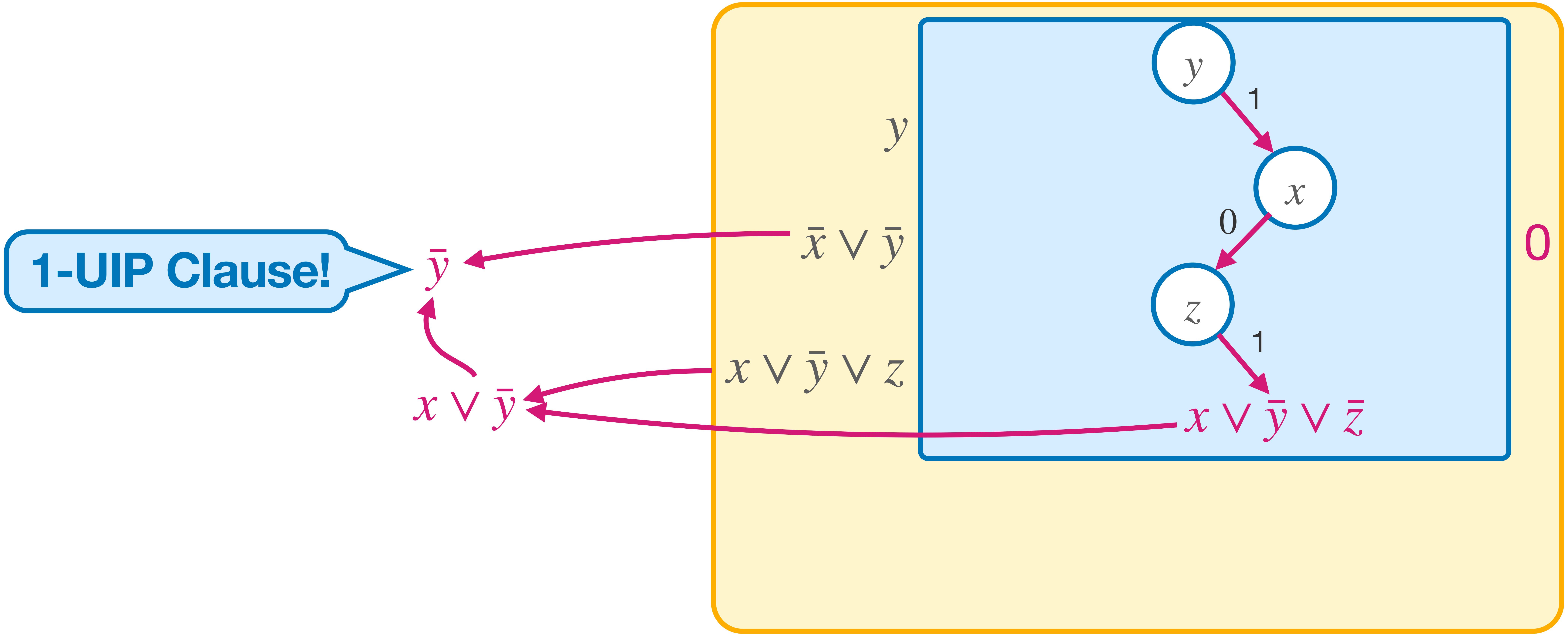

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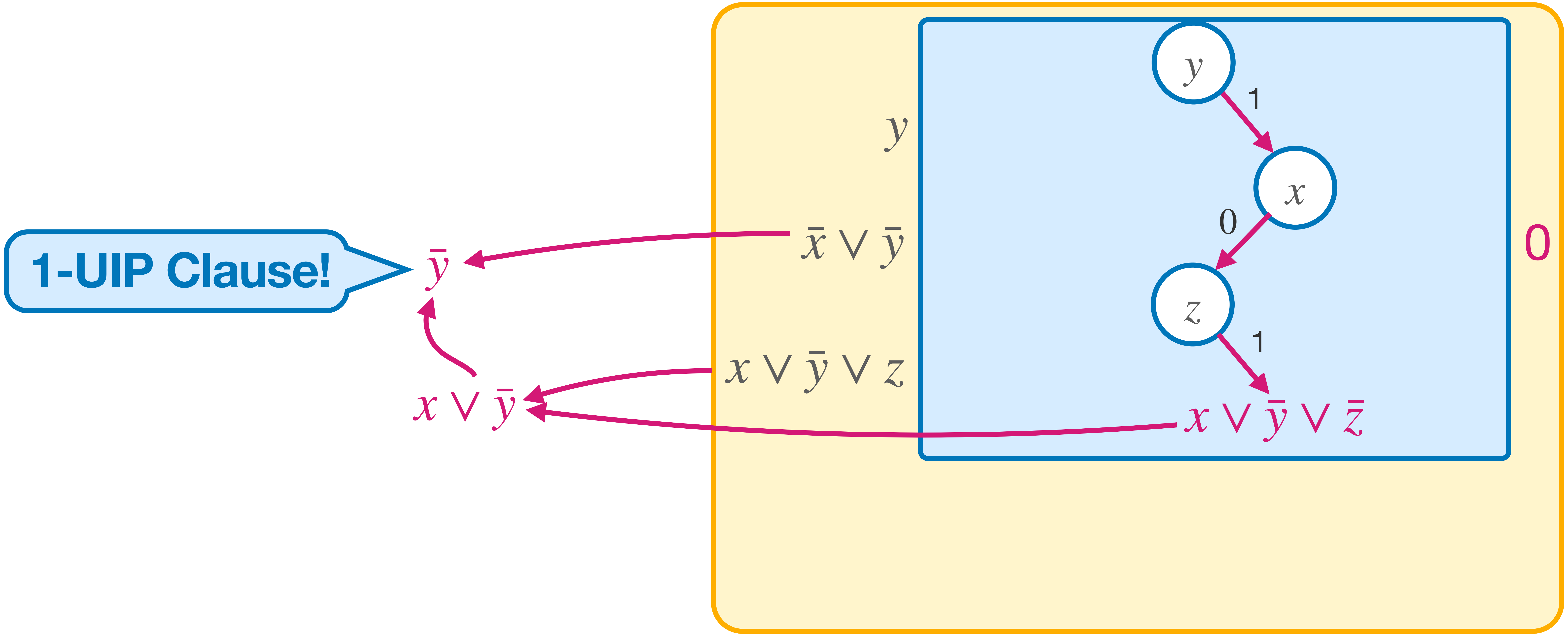

Analyzing CDCL

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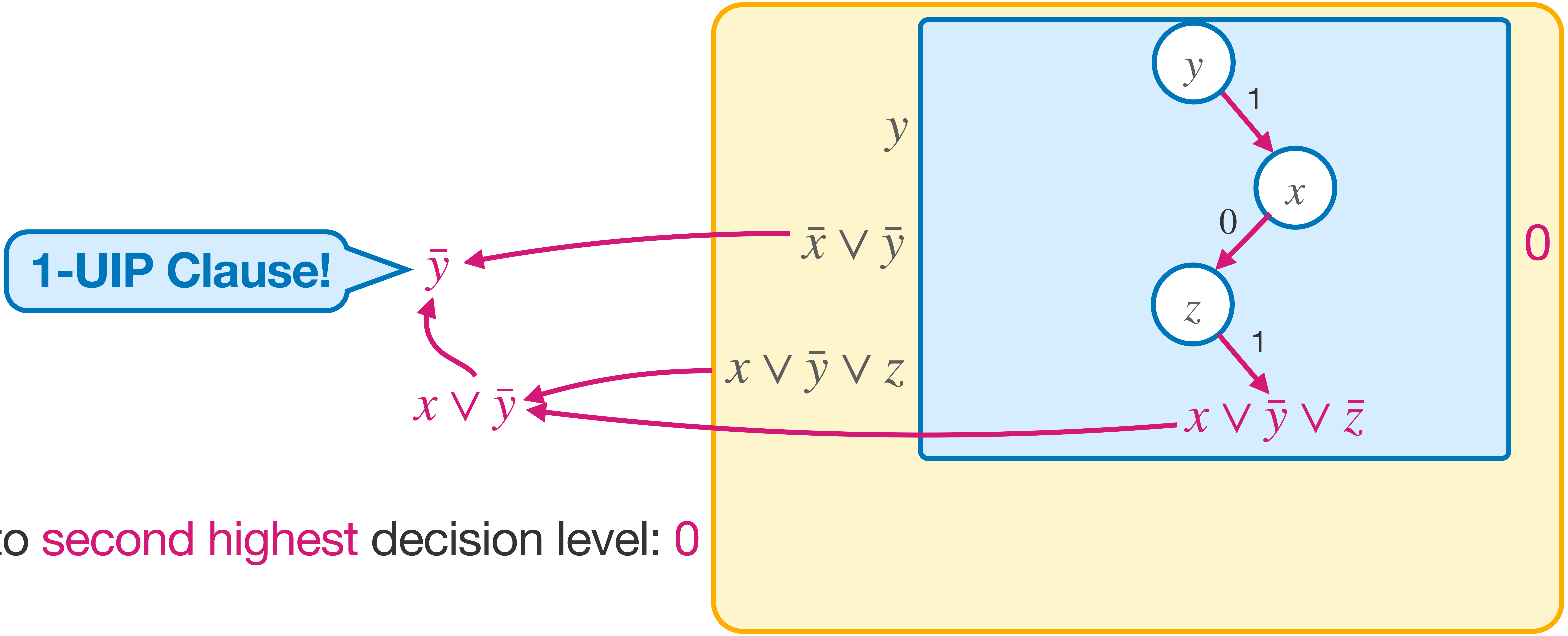

Backjump to second highest decision level: 0

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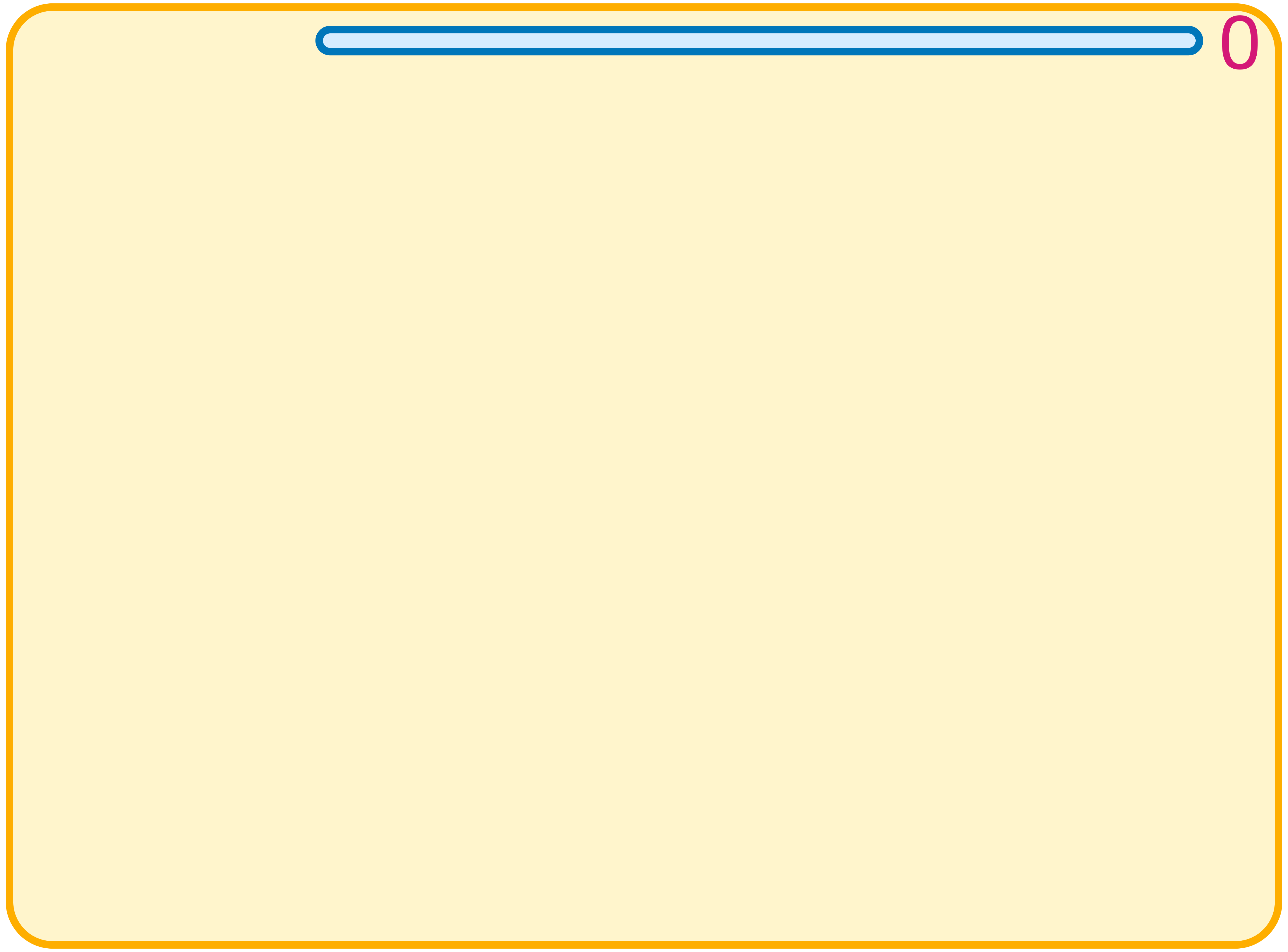

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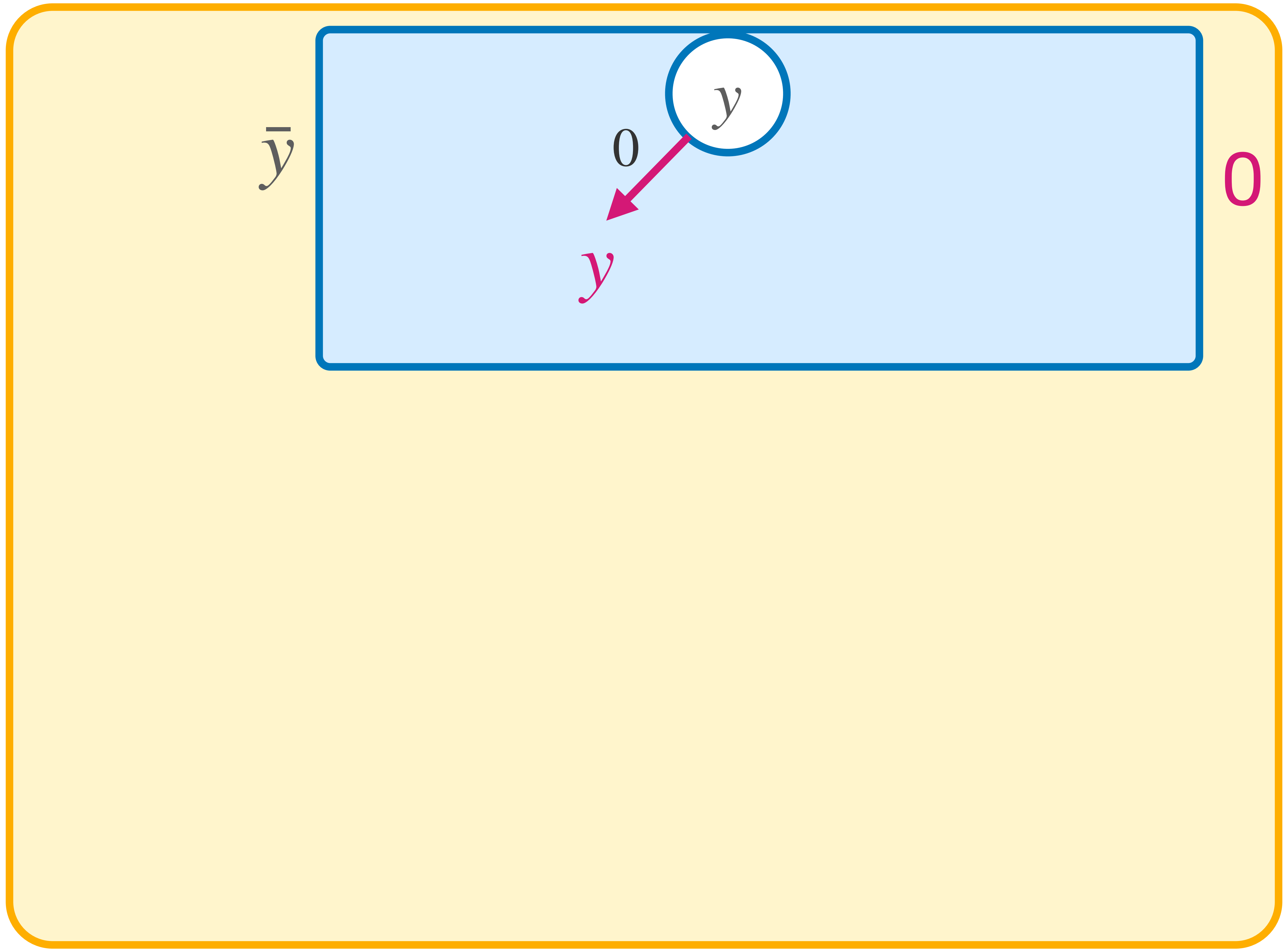

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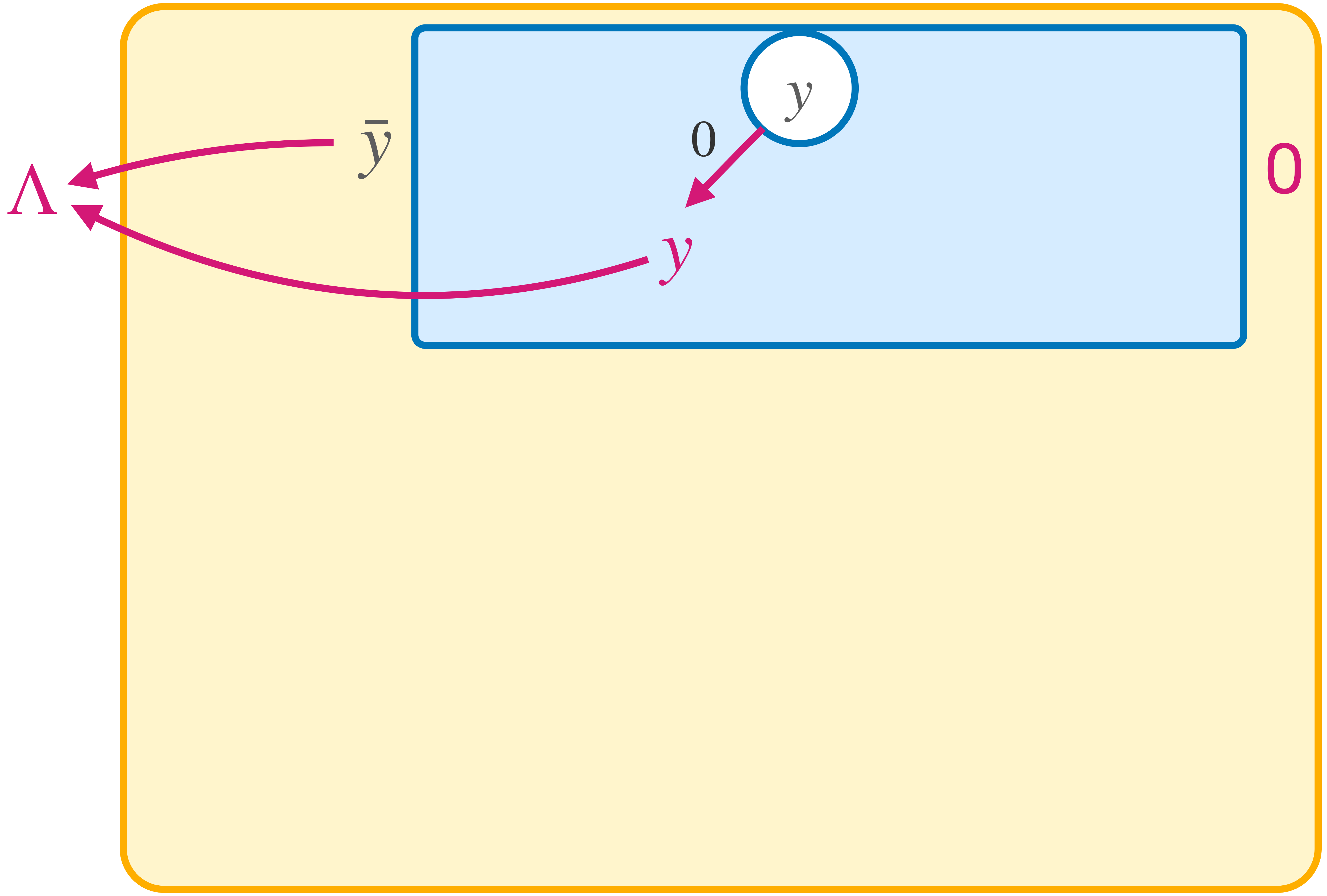

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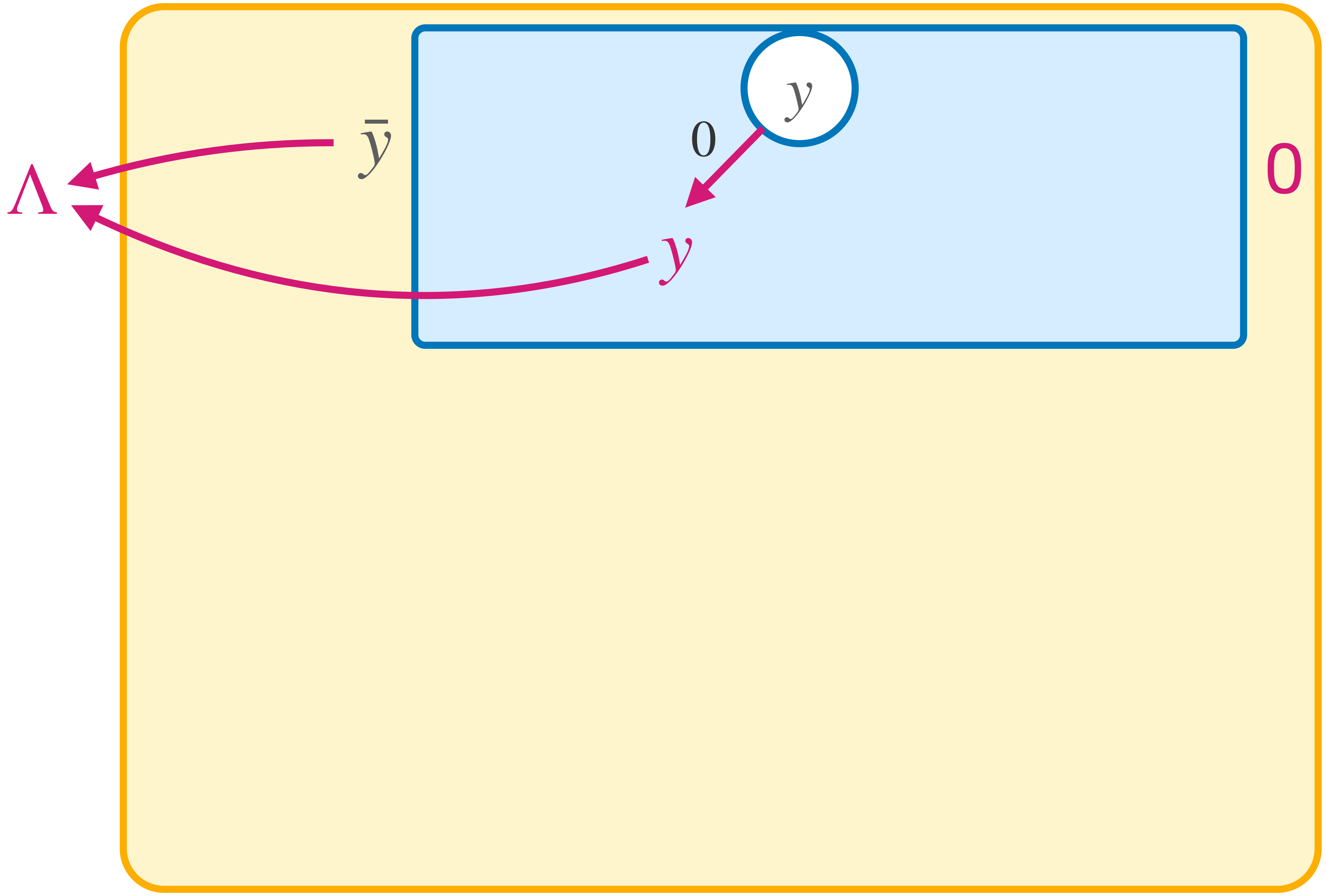
Learned empty clause Λ.

Halt: Unsatisfiable!



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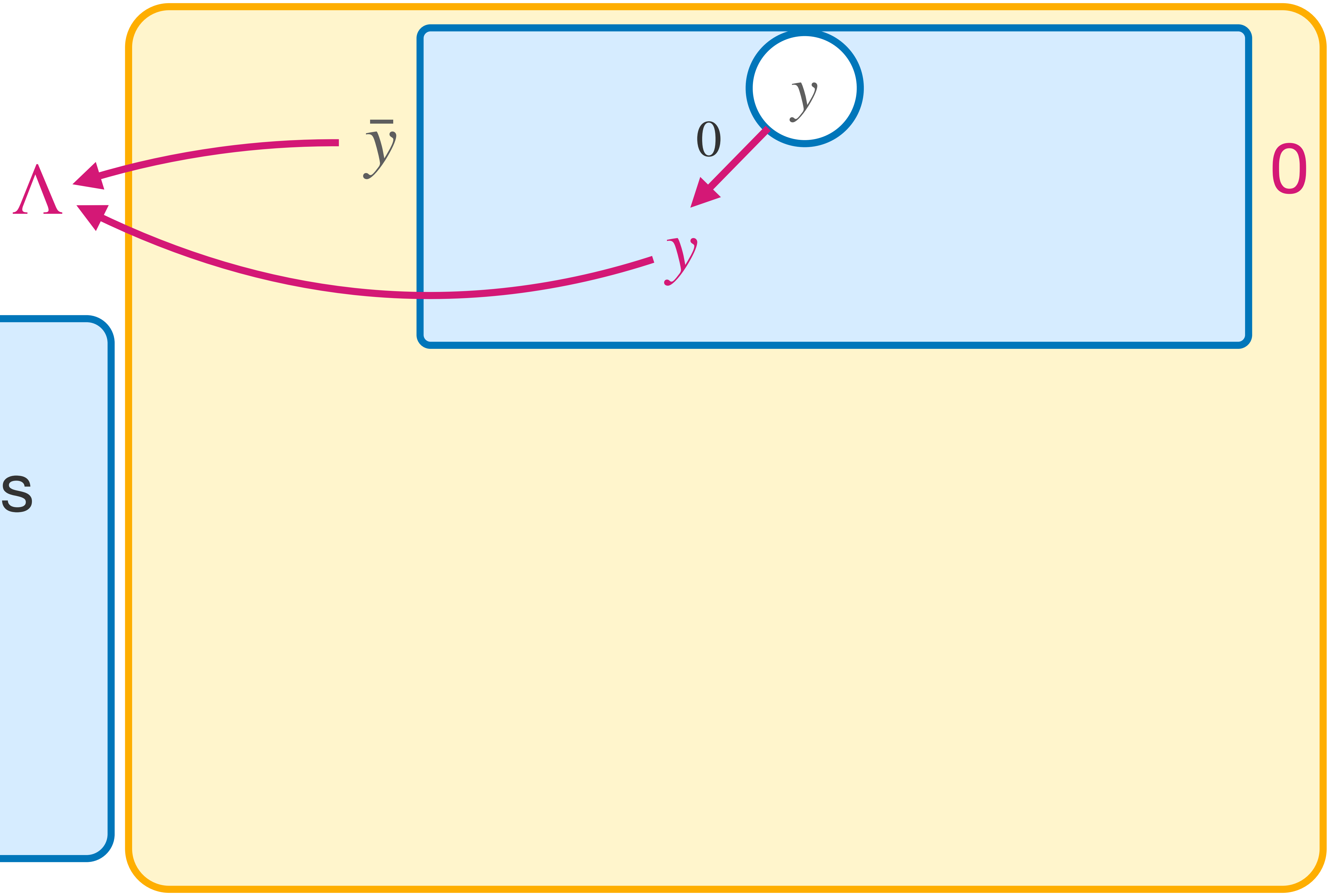

Takeaway:

CDCL run on an unsatisfiable formula halts when Λ is derived from clause learning

Analyzing CDCL

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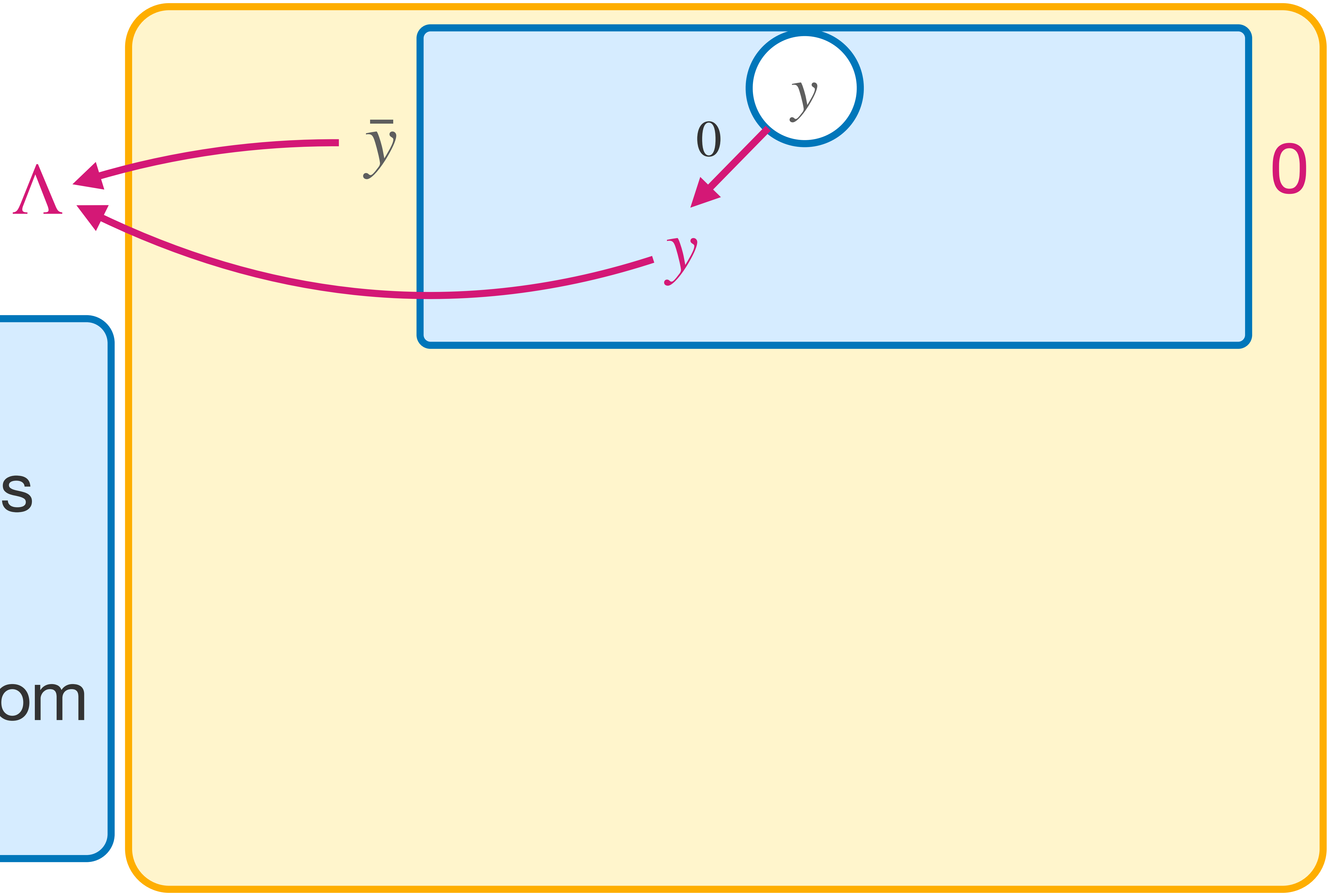

Takeaway:

CDCL run on an unsatisfiable formula halts when Λ is derived from clause learning

$\rightarrow$ Clause learning derives new clauses from old ones using Resolution

Analyzing CDCL

Theorem: Let F be an unsatisfiable CNF formula. If CDCL takes time s to solve F, then there is a size-s Resolution proof of F

Proof: Every time CDCL learns a clause, derive that clause in Resolution

Analyzing CDCL

Theorem: Let F be an unsatisfiable CNF formula. If CDCL takes time s to solve F, then there is a size-s Resolution proof of F

Proof: Every time CDCL learns a clause, derive that clause in Resolution

$\rightarrow$ Because CDCL halts when Λ is derived, we have a Resolution proof!

Analyzing CDCL

$$(y \vee z) \wedge (y \vee \bar{z}) \wedge (x \vee \bar{y} \vee z) \wedge (x \vee \bar{y} \vee \bar{z}) \wedge (\bar{x} \vee \bar{y})$$

Resolution

$(y \vee z), (y \vee \bar{z}), (x \vee \bar{y} \vee z), (x \vee \bar{y} \vee \bar{z}), (\bar{x} \vee \bar{y})$

0

Analyzing CDCL

$$(y \lor z) \land (y \lor \bar{z}) \land (x \lor \bar{y} \lor z) \land (x \lor \bar{y} \lor \bar{z}) \land (\bar{x} \lor \bar{y})$$

Resolution

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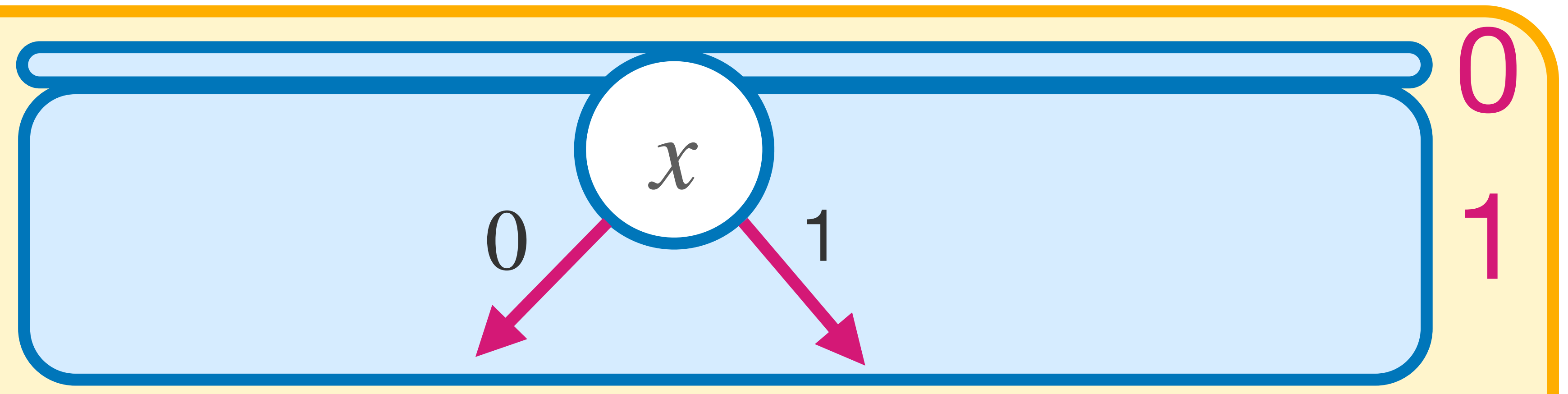

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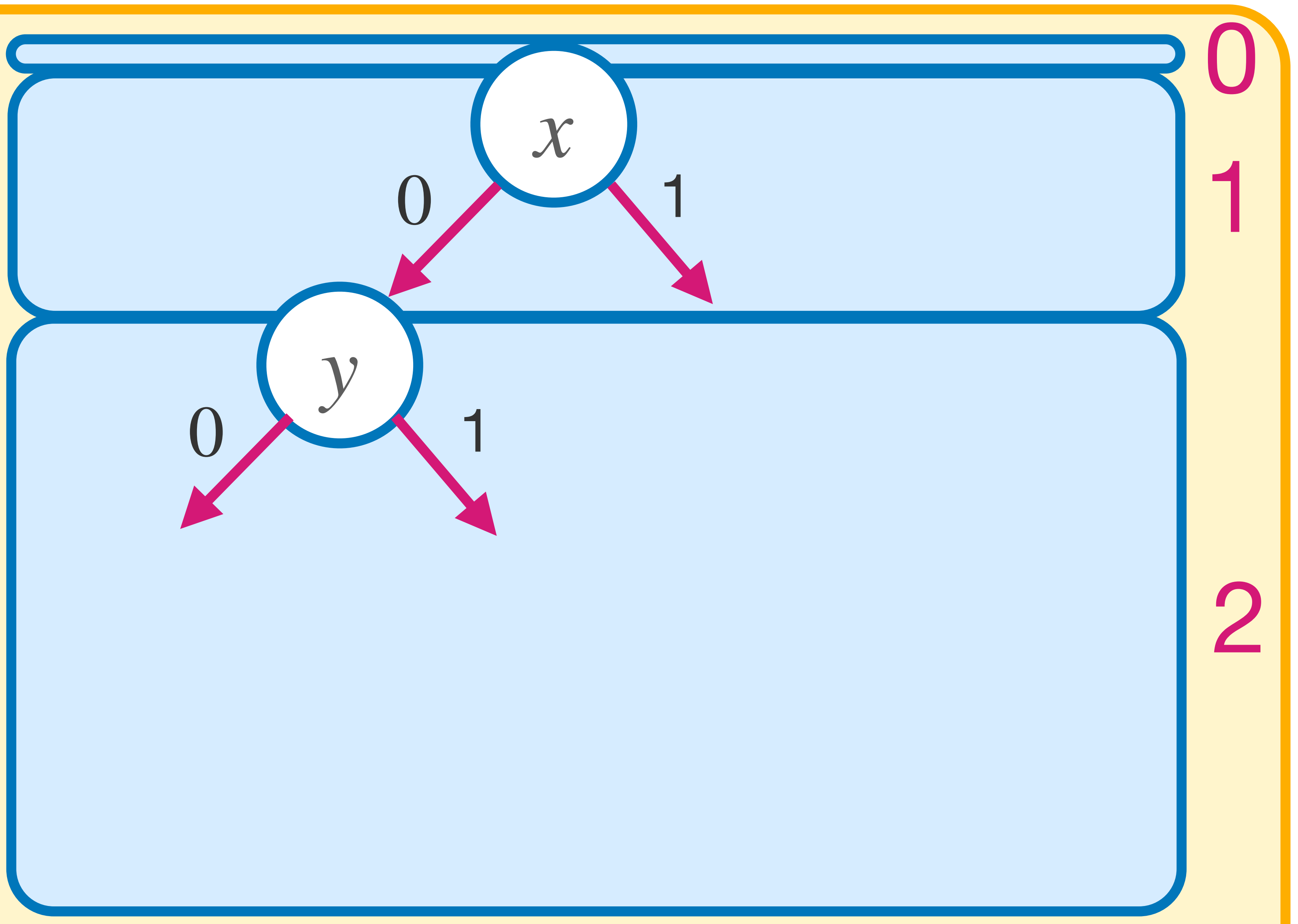

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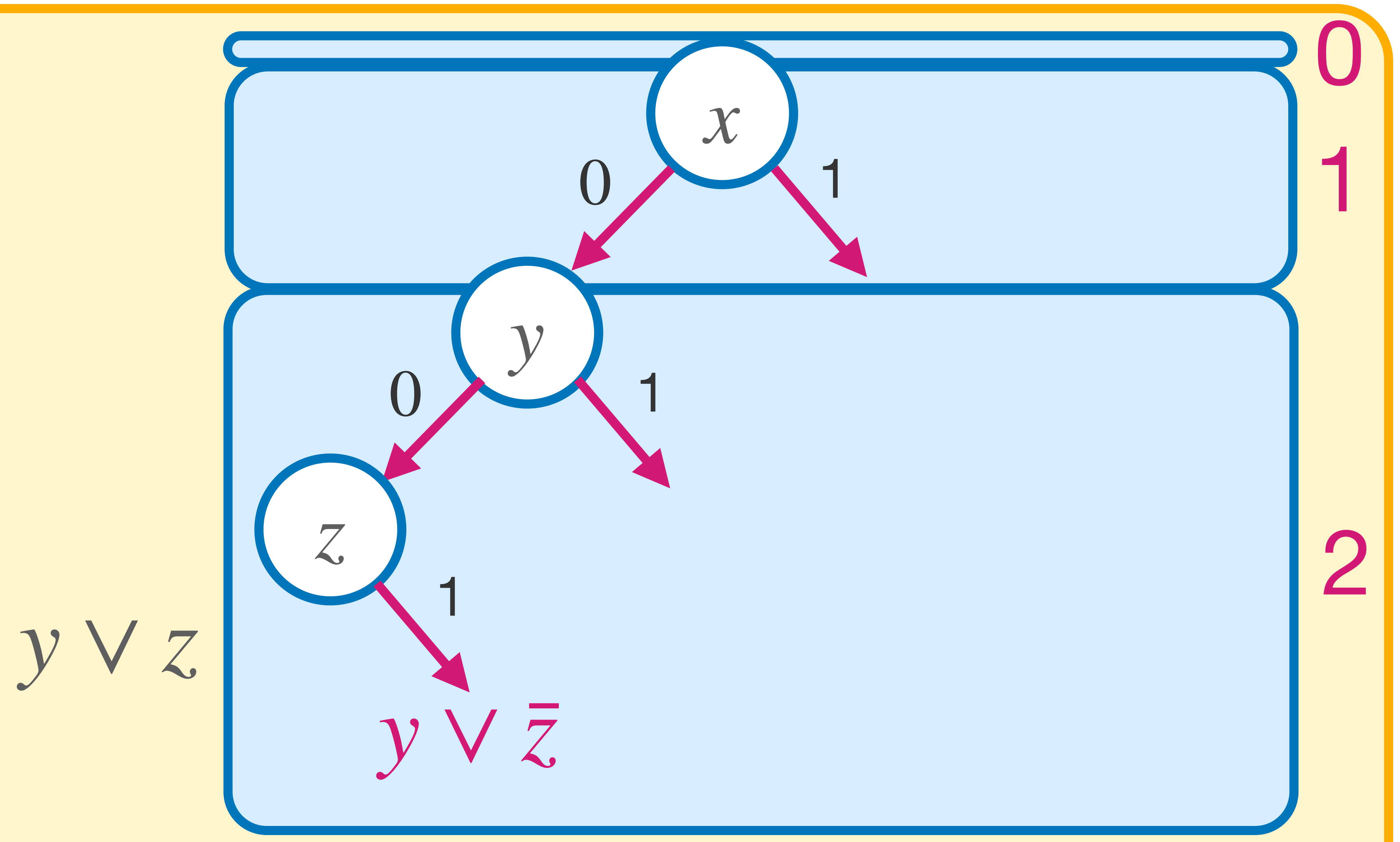

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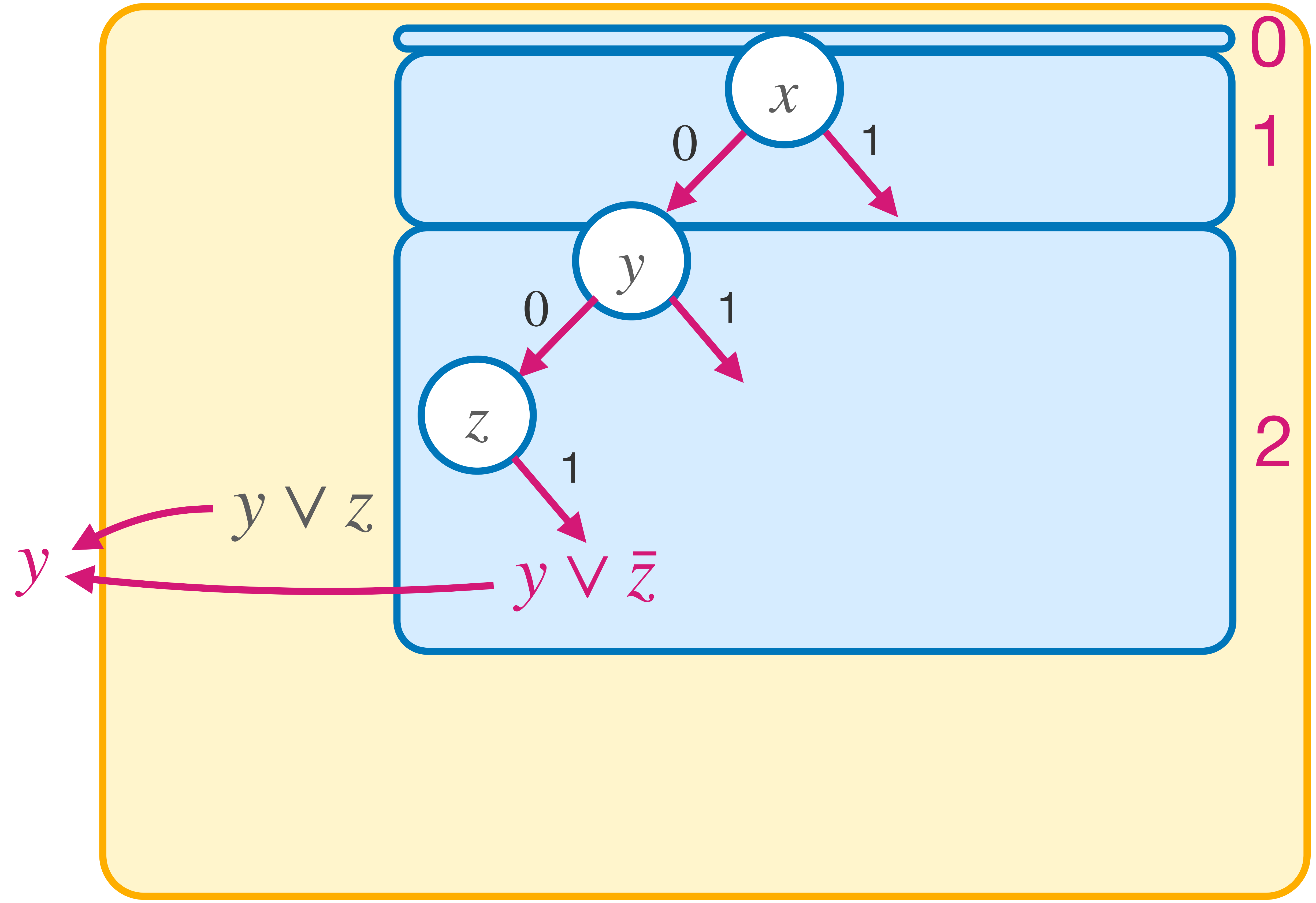

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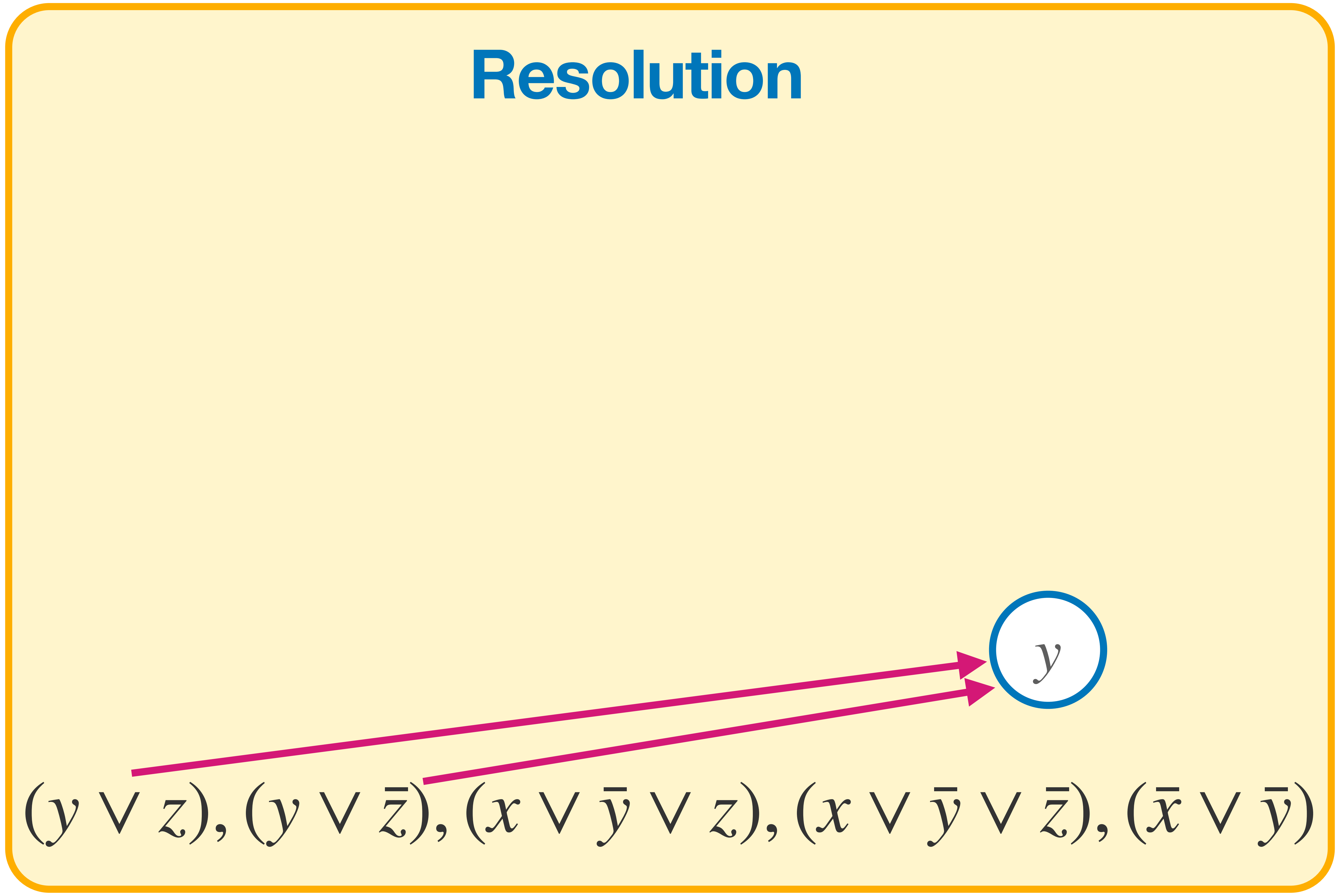
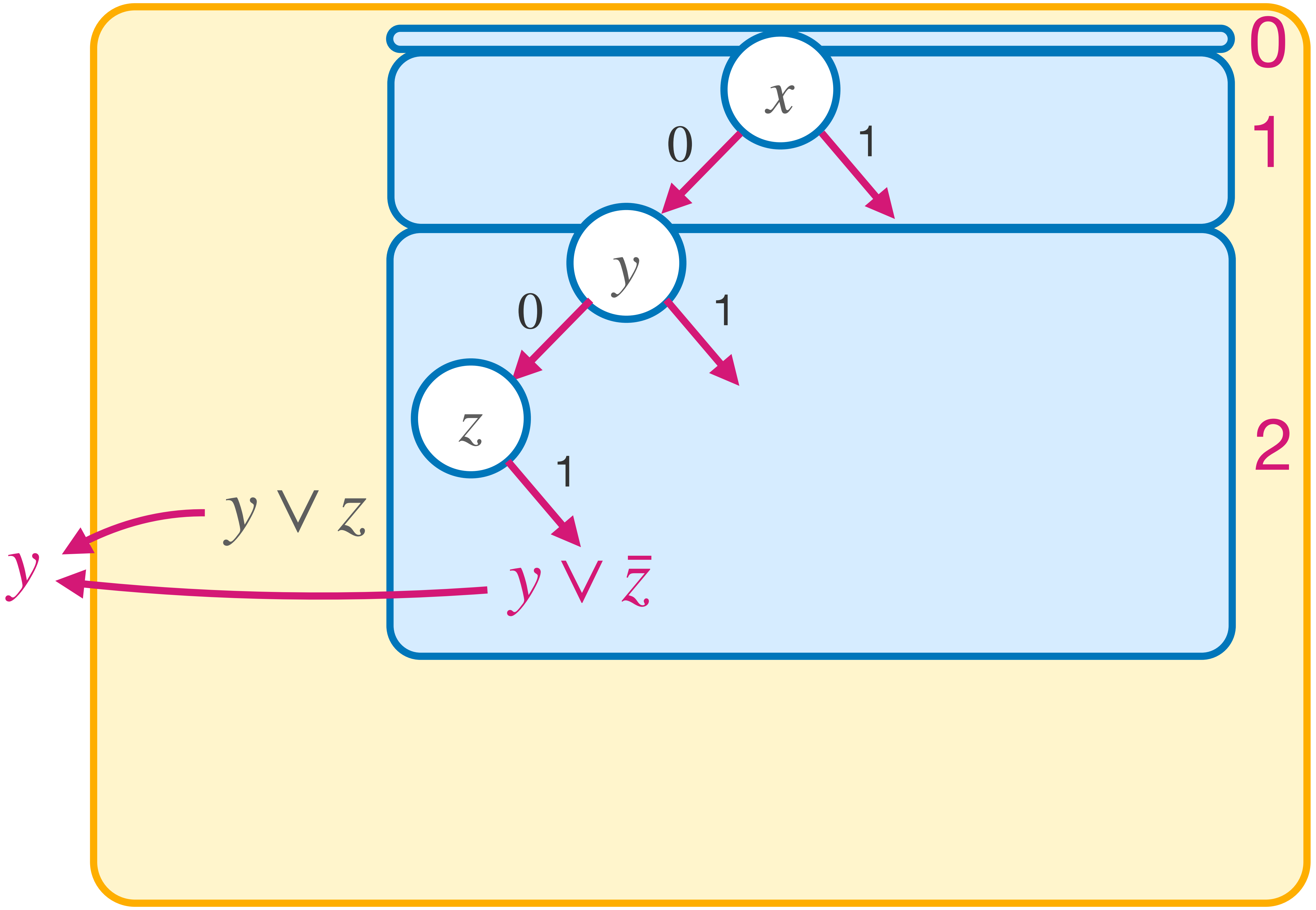
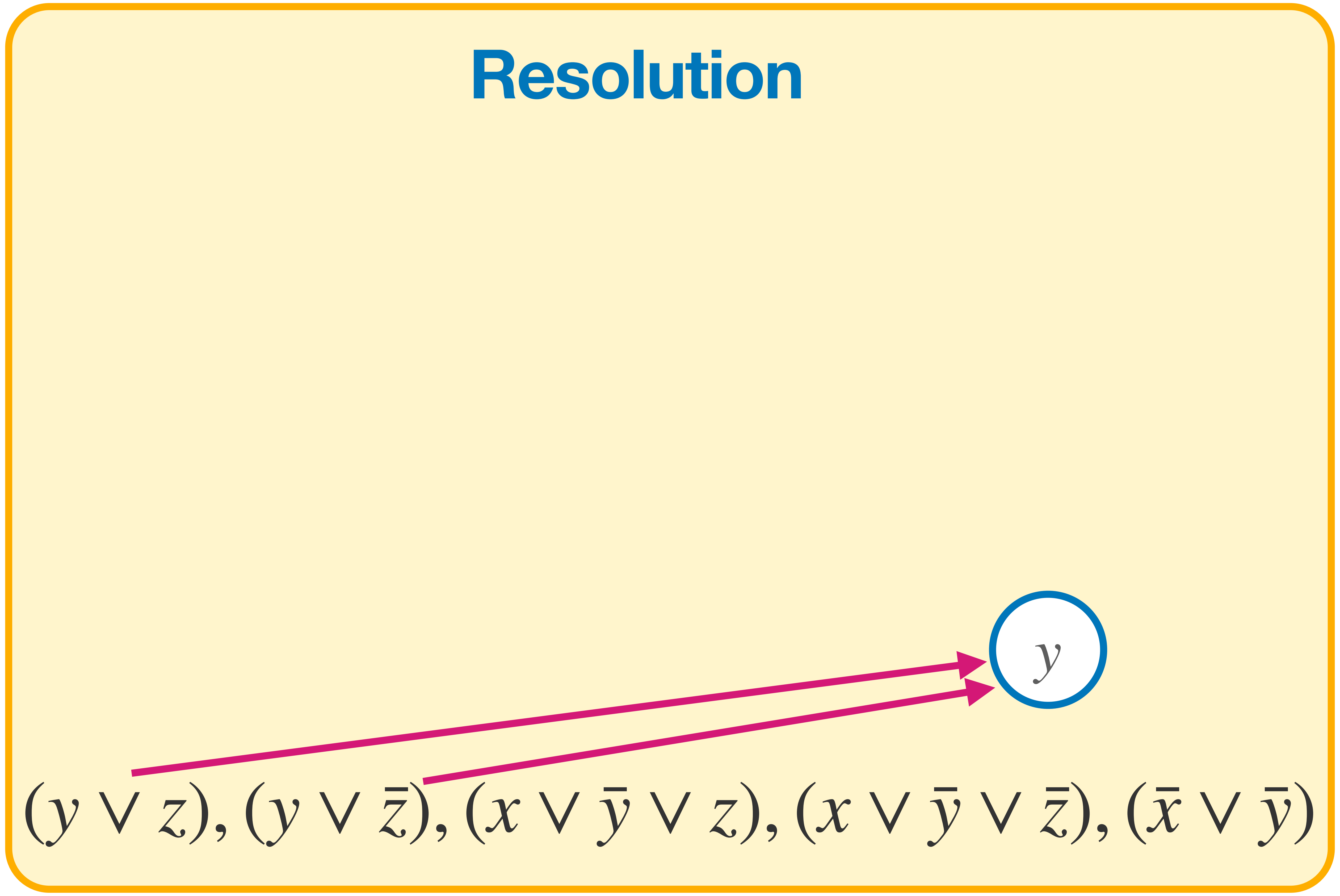
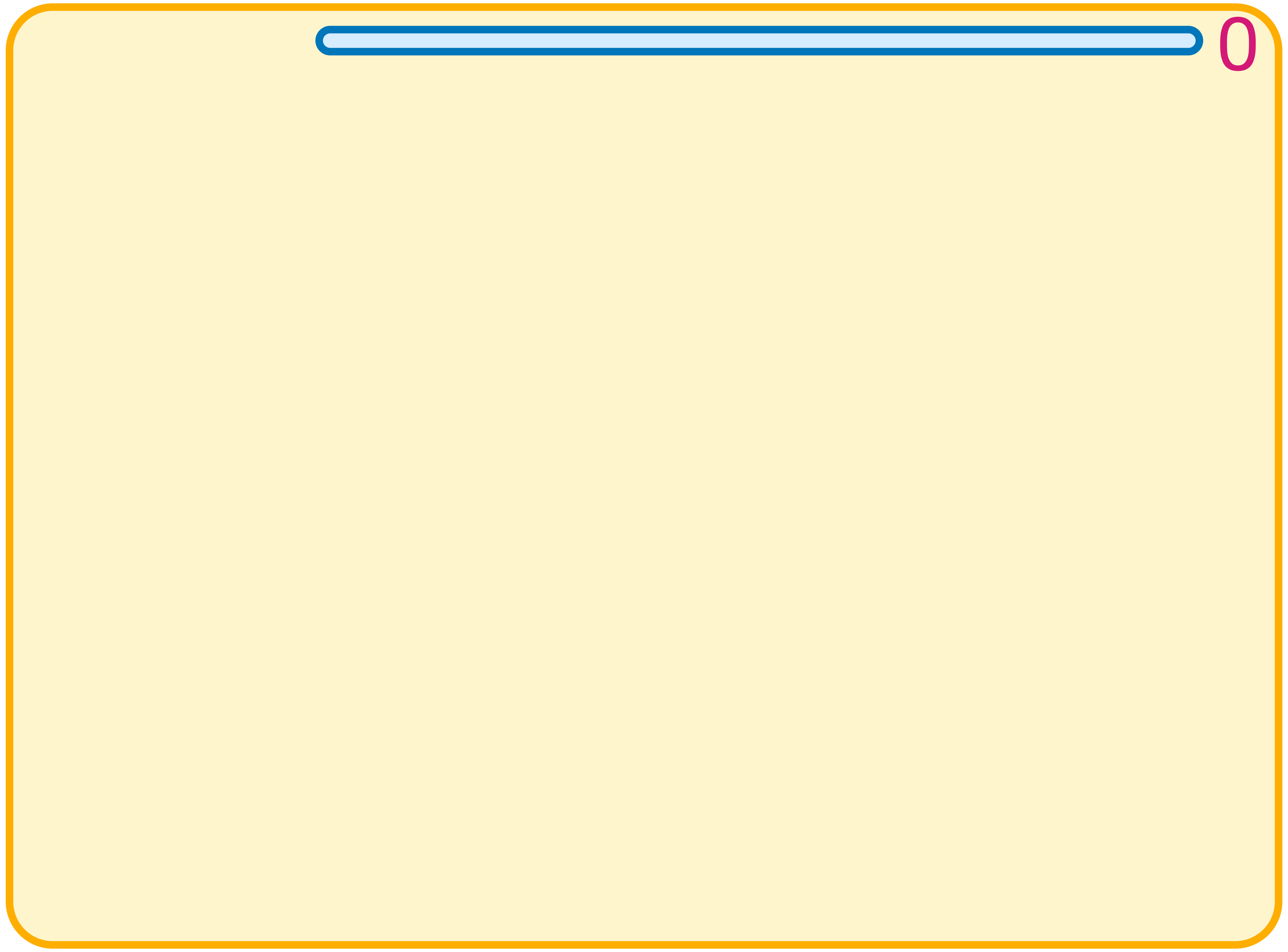




Analyzing CDCL

$$(y) \land (y \lor z) \land (y \lor \bar{z}) \land (x \lor \bar{y} \lor z) \land (x \lor \bar{y} \lor \bar{z}) \land (\bar{x} \lor \bar{y})$$





Analyzing CDCL

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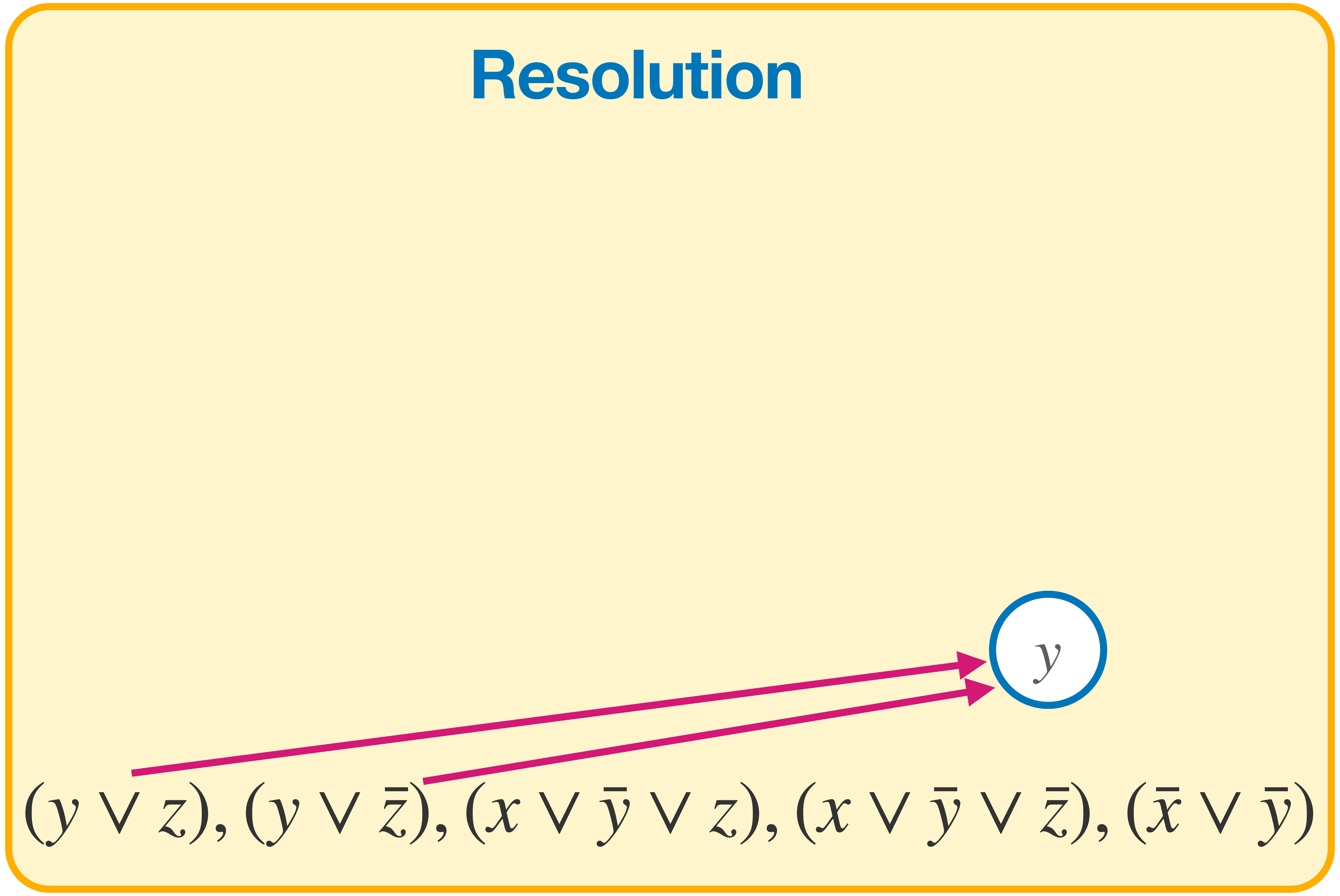

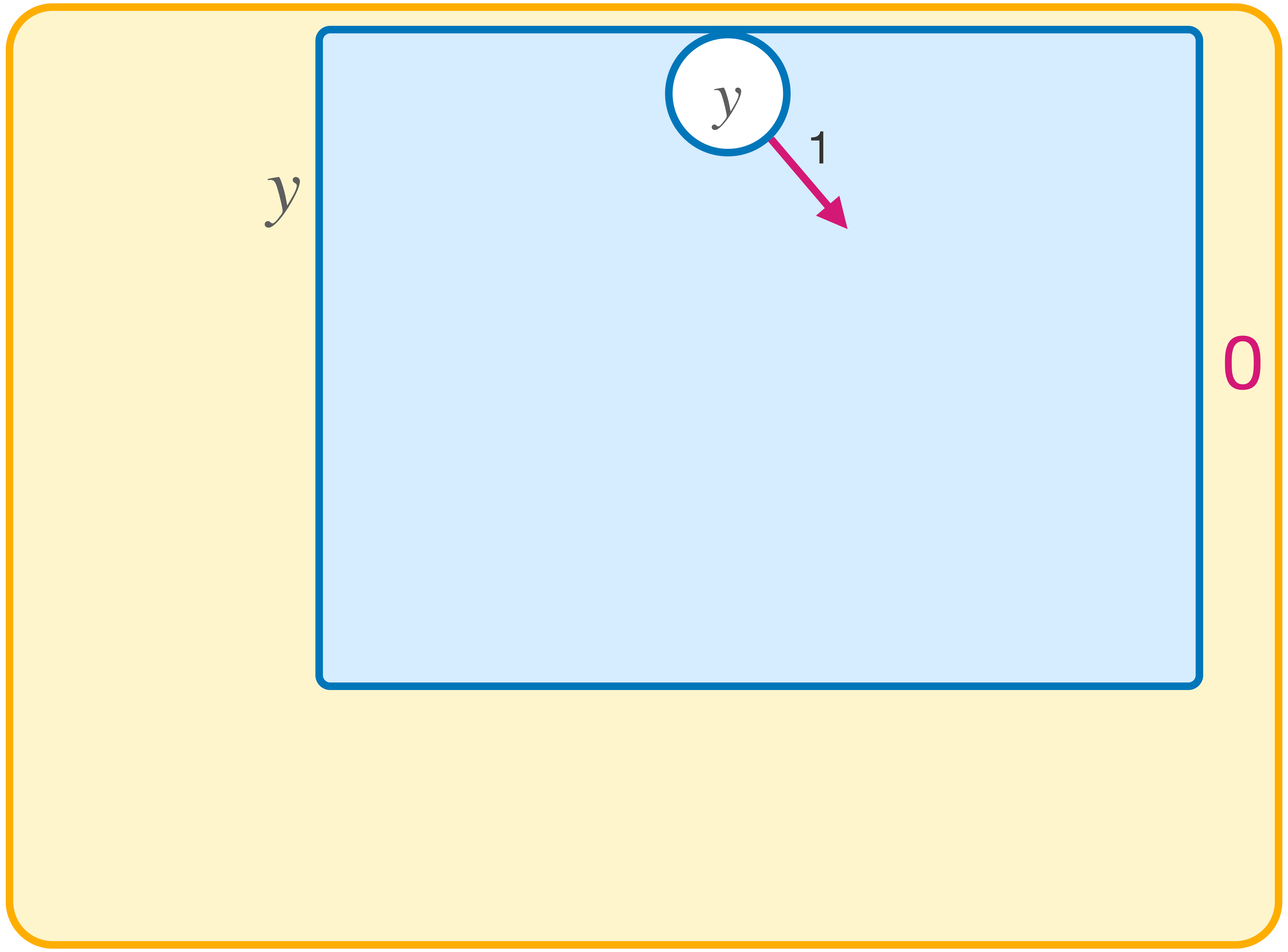

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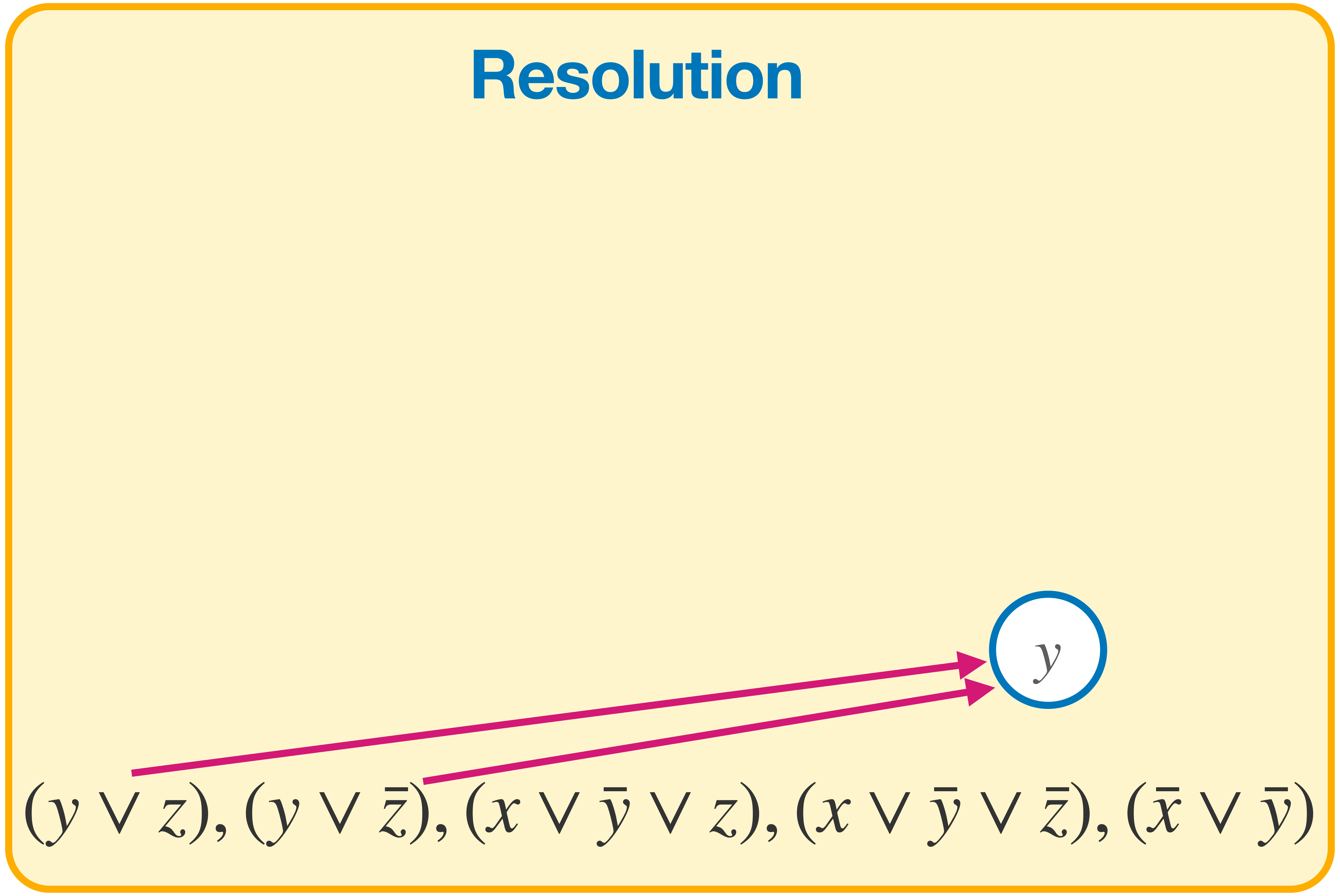

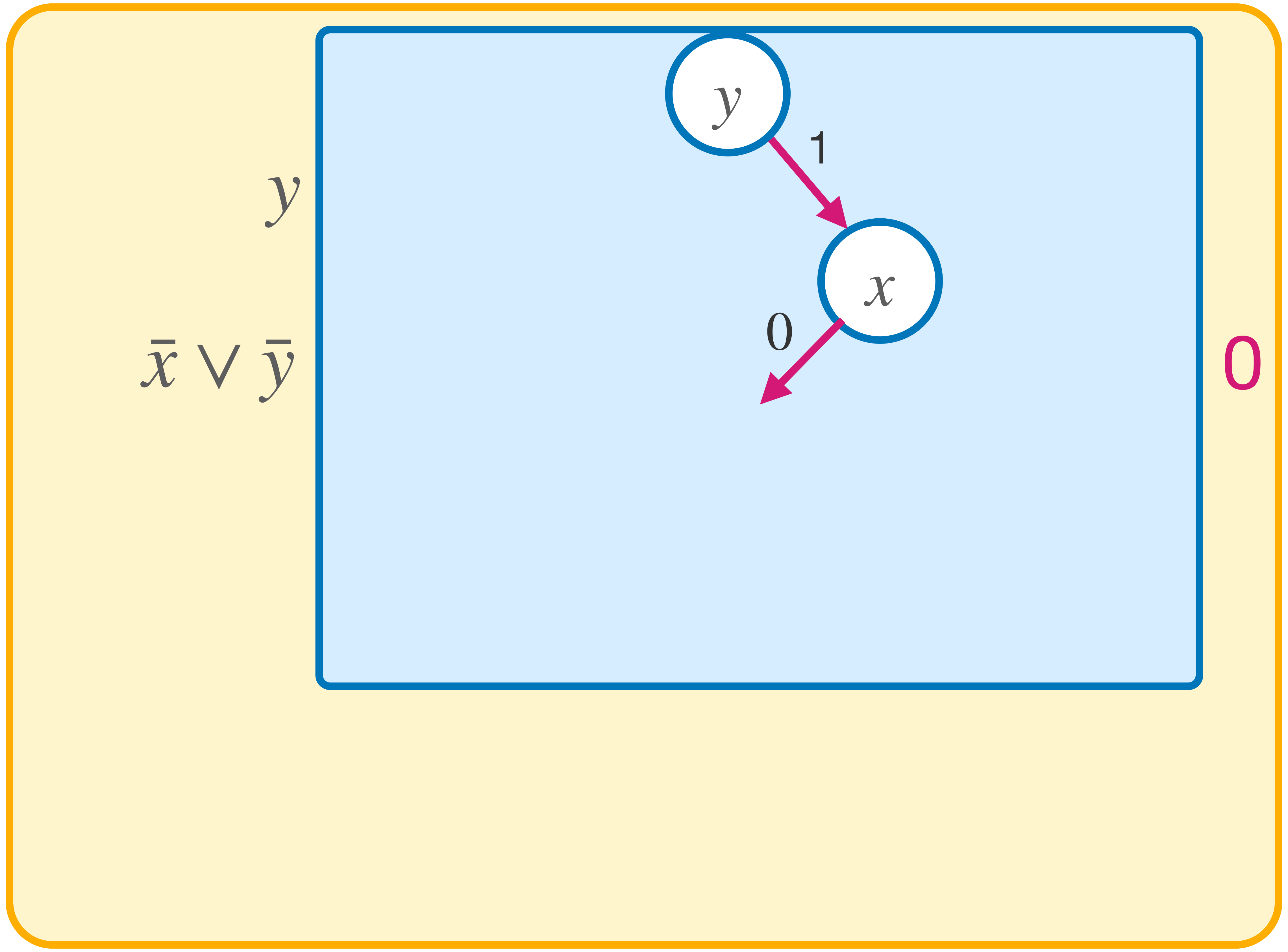

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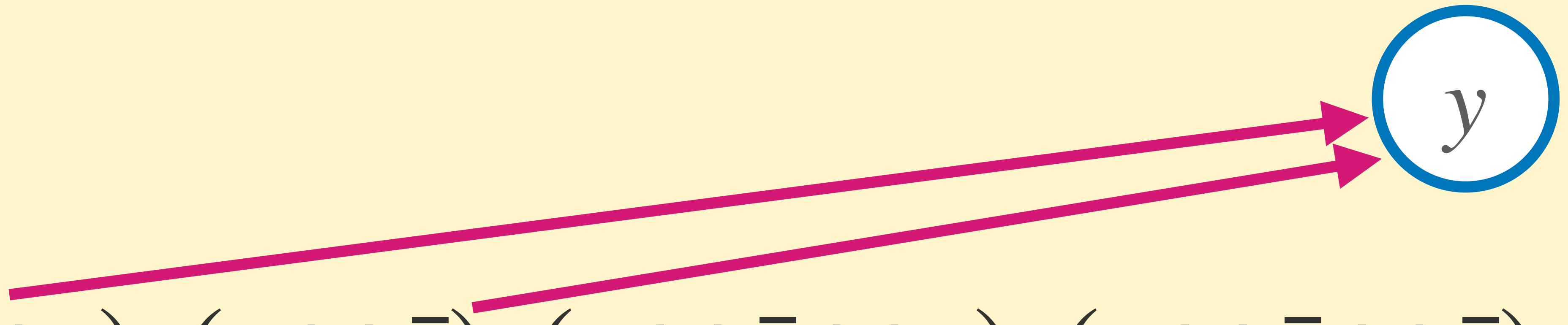

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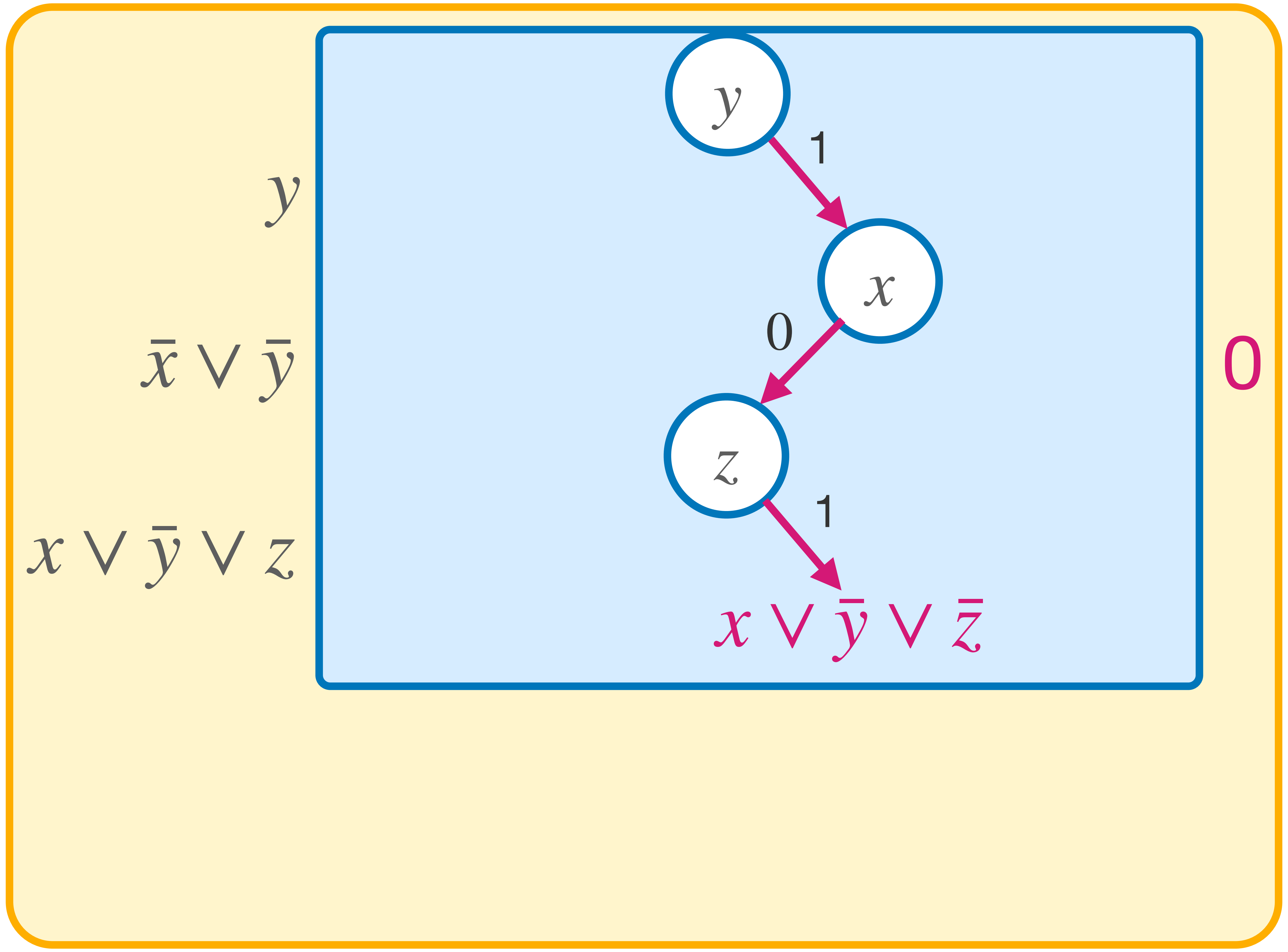

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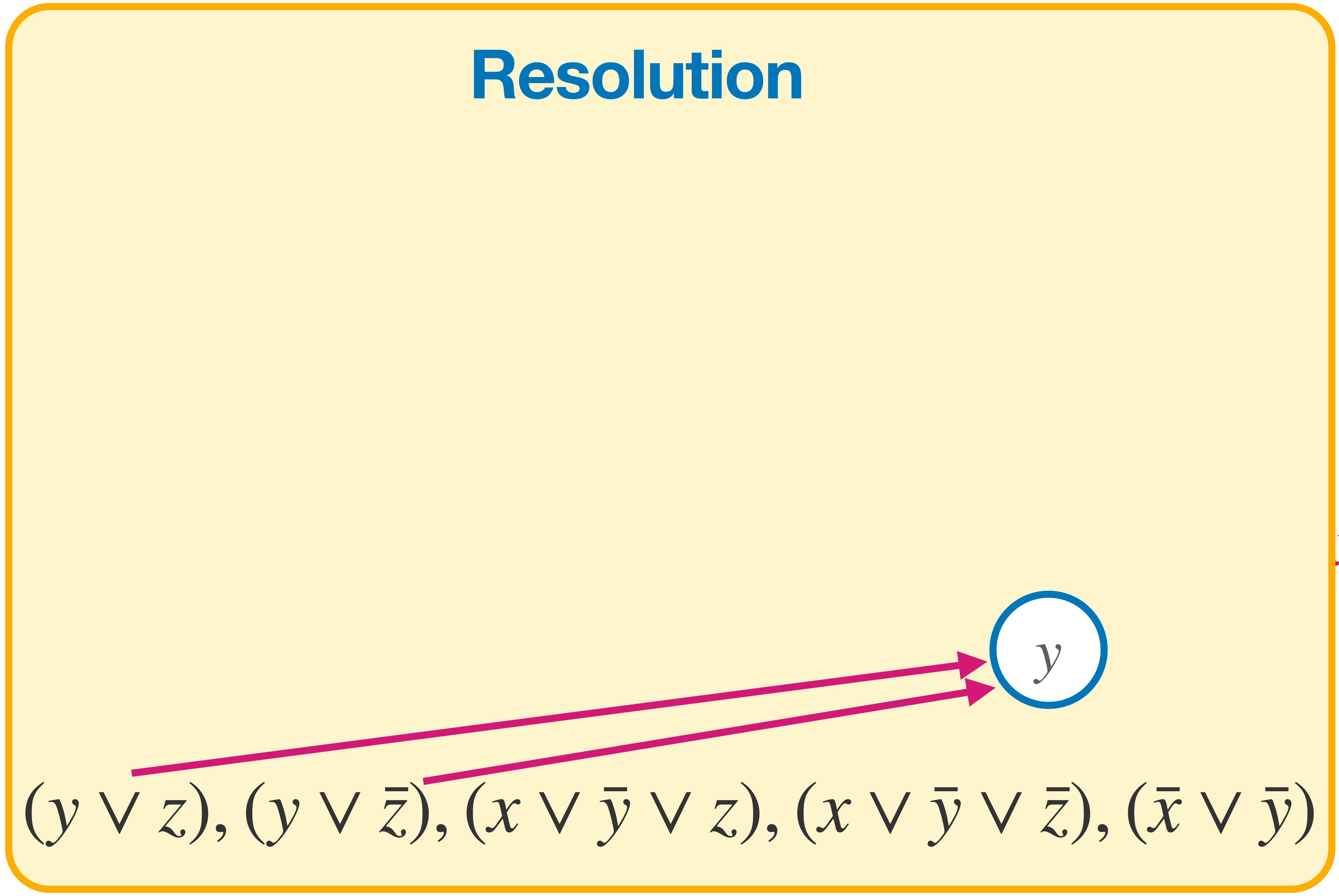

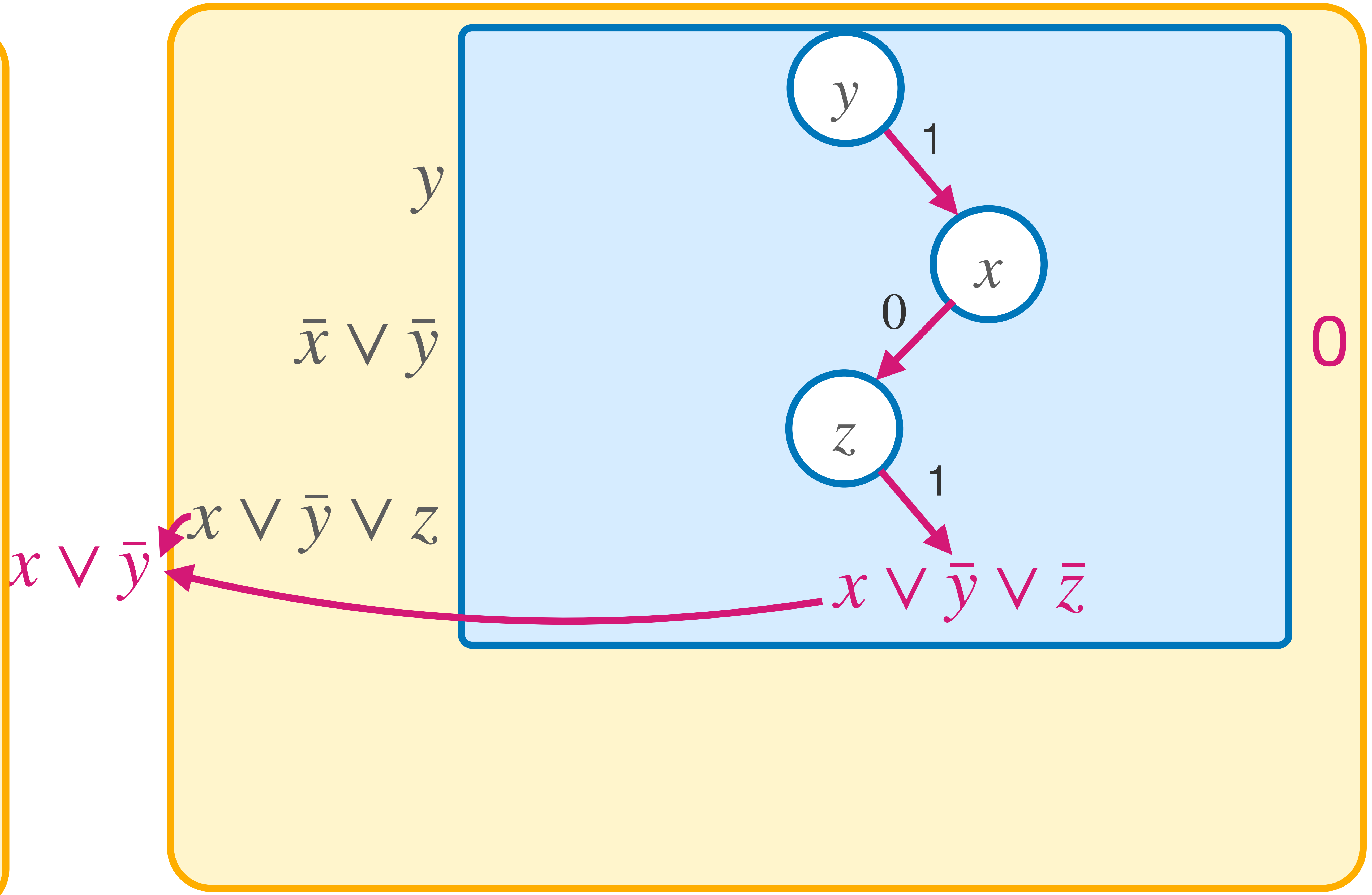

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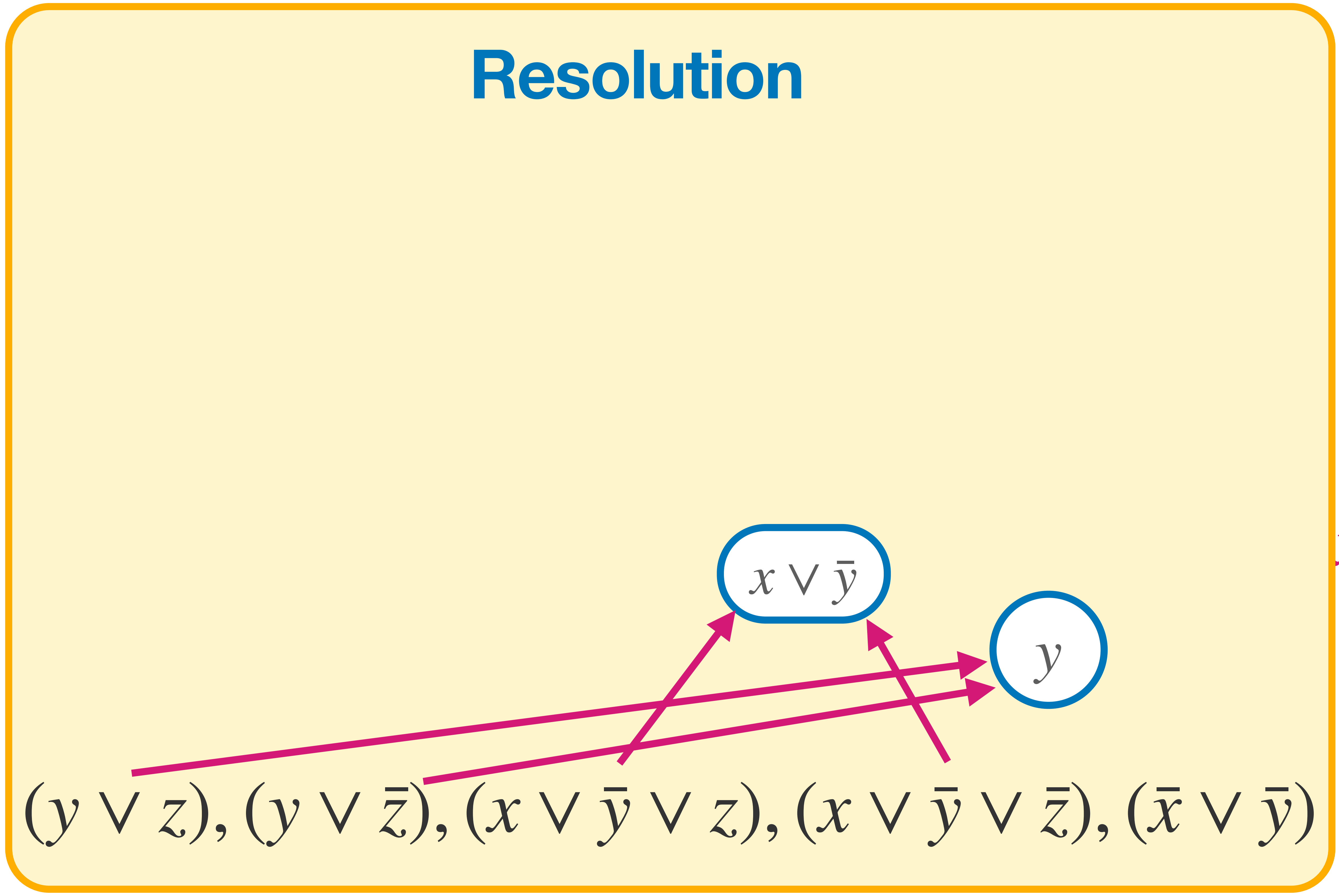

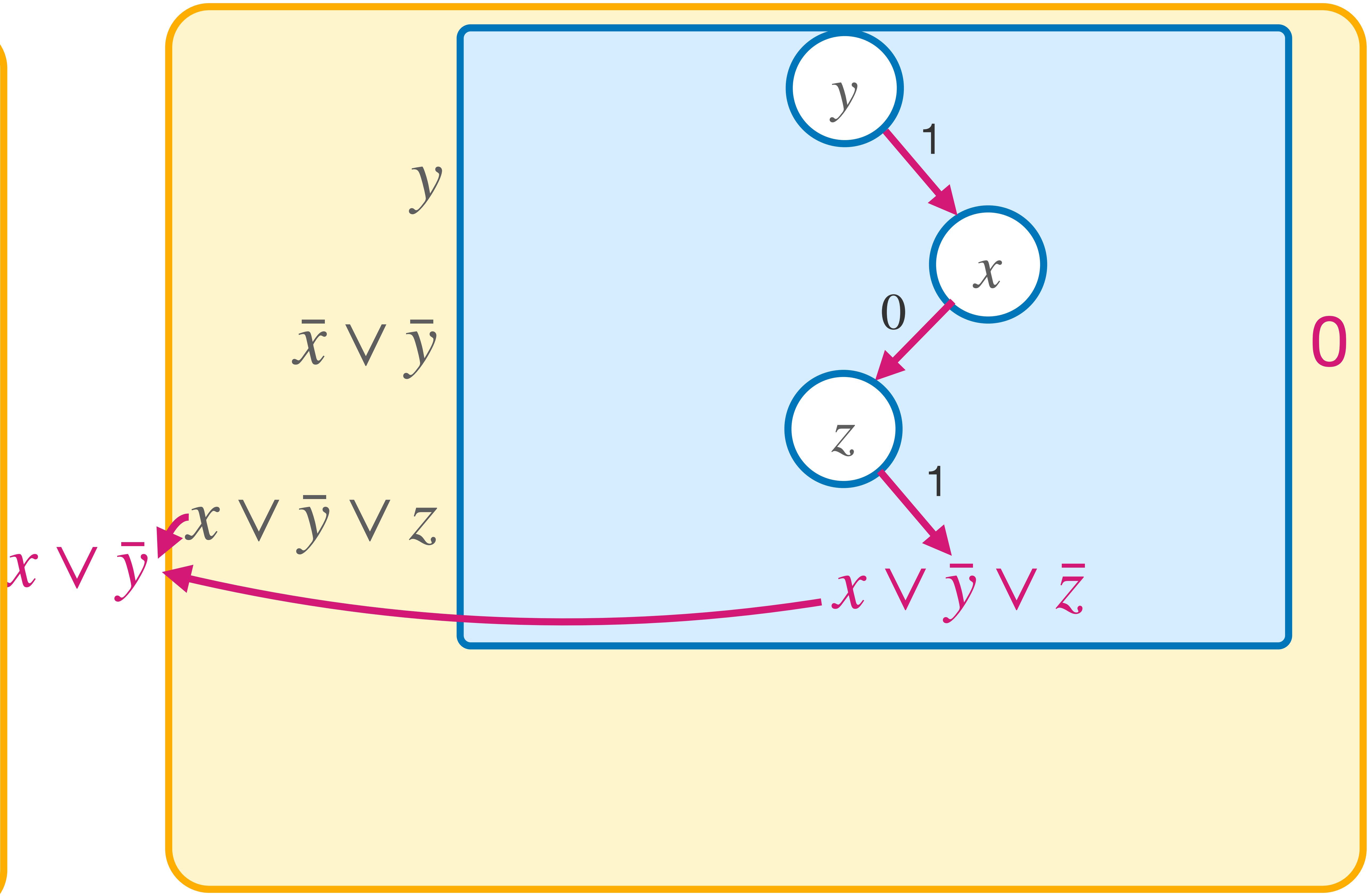

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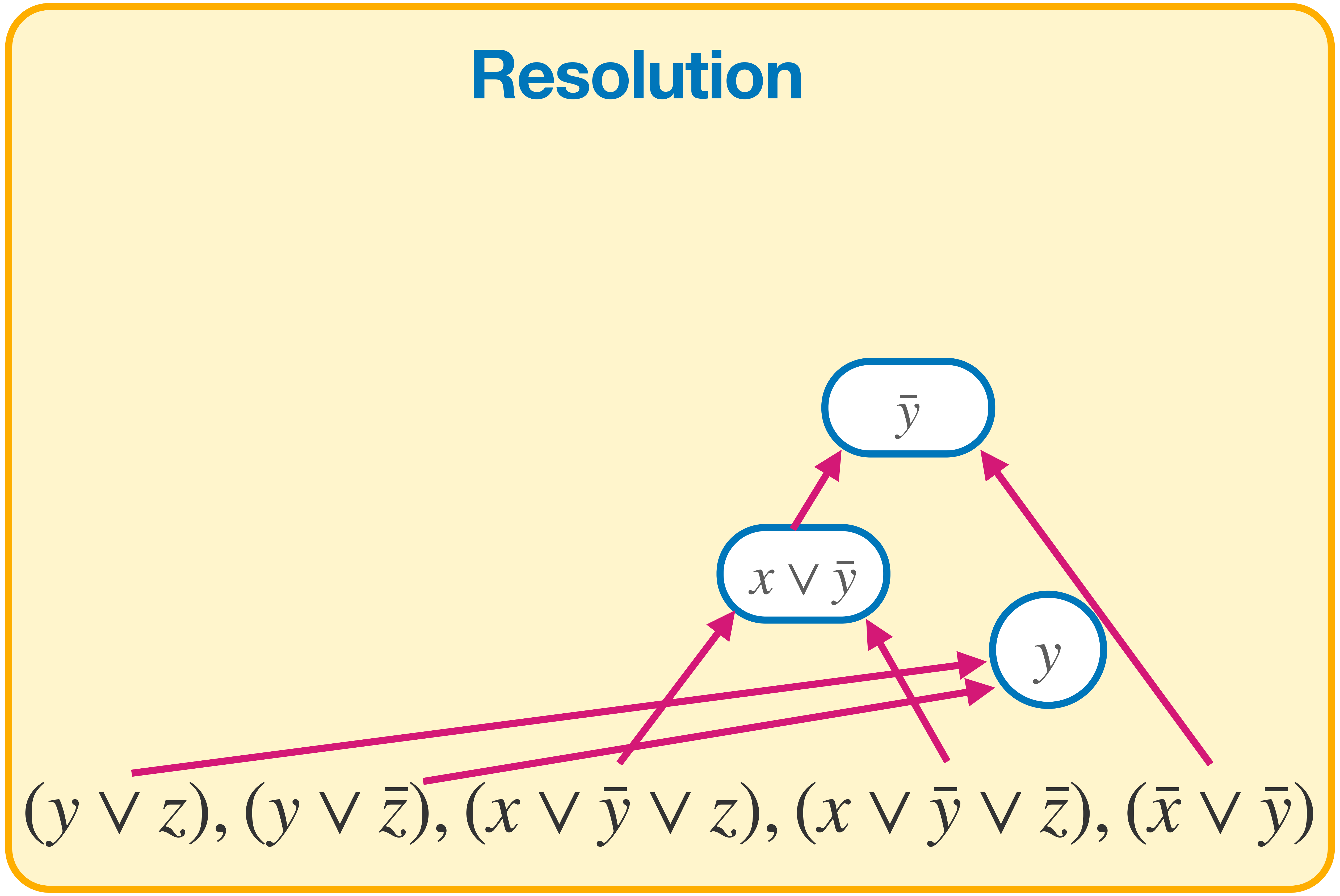

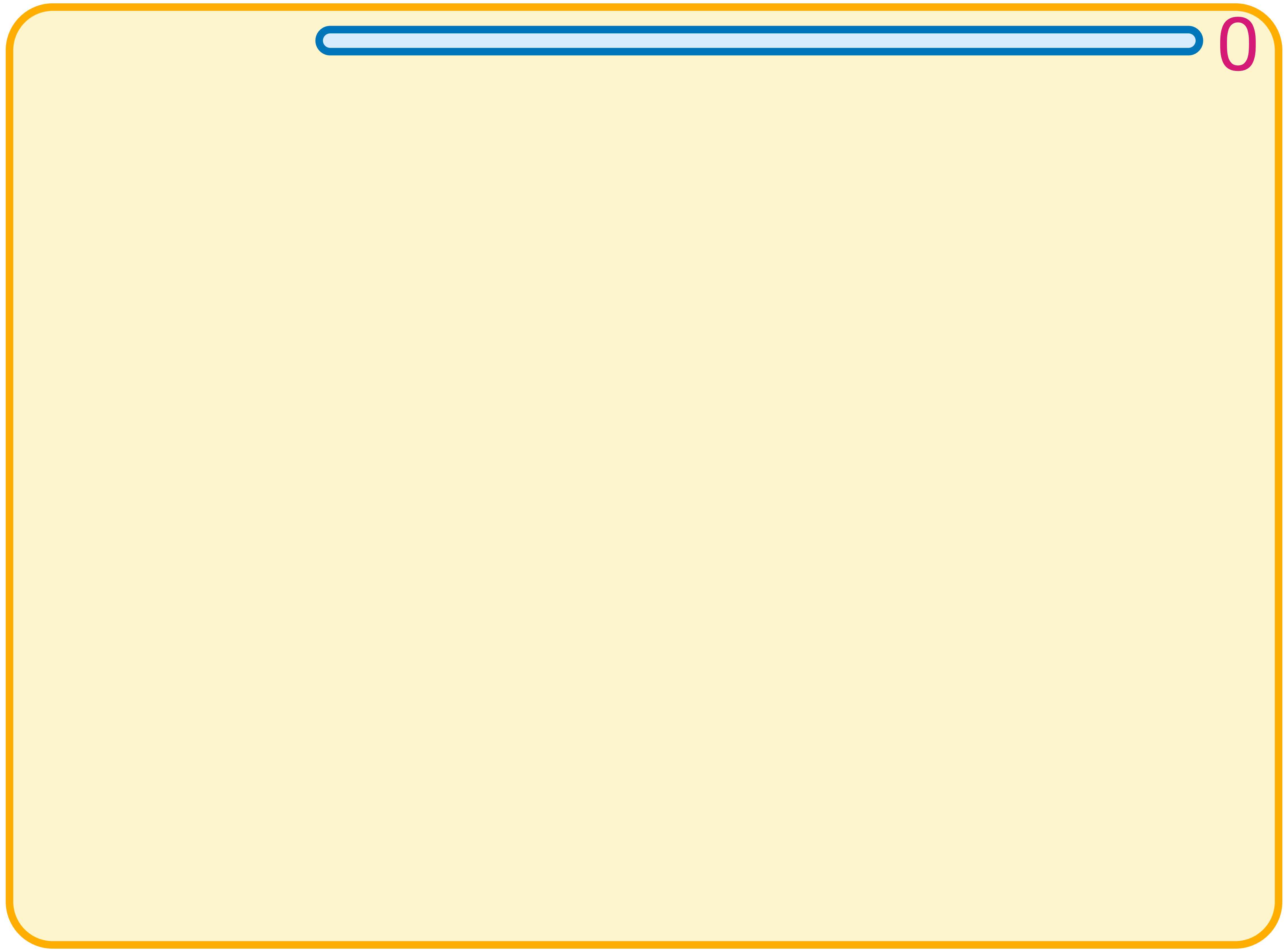

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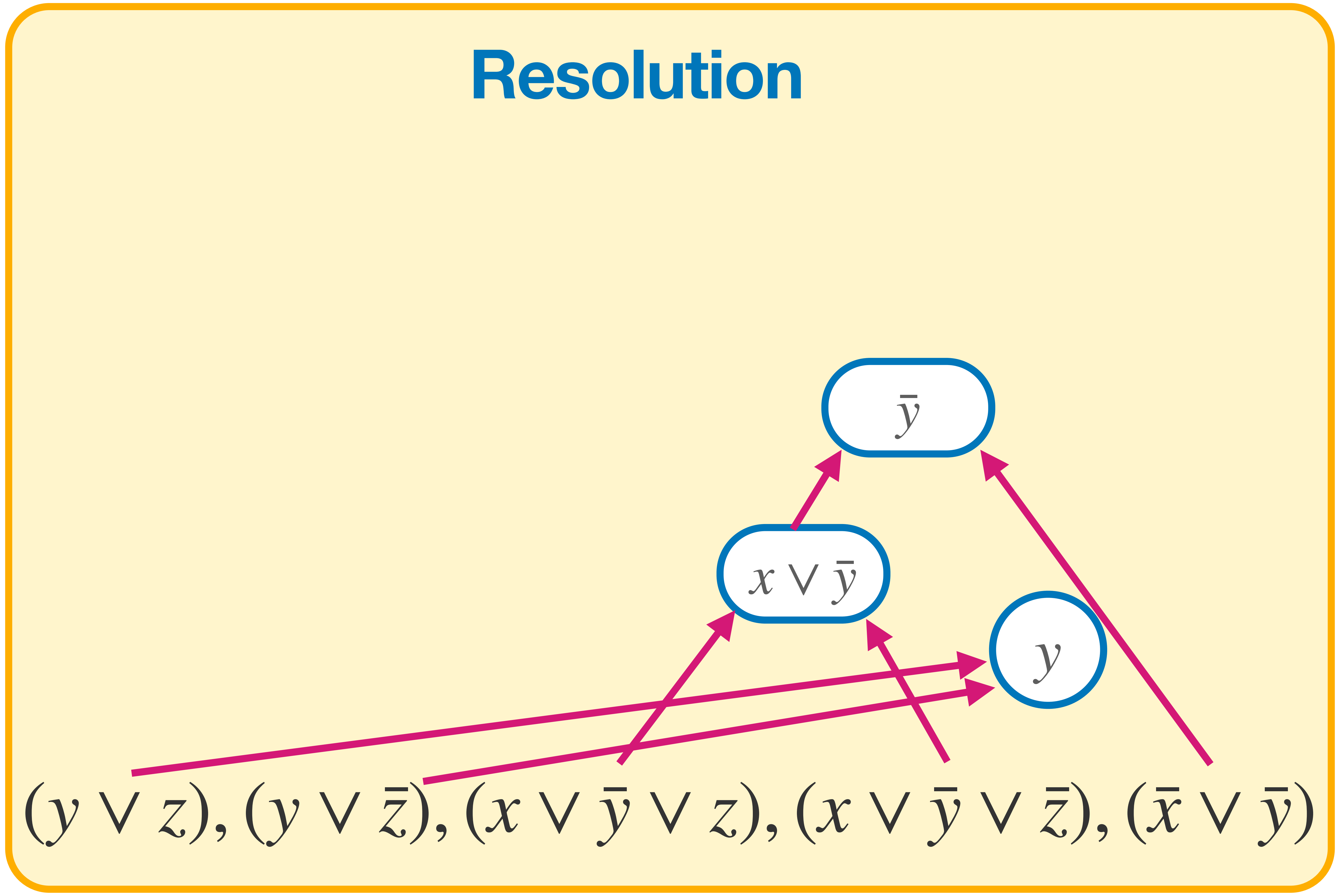

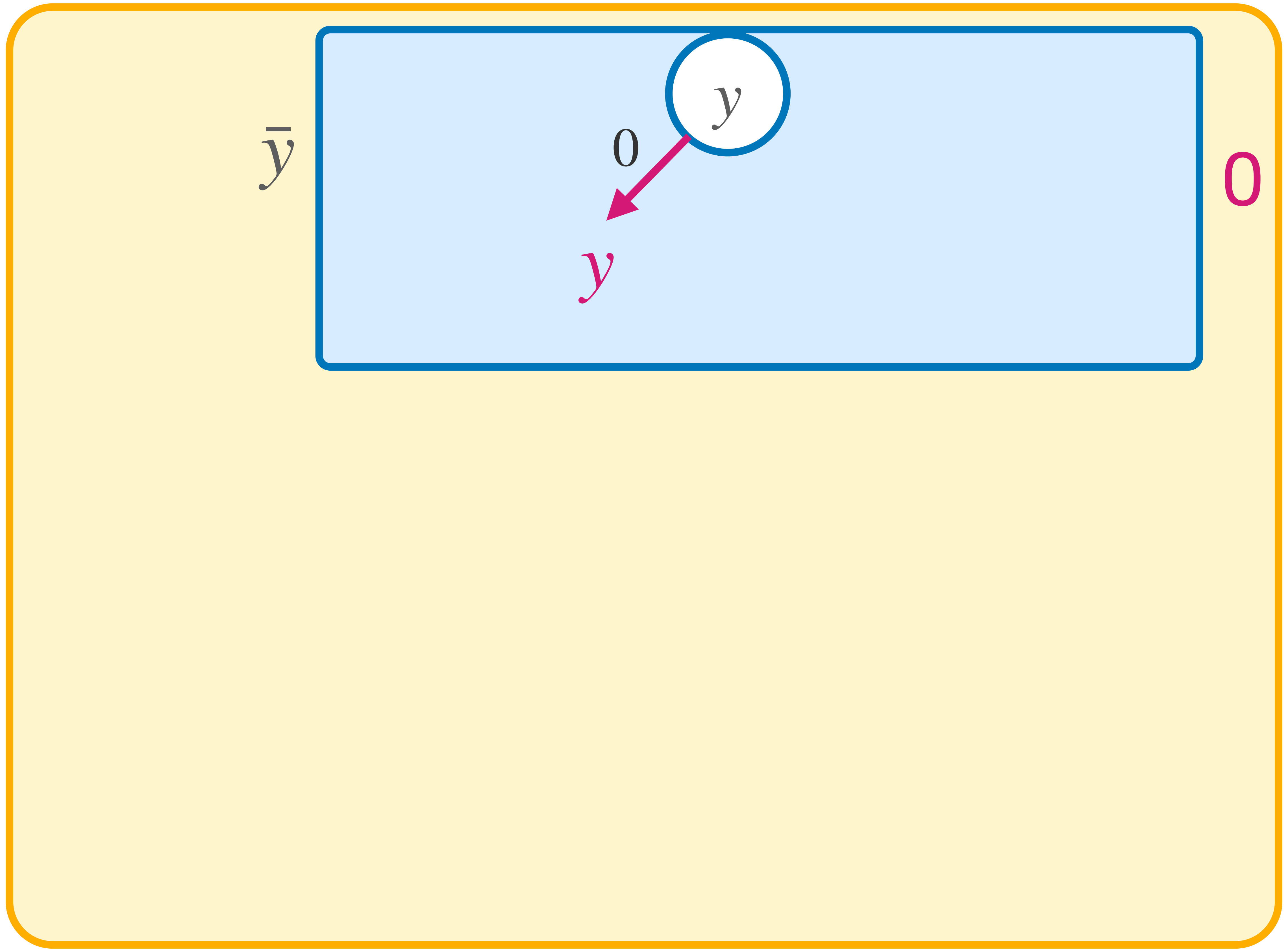

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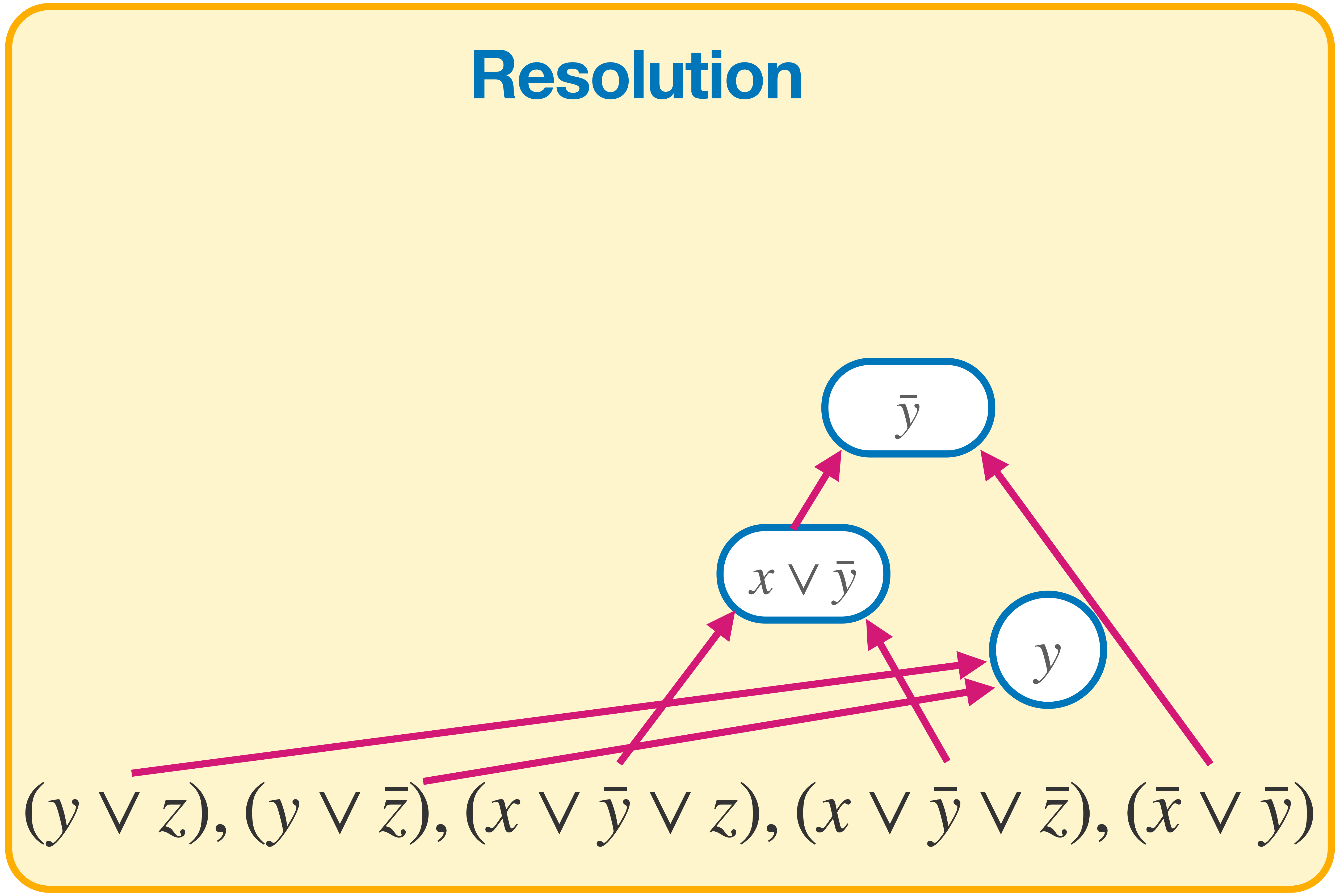

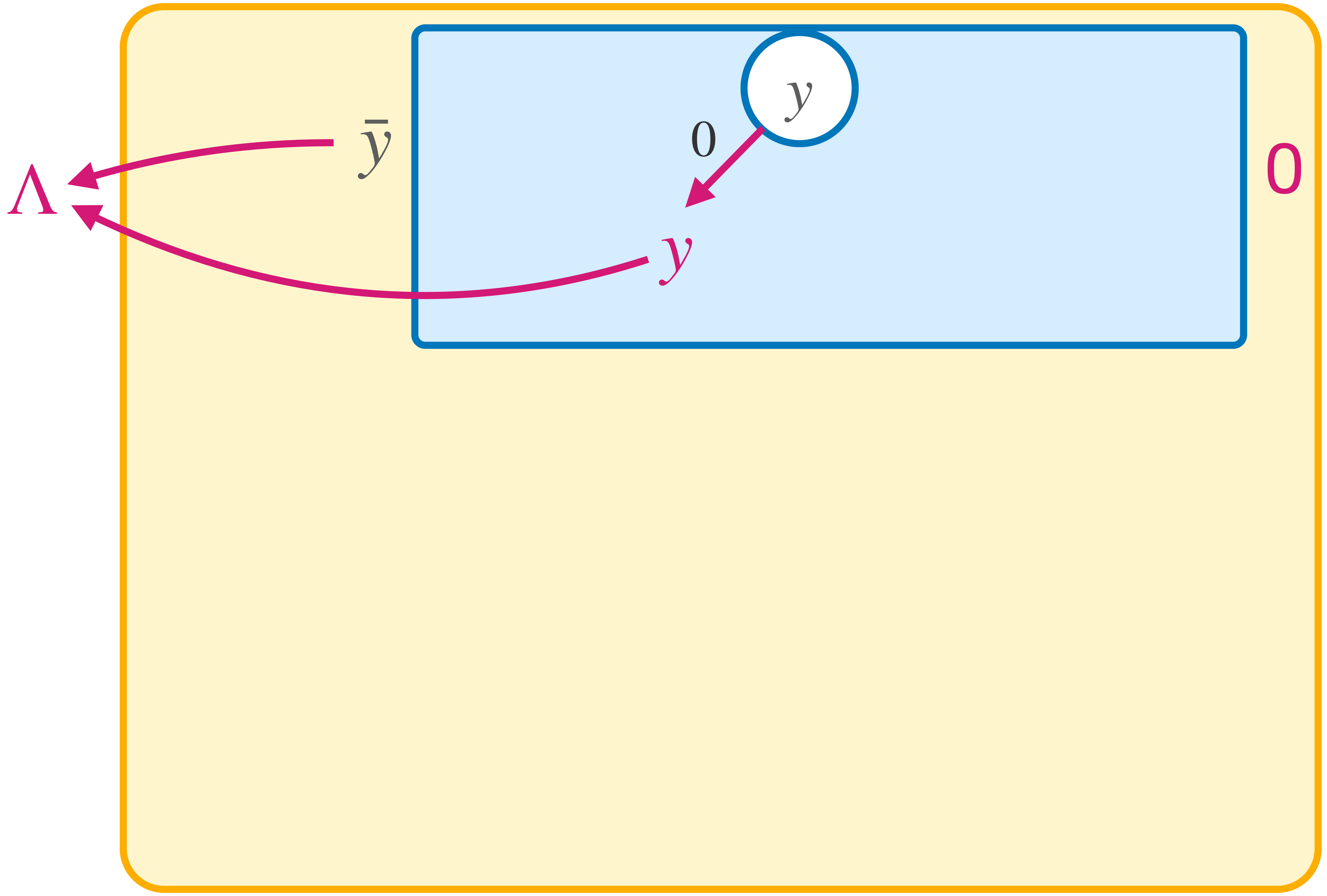

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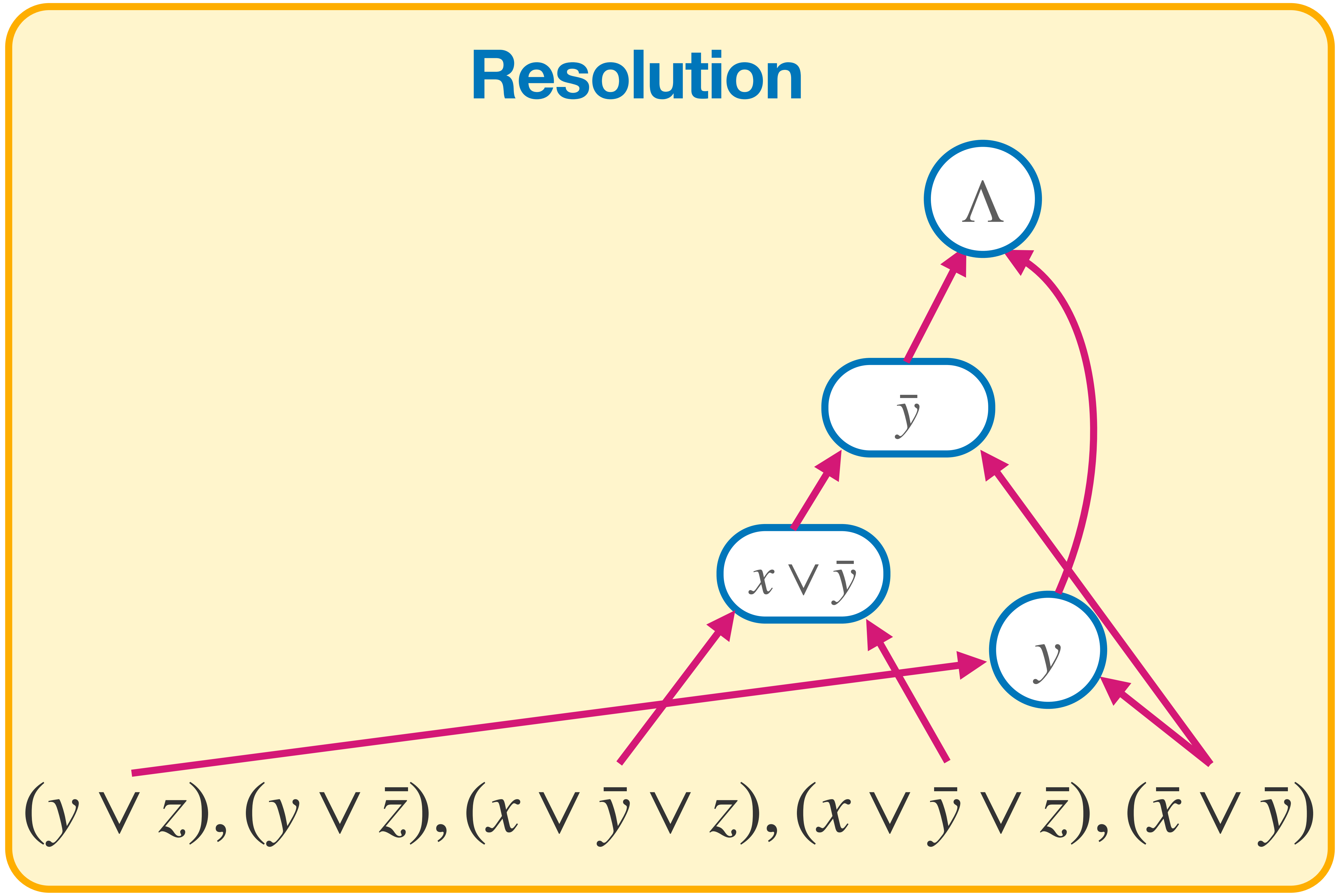

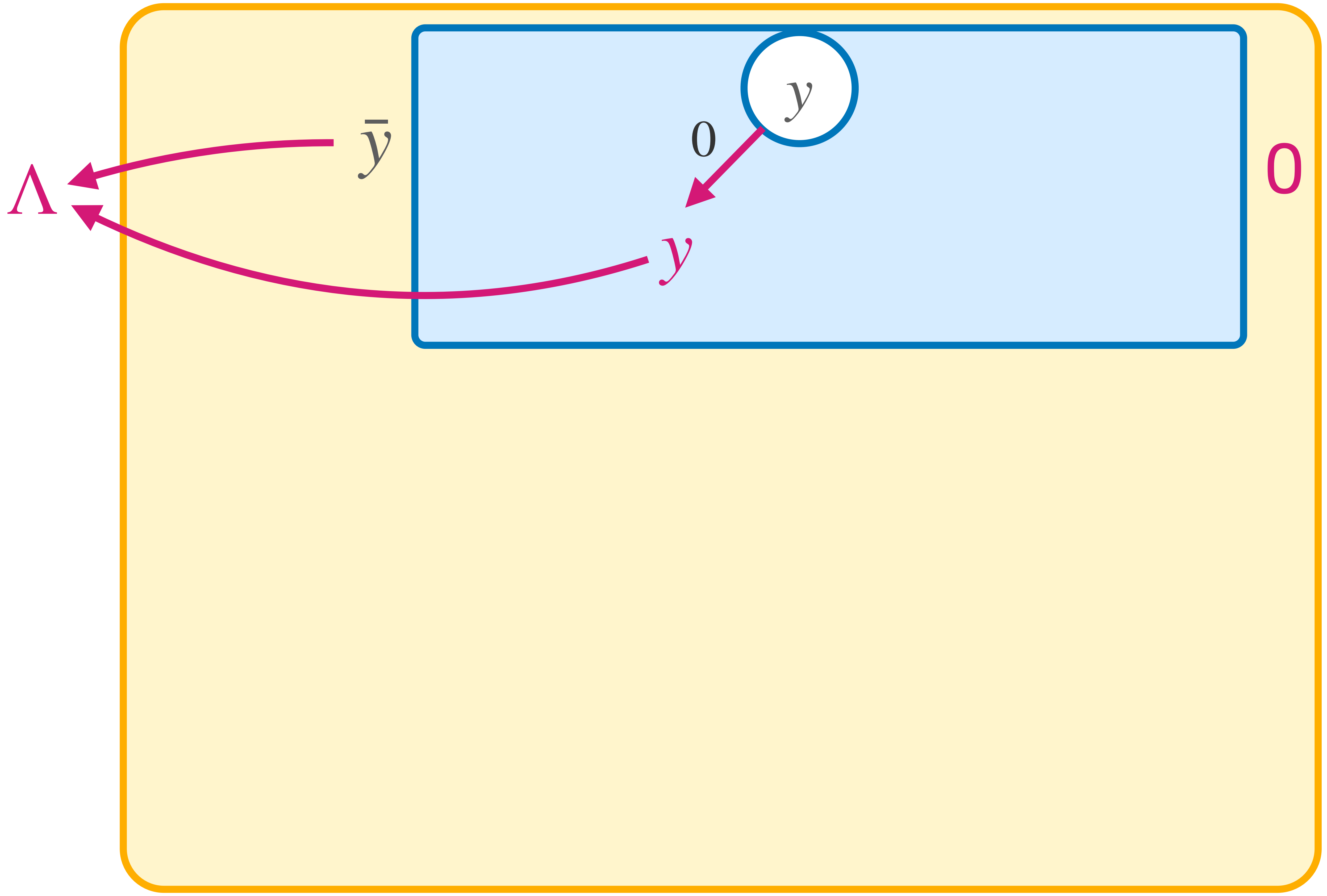

Analyzing CDCL

Theorem: Let F be an unsatisfiable CNF formula. If CDCL takes time s to solve F, then there is a size-s Resolution proof of F

In order to prove bounds on the runtime of CDCL it suffices to analyze Resolution proof size

Resolution Lower Bounds — Some History

First lower bound proved by Armin Haken in `85

Technique: Bottleneck Counting

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In every Resolution proof of F,

1. Every $x \in \{0,1\}^n$ falsifies a wide clause in the proof

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1. Every $x \in \{0,1\}^n$ falsifies a wide clause in the proof
2. Every wide clause is falsified by only a small number of $x \in \{0,1\}^n$

$\Longrightarrow$ Proof must have many wide clauses! (Size lower bound!)

Resolution Lower Bounds — Some History

Connection between width and size formalized by Ben-Sasson Wigderson `99

Resolution Lower Bounds — Some History

Connection between width and size formalized by Ben-Sasson Wigderson `99

Theorem:

For any unsatisfiable CNF formula F on n variables with clauses of width $\leq w$,

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$$= ((\bar{y}_1 \vee \bar{z}_1) \wedge (y_1 \vee z_1)) \vee ((y_2 \wedge z_2) \vee (\bar{y}_2 \wedge z_2))$$

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For any unsatisfiable CNF formula F,

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$F(x_1, \ldots, x_n) \circ XOR$ obtained by substituting $x_i \leftarrow y_i \oplus z_i = (\bar{y}_i \vee \bar{z}_i) \wedge (y_i \vee z_i)$

e.g. $(x_1 \vee \bar{x}_2) \circ XOR = (y_1 \oplus z_1) \vee \neg(y_2 \oplus z_2)$

$= ((\bar{y}_1 \vee \bar{z}_1) \wedge (y_1 \vee z_1)) \vee ((y_2 \wedge z_2) \vee (\bar{y}_2 \wedge z_2))$

$=\ldots$ expand as CNF

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Step 2. Prove that some formula F (Pigeonhole formula) requires large width

Size to Width

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Proof: $w := Width_R(F)$, let Π be any Resolution proof of $F \circ XOR$

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2. If Π is small $\Longrightarrow \Pi \restriction \rho$ has width $< w$

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$\implies \Pi \restriction \rho$ is a Resolution proof of F

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If $|\Pi| \leq (4/3)^w \implies$ exists ρ such that $\Pi \restriction \rho$ has width $< w$ Contradiction!