Outline

• Introduction
  • Applications of voting
  • Motivating the distortion framework

• Utilitarian distortion framework
  • Model
  • Known results

• Metric distortion framework
  • Model
  • Known results

• Applications beyond voting
Voting

Algorithm for aggregating individual preferences to make collective decisions
Voting with Ranked Ballots

Voting Rule $f$
Randomized Voting with Ranked Ballots

Voting Rule $f$
Applications of Randomized Voting

• **Interpretation 1: Randomization**
  - Probably inappropriate for high-stakes political elections
  - Low stakes decisions like “which restaurant for lunch?”
  - Ensemble-leaning based recommendation engines

• **Interpretation 2: Resource division**
  - Foundation splitting its budget between grantees
  - Plan a workshop schedule (posters, talks, coffee, lunch, …)
  - Split a parliament between parties
  - Repeated decisions (seminar weekday, lunch restaurant)
Traditional Analysis: The Axiomatic Method

• Condorcet consistency
  • Whenever there exists an alternative $a$ such that for every other alternative $b$ a strict majority prefer $a$ to $b$, the voting rule must select $a$.

• Weak monotonicity
  • If the voting rule selects alternative $a$ in an instance and $a$ moves up in the rankings of some of the voters, the voting rule must continue to select $a$.

• Axioms are qualitative
  • A voting rule either satisfies an axiom or it does not
Axiomatic Method

Axiom 1

Axiom 2

Axiom 3

Impossibility Results

...disagreement about rules
Voting with Ranked Ballots

Voting Rule $f$
Utilitarian Voting

\[ \text{Utilitarian Social Welfare} \]

\[ f = 0.7 \]
\[ f = 0.2 \]
\[ f = 0.1 \]

\[ \text{Utilitarian Voting} \]

\[ f = 1.5 \]
\[ f = 1.0 \]
\[ f = 0.5 \]
No Access to Utilities

Even if voters have utilities, we may not know them, for many reasons.

- Easier elicitation
  - Higher cognitive effort to assign utilities than to rank alternatives
  - It may be costly to figure out utilities (e.g. computation time to simulate consequences)

- Less communication

- Utilities are simply unknown or unknowable

- Privacy

- leads to “implicit utilitarian voting”: voting rule only knows the ranking, but gets evaluated on the utilities.
Utilitarian Voting

$\frac{0.7}{0.2} = 3.5 \
\frac{0.4}{0.3} = 1.33 \
\frac{0.5}{0} = \infty$

Utility Vector

$\frac{1.5}{1.0}$

Utilitarian Social Welfare

$\text{Apx Ratio} (\frac{1.5}{1.0})$

"could have obtained 1.5x more welfare"
Optimal Voting Rules with Ranked Ballots

Minimize distortion
(Worst-case approximation ratio for utilitarian social welfare)
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Voting with Ranked Ballots

- \( N = \) set of \( n \) voters
- \( A = \) set of \( m \) alternatives
  - \( \Delta(A) = \) set of distributions over \( A \)
- \( \succ \) = observed ranked preference profile
  - \( \succ_i \) = preference ranking of voter \( i \)
  - \( a \succ_i b \) means the voter ranks \( a \) higher than \( b \)
- (Randomized) Voting rule \( f \)
  - Maps every preference profile \( \succ \) to a distribution over alternatives \( f(\succ) = x \in \Delta(A) \)
  - We say that \( f \) is deterministic if \( f(\succ) \) has singleton support for every \( \succ \)
Utilitarian Distortion

1. There exists an underlying utility profile $\vec{u}$ such that for each $i \in N$:
   • Consistency (denoted $u_i \succ_i u_j$): $\forall a, b : a \succ_i b \Rightarrow u_i(a) \geq u_i(b)$
   • Unit-sum: $u_i(a) \geq 0$, $\sum_a u_i(a) = 1$
     • [Aziz 2019] provides seven justifications!
   • Linear extension to distributions: For $x \in \Delta(A)$, $u_i(x) = \sum_a u_i(a) \cdot x(a)$

2. If we knew the utilities, we would want to maximize the (utilitarian) social welfare
   • $sw(x, \vec{u}) = \sum_{i \in N} u_i(x)$ [by linearity, this optimum is attained by an alternative]

3. Because this is impossible given the limited ranked information, we want to best approximate the social welfare in the worst case.
Utilitarian Distortion

- Distortion

\[
\text{dist}(x, \succ) = \sup_{\vec{u} \succ x} \max_{a \in A} \frac{\text{sw}(a, \vec{u})}{\text{sw}(x, \vec{u})}
\]

- Given voting rule \(f\)

\[
\text{dist}(f) = \max_{\succ} \text{dist}(f(\succ), \succ)
\]

What is the lowest possible \(\text{dist}(f)\)? Which voting rule achieves it?
Example (deterministic)

<table>
<thead>
<tr>
<th>Voters</th>
<th>Alternatives</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 : a &gt; b &gt; c</td>
<td>1/3 1/3 1/3</td>
</tr>
<tr>
<td>2 : b &gt; a &gt; c</td>
<td>1 0 0</td>
</tr>
<tr>
<td>3 : a &gt; c &gt; b</td>
<td>1/3 1/3 1/3</td>
</tr>
</tbody>
</table>

• Suppose we choose \( a \):
  • How much better can \( b \) be?

\[
\frac{\max_{\vec{u} \succ a} \text{sw}(b, \vec{u})}{\text{sw}(a, \vec{u})} = \frac{1/3 + 1 + 1/3}{1/3 + 0 + 1/3} = \frac{5}{2}
\]

• How much better can \( c \) be?

\[
\frac{\max_{\vec{u} \succ a} \text{sw}(c, \vec{u})}{\text{sw}(a, \vec{u})} = \frac{1/3 + 0 + 1/3}{1/3 + 0 + 1/3} = 1
\]

• Hence, \( \text{dist}(a, \succ) = \frac{5}{2} = 2.5 \)

• Similarly, compute \( \text{dist}(b, \succ) = 7 \) and \( \text{dist}(c, \succ) = \infty \)
  • \( a \) has lower distortion than \( b \) and \( c \)
Example (randomized)

1 : $a > b > c$
   
   1 0 0

2 : $b > a > c$
   
   $\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$

3 : $a > c > b$
   
   1 0 0

- Among deterministic choices, $a$ is best with distortion 2.5
- With randomization, we can achieve lower distortion.
- On this profile, $x = (a: 0.5882, b: 0.4118, c: 0)$ has distortion 1.54 (best possible).
Utilitarian Distortion

- Instance-optimal rules
  - Deterministic $f_{det}^*$: Maps every preference profile $\succ$ to $a^* \in \arg \min_{a \in A} \text{dist}(a, \succ)$
  - Randomized $f_{rand}^*$: Maps every preference profile $\succ$ to $x^* \in \arg \min_{x \in \Delta(A)} \text{dist}(x, \succ)$
  - Have the lowest distortion on each $\succ$, and therefore in the worst case over all $\succ$

Are the instance-optimal rules polytime computable? Do they have a nice analytical structure?
Optimal Deterministic Distortion

• **Theorem** [Caragiannis, Procaccia, 2011; Caragiannis, Nath, Procaccia, Shah, 2017]
  • For deterministic aggregation of ranked ballots, the optimal distortion is $\Theta(m^2)$ and the instance-optimal rule $f_{det}^*$ is polytime computable.

• **Proof (lower bound):**
  • High-level approach:
    • Take an arbitrary voting rule $f$
    • Construct a preference profile $\succ$
    • Let $f$ choose a winner $a$ on $\succ$
    • Reveal a bad utility profile $\bar{u}$ consistent with $\succ$ in which $a$ is $\Omega(m^2)$ factor worse than the optimal alternative
Deterministic Rules

• Proof (lower bound):
  • Let $f$ be any deterministic voting rule
  • Consider $\succ$ on the right
  • Case 1: $f(\succ) = a_m$
    • Infinite distortion. Why?
  • Case 2: $f(\succ) = a_i$ for some $i < m$
    • Bad utility profile $\vec{u}$ consistent with $\succ$
      • Voters in column $i$ have utility $1/m$ for every alternative
      • All other voters have utility $1/2$ for their top two alternatives
    • $sw(a_i, \vec{u}) = \frac{n}{m-1} \cdot \frac{1}{m}$, $sw(a_m, \vec{u}) \geq \frac{n-n/(m-1)}{2} = \Omega(n)$
    • Distortion $= \Omega(m^2)$
Deterministic Rules

• Proof (upper bound):
  • Plurality rule: Select an alternative $a$ that is the top choice of the most voters
  • For this plurality winner:
    • At least $n/m$ voters have $a$ as their top choice (pigeonhole principle)
    • Every voter has utility at least $1/m$ for their top choice (pigeonhole principle)
  • Hence, for every consistent utility profile $\bar{u}$:
    • $sw(a, \bar{u}) \geq n/m^2$
    • $sw(a^*, \bar{u}) \leq n$ for every alternative $a^*$
  • $\text{dist}(a, \succ) = O(m^2)$
Optimal Randomized Distortion

• **Theorem** [Boutilier, Caragiannis, Haber, Lu, Procaccia, and Sheffet, 2015]
  • For randomized aggregation of ranked ballots:
    • There is a voting rule with distortion $O(\sqrt{m} \cdot \log^* m)$.
    • Every voting rule has distortion at least $\Omega(\sqrt{m})$.
    • The instance-optimal rule $f_{\text{rand}}^*$ is computable in polynomial time.

• **Proof (lower bound):**
  • Same high-level approach:
    • Take an arbitrary *randomized* voting rule $f$
    • Construct a preference profile $\succ$
    • Let $f$ choose a distribution $x$ over alternatives
    • Reveal a bad utility profile $\widehat{u}$ consistent with $\succ$ in which the expected social welfare under $x$ is $\Omega(\sqrt{m})$ factor worse than the optimal social welfare
Randomized Rules

• Proof (lower bound):
  • Let \( f \) be an arbitrary rule
  • Consider \( \succ \) on the right with \( \sqrt{m} \) special alternatives
  • \( f \) returns distribution \( x \) in which at least one special alternative (say \( a_i \)) must be chosen w.p. at most \( \frac{1}{\sqrt{m}} \)
  • Bad utility profile \( \vec{u} \) consistent with \( \succ \):
    • All voters ranking \( a_i \) first have utility 1 for \( a_i \)
    • All other voters have utility \( \frac{1}{m} \) for every alternative
  • \( sw(a_i, \vec{u}) = \Theta \left( \frac{n}{\sqrt{m}} \right) \) but \( sw(a, \vec{u}) \leq \frac{n}{m} \) for every other alternative \( a \)
  • \( sw(x, \vec{u}) \leq \left( \frac{1}{\sqrt{m}} \right) \cdot \Theta \left( \frac{n}{\sqrt{m}} \right) + \left( 1 - \frac{1}{\sqrt{m}} \right) \cdot \frac{n}{m} = O \left( \frac{n}{m} \right) \)
  • Hence, \( dist(x, \vec{u}) = \Omega(\sqrt{m}) \)
Optimal Randomized Distortion

• **Harmonic Rule**
  • The rule that achieves $O(\sqrt{m} \cdot \log^* m)$ distortion is complicated and artificial (it only makes sense if you want low distortion) and is unlikely to generalize
  • [Boutilier et al. 2015] propose a simpler rule that achieves $O(\sqrt{m} \cdot \log m)$ distortion

<table>
<thead>
<tr>
<th>Harmonic Rule</th>
</tr>
</thead>
</table>
| • Each voter $i$ awards $1/r$ points to her $r^{th}$ ranked alternative for every $r \in \{1, \ldots, m\}$
| • Harmonic score of alternative $a$, denoted $hsc(a, \succ)$, is the total point awarded to $a$
| • W.p. $\frac{1}{2}$, choose each $a \in A$ with probability proportional to $hsc(a, \succ)$
| • W.p. $\frac{1}{2}$, choose each $a \in A$ uniformly at random |

• Key proof idea:
  • $hsc(a, \succ) \geq sw(a, \bar{u})$ for every $a$, while $\sum_a hsc(a, \succ) = O(\log m) \cdot \sum_a sw(a, \bar{u})$
Optimal Randomized Distortion

• **Theorem** [Ebadian, Kahng, Peters, Shah, 2022]
  - For randomized aggregation of ranked ballots, the optimal distortion is $\Theta(\sqrt{m})$.

• **Proof via three steps:**
  I. Define “stable lotteries”
  II. Prove the existence (and efficient computation) of stable lotteries via the minimax theorem
  III. Derive $O(\sqrt{m})$ distortion using stable lotteries
Step I: Define Stable Lotteries

Voter 1

Voter 2

Voter 3

• For a set of alternatives $S = \{\text{ }, \text{ }, \text{ }\}$ and an alternative $a = \text{ }$

$$V(a, S) = |\{i \in N : a >_i b, \forall b \in S\}| = 2$$

• Lottery $S$ over sets of size $k$ is stable if $\mathbb{E}_{S \sim S}[V(a, S)] \leq \frac{n}{k}$ for every $a \in A$
Step II: Prove Stable Lotteries Exist

• **Theorem:** For every $k$, a stable lottery over committees of size $k$ exists.

• **Proof (skip):**
  
  $\min_S \max_{a \in A} \mathbb{E}_{S \sim S}[V(a, S)] \leq \min_S \max_{x \in \Delta(A)} \mathbb{E}_{S \sim S, a \sim x}[V(a, S)]$
  
  $= \max_{x \in \Delta(A)} \min_S \mathbb{E}_{S \sim S, a \sim x}[V(a, S)] \leq \frac{n}{k}$

• For any $x \in \Delta(A)$, consider the lottery $S^*$, where we sample $k$ alternatives i.i.d. according to $x$ and replace any duplicates with arbitrary other alternatives

• For each voter $i$:

  $$\Pr_{S \sim S^*, a \sim x}[a >_i b, \forall b \in S] \leq \frac{1}{k + 1}$$

• Hence:

  $$\mathbb{E}_{S \sim S^*, a \sim x}[V(a, S)] \leq \frac{n}{k + 1} < \frac{n}{k} \quad \blacksquare$$
Step III: Proof of $O(\sqrt{m})$ Distortion

**Stable Lottery Rule**

- W.p. $\frac{1}{2}$, find a stable lottery $S$ over sets of size $\sqrt{m}$, sample $S \sim S$, choose $a \in S$ uniformly at random
- W.p. $\frac{1}{2}$, choose $a \in A$ uniformly at random

**Theorem:** Stable lottery rule achieves $O(\sqrt{m})$ distortion.

- Let $a^*$ be an alternative maximizing social welfare
- For any $S$: $sw(a^*, \bar{u}) \leq V(a^*, S) + \sum_{b \in S} sw(b, \bar{u})$
- Taking expectation over $S \sim S$:
  $$sw(a^*, \bar{u}) \leq \mathbb{E}_{S \sim S}[V(a^*, S)] + \mathbb{E}_{S \sim S}[\sum_{b \in S} sw(b, \bar{u})]$$
  $$\leq 2\sqrt{m} \cdot \left(\frac{1}{m} \cdot \frac{n}{m} + \frac{1}{2} \cdot \mathbb{E}_{S \sim S} \left[\frac{1}{|S|} \cdot \sum_{b \in S} sw(b, \bar{u})\right]\right)$$
  $$= 2\sqrt{m} \cdot sw(f(\succ), \bar{u}) \blacksquare$$
Notes

• Stable lotteries
  • Introduced by [Cheng, Jiang, Munagala, Wang, 2020], who show the existence of a stronger form of stable lotteries which bounds $V(S', S)$ for all $S' \subseteq A$
  • Requires a much more intricate proof

• Stable committees
  • 16-stable committees exist [Jiang, Munagala, Wang, 2020]: $V(a, S) \leq 16 \cdot \frac{n}{k}$ for all $a \in A$
  • Factor 16 cannot be improved to any lower than 2
  • Open question: Do 2-approximately stable committees exist?

• Lower bound
  • The lower bound from before is $\frac{\sqrt{m}}{2}$
  • Open question: A gap of factor 4 between this lower bound and the $2\sqrt{m}$ upper bound by stable lottery rule
Extensions

- Other utility classes and objective functions
- Incentives
- Ballot formats other than ranked ballots
- Committee selection
- Optimal ballot design
- Participatory budgeting
- Social welfare functions
Other Objective Functions

• Nash social welfare
  • \( sw(x, \vec{u}) = \sum_{i \in N} u_i(x) \)
  • \( nsw(x, \vec{u}) = \left( \prod_{i \in N} u_i(x) \right)^{1/n} \)
  • Nash social welfare is independent of individual scales
    • Any distortion upper bound with respect to unit-sum utilities holds for arbitrary utilities

• Theorem [Ebadian, Kahng, Peters, Shah, 2022]:
  • With respect to the Nash social welfare:
    • The distortion of harmonic rule is \( \Theta(\sqrt{m \cdot \log m}) \)
    • The distortion of stable committee rule (similar to stable lottery rule) is \( \Theta(\sqrt{m}) \)
    • There is a randomized rule with distortion \( O(\log m) \)
    • No randomized rule has distortion better than \( \left( \frac{m^m}{m!} \right)^{1/m} \rightarrow e \)

• Open question: Close the gap between \( O(\log m) \) and \( e \)
Other Objective Functions

- **Additive distortion**
  - \( sw(x, \bar{u}) = \left( \frac{1}{n} \right) \cdot \sum_{i \in N} u_i(x) \)
  - \( dist^+(x, \succeq) = \max_{\bar{u} \succ \succeq} \left[ \max_{a \in A} sw(a, \bar{u}) - sw(x, \bar{u}) \right] \)

- **Theorem** [Caragiannis, Nath, Procaccia, Shah, 2017]:
  - For deterministic rules, the optimal additive distortion is \( \frac{1}{2} \).
  - For randomized rules, the optimal additive distortion is between \( \frac{1}{4} \) and \( \frac{1}{2} \cdot \left( 1 - \frac{1}{m^2} \right) \).

- **Theorem** [Kahng, Kehne, 2022]:
  - For randomized rules, the optimal additive distortion is between \( \frac{5}{18} \) and \( \frac{11}{27} \).

- **Open question**: Close the gap for randomized rules
Other Objective Functions

- If we knew the utility profile $\mathbf{u}$:
  - Efficiency would ask us to select $x^* \in \arg\max_x sw(x, \mathbf{u})$
  - What about fairness? Particularly attractive in budget division.

- **Proportional Fairness**: $PF(x, \mathbf{u}) = \sup_y \frac{1}{n} \sum_i \frac{u_i(y)}{u_i(x)}$
  - Average % change in utilities when moving to any other distribution $y$
  - **Folklore**: If we knew $\mathbf{u}$, choosing $x^* \in \arg\max_x \prod_i u_i(x)$ would guarantee $PF(x^*, \mathbf{u}) = 1$
    - Optimal, consider $y = x$
  - **Folklore**: $PF = \alpha$ implies $\alpha$-approximation to the core
    - Any subgroup of $x$ % of voters cannot find an $\alpha$ factor Pareto improvement over $x$ by allocating $x$ % of the probability mass (or budget), for any $x$

- **Theorem** [Ebadian, Kahng, Peters, Shah, 2022]:
  - The optimal randomized rule achieves $\Theta(\log m)$ proportional fairness.

- **Open question**: Can the core approximation be improved to a constant?
Other Utility Classes

• Unit range utilities:
  • $u_i(a) \in [0,1]$ for all $a \in A$, $\max_a u_i(a) = 1$, $\min_a u_i(a) = 0$

• Theorem [Ebadian, Kahng, Peters, Shah, 2022]:
  • With respect to unit range utilities:
    • The distortion of harmonic rule increases to $O(m^{2/3} \cdot \log^{1/3} m)$
    • The distortion of stable lottery rule remains $O(\sqrt{m})$
    • Every randomized rule has distortion $\Omega(\sqrt{m})$
Incentives

• **Strategyproofness**
  • A randomized rule is strategyproof if a voter cannot increase her expected utility by misreporting her preference ranking in any instance.

• **Theorem** [Bhaskar, Dani, Ghosh, 2018]:
  • With respect to unit-sum utilities, the best distortion subject to strategyproofness is $\Theta(\sqrt{m \cdot \log m})$.
  • Upper bound is achieved by harmonic rule, which is strategyproof.

• **Theorem** [Filos-Ratsikas, Bro Miltersen, 2014; Lee 2019]:
  • With respect to unit-range utilities, the best distortion subject to strategyproofness is $\Theta(m^{2/3})$.
  • **Note**: This explains why the distortion of harmonic rule, which is strategyproof, increases to $\tilde{O}(m^{2/3})$ for unit-range utilities
    • Harmonic rule achieves near-optimal distortion subject to strategyproofness with respect to both unit-sum and unit-range utilities!
Committee Selection

• **Goal:** Select a set of alternatives of given size $k$
  • Representation utilities: $u_i(S) = \max_{a \in S} u_i(a)$
  • A priori, it is not clear if the best possible distortion increases or decreases with $k$

• **Theorem** [Caragiannis, Nath, Procaccia, Shah, 2017]
  • The optimal distortion of deterministic rules is $\Theta\left(1 + \frac{m \cdot (m-k)}{k}\right)$.
  • Optimal distortion of randomized rules:
    • Upper bound not monotone in $k$
    • Left an $m^{1/6}$ gap

![Distortion graph](image)
Committee Selection

**Stable Lottery Rule for Committees**

- If $k \leq \sqrt{m}$:
  - W.p. $\frac{1}{2}$, find a stable lottery $S$ over sets of size $k \cdot \sqrt{m}$, sample $S \sim S$, and choose $S' \subseteq S$ of size $|S'| = k$ uniformly at random
  - W.p. $\frac{1}{2}$, choose $S \subseteq A$ of size $|S| = k$ uniformly at random
- If $k \geq \sqrt{m}$
  - Choose $S \subseteq A$ of size $|S| = k$ uniformly at random

**Theorem** [Borodin, Halpern, Latifian, Shah, ‘22]:
- Among randomized rules, the stable lottery rule for committees of size $k$ achieves the optimal distortion of $\Theta \left( \min \left( \sqrt{m}, \frac{m}{k} \right) \right)$

**Corollary:**
- The best possible distortion (asymptotically) weakly decreases in $k$
Other Ballot Formats

• **Top-t preferences** (less information than ranked ballots)
  - Each voter ranks her $t$ most favorite alternatives

• **Theorem** [Borodin, Halpern, Latifian, Shah, ‘22]:
  - Stable lottery rule for committees has distortion $O\left(\min\left(\max\left(\sqrt{m}, \frac{m}{t}\right), \frac{m}{k}\right)\right)$
  - Apply the rule after arbitrarily completing partial preferences to ranked ballots!
  - Every randomized voting rule has distortion $\Omega\left(\min\left(\max\left(\sqrt{m}, \frac{m}{k\cdot t}\right), \frac{m}{k}\right)\right)$
  - Open question: Close this gap!

• **Corollary:**
  - For $k = 1$ (single-winner), the bound is $\Theta\left(\max\left(\sqrt{m}, \frac{m}{t}\right)\right)$
  - Optimal $O(\sqrt{m})$ distortion is already achieved at $t = \sqrt{m}$
    - So only ask voters to rank their top $\sqrt{m}$ alternatives!
  - For deterministic rules, $t = 1$ gives optimal $\Theta(m^2)$ distortion
Other Ballot Formats

• **Ranked ballots + additional queries** (more information than ranked ballots)
  • Value query: What is $u_i(a)$?
  • Comparison query: Is $u_i(a) \geq \alpha \cdot u_i(b)$?
  • We measure the number of queries *per voter*

• **Theorem** [Amanatidis, Birmpas, Filos-Ratsikas, Voudouris, 2021]:
  • For any $k$, it is possible to achieve distortion $O\left(\frac{k+1}{\sqrt{m}}\right)$ with $O(k \cdot \log m)$ value queries
  • It is possible to achieve $O(1)$ distortion using $O(\log^2 m)$ comparison queries
  • The best distortion with $\lambda$ value queries is $\Omega\left(\frac{1}{\lambda+1} \cdot \frac{1}{m^{2(\lambda+1)}}\right)$
  • ...

• **Many open questions:**
  • E.g., $O(1)$ distortion with $O(\log m)$ value queries?
Utilitarian Voting with Ranked Ballots

Vote Aggregation Rule

= 0.7
= 0.2
= 0.1
= 0.4
= 0.3
= 0.3
= 0.5
= 0.5
= 0.0
Utilitarian Voting with Generic Ballots

\[
\begin{align*}
\rho_1 &= 0.7 \\
\rho_2 &= 0.2 \\
\rho_3 &= 0.1
\end{align*}
\]
# Examples of Ballots

## Ranked Ballot

<table>
<thead>
<tr>
<th>Ranked Ballot</th>
<th>1&lt;sup&gt;st&lt;/sup&gt;</th>
<th>2&lt;sup&gt;nd&lt;/sup&gt;</th>
<th>3&lt;sup&gt;rd&lt;/sup&gt;</th>
<th>4&lt;sup&gt;th&lt;/sup&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td>⬤</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>⬤</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td></td>
<td></td>
<td>⬤</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td></td>
<td>⬤</td>
</tr>
</tbody>
</table>

## Top-<i>t</i> Ballot

<table>
<thead>
<tr>
<th>Top-&lt;i&gt;t&lt;/i&gt; Ballot</th>
<th>1&lt;sup&gt;st&lt;/sup&gt;</th>
<th>2&lt;sup&gt;nd&lt;/sup&gt;</th>
<th>3&lt;sup&gt;rd&lt;/sup&gt;</th>
<th>4&lt;sup&gt;th&lt;/sup&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td>⬤</td>
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<td></td>
</tr>
<tr>
<td>B</td>
<td>⬤</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td></td>
<td></td>
<td>⬤</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
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<td>⬤</td>
</tr>
</tbody>
</table>

## Range Voting

<table>
<thead>
<tr>
<th>Range Voting</th>
<th>1 (Worst)</th>
<th>2</th>
<th>3</th>
<th>4 (Best)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td></td>
<td>⬤</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td></td>
<td></td>
<td>⬤</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>⬤</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td>⬤</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

## Approval Ballot

<table>
<thead>
<tr>
<th>Approval Ballot</th>
<th>1&lt;sup&gt;st&lt;/sup&gt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>⬤</td>
</tr>
<tr>
<td>B</td>
<td>⬤</td>
</tr>
<tr>
<td>C</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
</tr>
</tbody>
</table>
Optimal Voting with Optimal Ballot Design

• Tradeoff

<table>
<thead>
<tr>
<th>Distortion</th>
<th>Communication</th>
</tr>
</thead>
</table>
| Lowest distortion allowed by the ballot design when using its best aggregation rule | “Expressiveness” / “cognitive difficulty” imposed
| Crude measure: #bits communicated by each voter |

How many bits of information does each voter need to communicate for us to achieve distortion $d$?
Theorem [Mandal, Procaccia, Shah, Woodruff, 2019; Mandal, Shah, Woodruff, 2020]

For any $d$, the optimal ballot (combined with its optimal randomized aggregation rule) elicits the following number of bits of information from each voter to achieve distortion $d$:

- Deterministic ballot: $\tilde{\Theta}(m/kd)$
- Randomized ballot: $\tilde{\Theta}(m/ka^3)$

Comparison to ranked ballots

- Ranked ballots achieve $d = \Theta(\min(\sqrt{m}, m/k))$ distortion by eliciting $\Theta(m \cdot \log m)$ bits
- Optimal ballot achieves $d = O(1)$ distortion already by eliciting only $\tilde{\Theta}(m/k)$ bits

Optimal Voting with Optimal Ballot Design
Participatory Budgeting

- Ranking by value

- Ranking by VFM

- Knapsack voting (budget = 4)

- Threshold approval (threshold = 3)

[Benade, Procaccia, Nath, Shah, 2021]
Participatory Budgeting

• Additive utilities
  • \( u_i(S) = \sum_{a \in S} u_i(a) \)
  • Previously mentioned results were for representation utilities: \( u_i(S) = \max_{a \in S} u_i(a) \)

• Theorem [Benade, Nath, Procaccia, Shah, 2017]:
  • The best possible distortion using randomized aggregation rule is as follows:
    • Knapsack ballot: \( \Theta(m) \)
    • Ranking by value: \( \tilde{\Theta}(\sqrt{m}) \)
    • Ranking by VFM: \( \tilde{\Theta}(\sqrt{m}) \)
    • Threshold approval votes: \( O(\log^2 m), \Omega\left(\frac{\log m}{\log \log m}\right) \)
Social Welfare Functions

- **Output**: a ranking of the alternatives $\succ^*$
  - How do we define the utility of a voter for a ranking?
  - Each voter $i$ has non-increasing weights $w_{i,j}$ such that $w_{i,j} \geq 0$ for all $j$ and $\sum_{j=1}^{m} w_{i,j} = 1$
    - $w_{i,j}$ = how much voter $i$ cares about which alternative gets ranked $j^{th}$ in $\succ^*$
    - $u_i(\succ^*) = \sum_{j=1}^{m} w_{i,j} \cdot u_i(a_j)$, where $a_j$ is the $j^{th}$ ranked alternative in $\succ^*$
  - Distortion $\rightarrow$ worst case over the choice of both voter utilities and voter weights
    - Strictly harder than single-winner selection ($w_{i,1} = 1$)

- **Theorem** [Benade, Procaccia, Qiao, 2019]:
  - The best distortion of any randomized social welfare function is $O(\sqrt{m \cdot \log^3 m})$.
  - Only polylogarithmically higher than single-winner selection!
Many, Many Open Questions

• Combining extensions
  • Strategyproofness +
    • Nash welfare distortion, additive distortion, other ballots, committee selection, ...
  • Committee selection or participatory budgeting +
    • Nash welfare distortion, additive distortion, ...
  • Unit-range utilities +
    • Additive distortion, other ballots, committee selection, participatory budgeting, ...
  • Social welfare functions?
  • ...

• Social welfare functions?
Outline

• Introduction
  • Applications of voting
  • Motivating the distortion framework

• Utilitarian distortion framework
  • Model
  • Known results

• Metric distortion framework
  • Model
  • Known results

• Applications beyond voting
Metric Distortion

Assess quality using the underlying metric

Metric Space

Voters

Preferences

Voting Rule

Winner

A

[Anshelevich, Bhardwaj, Elkind, Postl, Skowron, 2018]
Why The Metric?
Why The Metric?

2D Models

3D Models

Popular Tools
Metric Distortion

1. There exists an underlying metric $d$ over voters and alternatives such that:
   - Consistency (denoted $d \triangleright \succ$): $\forall a, b : a \succ_i b \Rightarrow d(i, a) \leq d(i, b)$
   - Triangle inequality: $\forall x, y, z, d(x, y) + d(y, z) \geq d(x, z)$
   - Linear extension to distributions: For $x \in \Delta(A), c_i(x) = d(i, x) = \sum_a d(i, a) \cdot x(a)$

2. If we knew the costs, we would minimize the social cost
   - $sc(x, d) = \sum_{i \in N} d(i, x)$

3. Because this is impossible given the limited ranked information, we want to best approximate the social cost in the worst case.
Metric Distortion

• Distortion

\[
\text{dist}(x, \succ) = \sup_{d \succ} \frac{sc(x, d)}{\min_{a \in A} sc(a, d)}
\]

• Given voting rule \(f\)

\[
\text{dist}(f) = \max_{\succ} \text{dist}(f(\succ), \succ)
\]

What is the lowest possible distortion of deterministic and randomized rules? Which voting rules achieves it?
• A simple lower bound of 3 (deterministic rules) with just two candidates

\[ \frac{n}{2} \text{ voters} : a > b \]

Rule selects w.l.o.g.

\[ \frac{n}{2} \text{ voters} : b > a \]

Bad instance

\[ sc(a, d) = 1 \frac{n}{2} + 2 \frac{n}{2} = 3 \frac{n}{2} \]

\[ sc(b, d) = 1 \frac{n}{2} + 0 \frac{n}{2} = \frac{n}{2} \]

\[ \text{distortion} \geq \frac{sc(a,d)}{sc(b,d)} \geq 3 \]

Can a deterministic rule achieve distortion 3?
Deterministic Rules

• **Theorem** [Anshelevich, Bhardwaj, Elkind, Postl, Skowron, 2018]:

<table>
<thead>
<tr>
<th>Rule</th>
<th>Distortion</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )-approval (( k &gt; 2 ))</td>
<td>Unbounded</td>
</tr>
<tr>
<td>Plurality, Borda count</td>
<td>( \Theta(m) )</td>
</tr>
<tr>
<td>Harmonic rule*</td>
<td>( O\left(\frac{m}{\sqrt{\log m}}\right), \Omega\left(\frac{m}{\log m}\right) )</td>
</tr>
<tr>
<td>Best positional scoring rule</td>
<td>( \Omega\left(\sqrt{\log m}\right) )</td>
</tr>
<tr>
<td>STV</td>
<td>( O\left(\log m\right), \Omega\left(\sqrt{\log m}\right) )</td>
</tr>
<tr>
<td>Copeland’s rule</td>
<td>5</td>
</tr>
<tr>
<td>Best deterministic rule</td>
<td>( \geq 3 )</td>
</tr>
</tbody>
</table>

*Deterministic version of the harmonic rule, which simply picks an alternative with the largest harmonic score*

• The instance-optimal deterministic rule can be computed in polynomial time by solving a number of linear programs.

• **Open question:** What is the best distortion achievable by any positional scoring rule?
Copeland’s Rule

• **Lemma** [Kempe 2020b]:
  - If \((a_1, a_2, \ldots, a_\ell)\) is a sequence of alternatives such that a (weak) majority of voters prefer \(a_i\) to \(a_{i+1}\) for each \(i = 1, \ldots, \ell - 1\), then \(sc(a_1, d) \leq (2\ell - 1) \cdot sc(a_\ell, d)\) for every metric \(d\) consistent with the preference profile.

• **Corollary:**
  - It is known that Copeland’s winner is in the uncovered set:
    - If \(a_1\) is Copeland’s winner, then for every other alternative \(a\), either sequence \((a_1, a)\) or \((a_1, a_2, a)\) for some \(a_2\) satisfies the condition above.
  - This explains distortion 5 of Copeland’s rule
  - Lemma quite powerful, later used by [Anagnostides, Fotakis, Patsilinakos, 2021]

• **Copeland’s rule is Condorcet consistent**
  - [Anshelevich, Bhardwaj, Elkind, Postl, Skowron, 2018]: Any voting rule can be made Condorcet consistent without losing distortion because the Condorcet winner is always a 3-approximation
Deterministic Rules

• **Theorem** [Kempe 2020a]:
  - The distortion of ranked pairs and Schulze’s rule is $\Theta(\sqrt{m})$.
  - Analysis via a powerful LP duality approach

• **Theorem** [Munagala, Wang, 2019]:
  - There exists a deterministic voting rule with distortion $2 + \sqrt{5} \approx 4.236$.

• **Theorem** [Gkatzelis, Halpern, Shah, 2020]:
  - There exists a deterministic voting rule, PluralityMatching, with distortion 3.
  - Proof by confirming a conjecture by [Munagala, Wang, 2019]

• **Theorem** [Kizilkaya, Kempe, 2022]:
  - There exists a deterministic voting rule, Plurality Veto, with distortion 3.
  - Proof by confirming a conjecture by [Munagala, Wang, 2019] in a 1-paragraph proof
Domination Graph of Candidate $\alpha$

Certificate that $\alpha$ is a good choice:
we can match each voter $j$ (with top choice $x$) to another voter $i = M(j)$ with $\alpha \succeq_i x$.

Edge $(i, j)$ exists when, in $i$’s vote, $\alpha$ weakly defeats the top choice of $j$.

Perfect Matching
Perfect Matching Gives Distortion 3

- **Lemma** [Munagala, Wang, 2019; Kempe 2020a]
  - If the domination graph of $a$ has a perfect matching, then $a$ has distortion at most 3.
  - Conjecture: For every profile, at least one candidate’s graph has a perfect matching.

- **Proof (skip):**
  \[
  SC(a) = \sum_{i \in V} d(i, a) \\
  \leq \sum_{i \in V} d(i, \text{top}(M(i))) \\
  \leq \sum_{i \in V} (d(i, b) + d(b, \text{top}(M(i)))) \\
  = \sum_{i \in V} (d(i, b) + d(b, \text{top}(i))) \\
  \leq \sum_{i \in V} (d(i, b) + d(b, i) + d(i, \text{top}(i))) \\
  \leq \sum_{i \in V} (d(i, b) + d(b, i) + d(i, b)) \\
  = 3 \cdot SC(b).
  \]
Plurality Veto

• Simple voting rule that selects a candidate with a perfect matching in the domination graph. [Kizilkaya, Kempe, 2022]
  • All alternatives start out being alive. Each voter $i$ gives 1 point to $i$’s top alternative.
  • Go through voters 1-by-1 in an arbitrary order.
  • Each voter $i$ subtracts 1 point from $i$’s least-favorite alive alternative. If that alternative’s score drops to 0, it dies.
  • The alternative $a$ surviving until the last round wins.

• Only two queries per voter!

• Note: there are $n$ points in total, and we take $n$ points away.

• In the domination graph of $a$:
  • For each $x$, we can match the $t$ voters who rank $x$ top with the $t$ voters who delete a point from $x$ during the execution of the rule.
  • For each such voter, $a \succeq_i x$ because $a$ is alive.
Randomized Rules

- **Theorem** [Anshelevich, Bhardwaj, Elkind, Postl, Skowron, 2018]:
  - No randomized rule has distortion better than 2.
    - Same example as before
    - Random Dictatorship has distortion $3 - \frac{2}{n}$.

- **Theorem** [Kempe 2020a]:
  - There is a randomized voting rule with access only to top choices with distortion $3 - \frac{2}{m}$.

- **Theorem** [Charikar, Ramakrishnan, 2022; Pulyassary, Swamy, 2021]:
  - No randomized rule has distortion better than 2.1126 for all $m$.
    - Weaker lower bounds for fixed, finite $m$

- **Open question**: What is the optimal metric distortion of randomized rules?
- **Open question**: Is the instance-optimal randomized rule polytime computable?
Extensions

- Other objective functions
- Ballot formats other than ranked ballots
- Committee selection
- Information-distortion tradeoff
Other Objective Functions

- **Bounding higher moments of distortion** [Fain, Goel, Munagala, Sakshuwong, 2017; Fain, Goel, Munagala, Prabhu, 2019; Fain, Fan, Munagala, 2020]
  - $k^{th}$ moment
    \[
    \text{dist}^k(x, \succ) = \sup_{d \succeq \succ} \left( \frac{\mathbb{E}_{a \sim x} \text{sc}(a, d)^k}{\min_{a^* \in A} \text{sc}(a^*, d)} \right)^{1/k}
    \]

- **Motivation:**
  - Bounding, e.g., the $2^{nd}$ moment ("squared distortion") bounds not only the expectation of the social cost approximation ratio, but also its variance
  - Filters out rules like Random Dictatorship that achieve terrible social cost with low probability
    - Unbounded squared distortion [Fain, Goel, Munagala, Sakshuwong, 2017]
  - By Markov’s inequality, one can obtain high-probability bounds on social cost approximation
  - By Jensen’s inequality, any upper bound on $\text{dist}^k$ is also an upper bound on $\text{dist}$

- **Open question:** What is the optimal $k^{th}$ moment distortion of randomized rules?
Other Ballot Formats

• **Top-\(t\) ballots**
  - Each voter ranks her \(t\) most favorite alternatives
  - \(t = 1 \Rightarrow\) Plurality is optimal with distortion \(2m - 1\)
  - \(t = m - 1 \Rightarrow\) Plurality Matching is optimal with distortion 3

• **Theorem** [Kempe 2020a, Kempe 2020b]:
  - The distortion of the optimal deterministic rule for top-\(t\) ballots is between \(\frac{2m}{t} - 1\) and \(\frac{12m}{t}\).

• **Theorem** [Anagnostides, Fotakis, Patsilinakos, 2021]:
  - The upper bound can be improved to \(\frac{6m}{t}\).

• **Open question**: Close the gaps!
Other Ballot Formats

• Top-$t$ ballots
  • Each voter ranks her $t$ most favorite alternatives
  • $t = 1 \Rightarrow$ Plurality is optimal with distortion $2m - 1$
  • $t = m - 1 \Rightarrow$ PluralityMatching is optimal with distortion 3

• Theorem [Gross, Anshelevich, Xia, 2017]:
  • The distortion of the optimal randomized rule for top-$t$ ballots is at least $3 - 2/\lceil m/t \rceil$ when $t \leq m/2$ and at least 2 when $t \geq m/2$.

• Open question: Design randomized rules with matching upper bounds!
Other Ballot Formats

• More information than ranked ballots
  • $\alpha$-decisive metric spaces (where $\alpha \in [0,1]$) [Anshelevich, Postl, 2016]:
    • Each voter’s distance to her top choice is at most $\alpha$ times her distance to her 2nd choice
    • $\alpha = 1$ provides no additional information
    • $\alpha = 0$ means every voter is co-located with her top choice

• Theorem [Gkatzelis, Halpern, Shah, 2020]:
  • Deterministic: No rule has distortion better than $\sim 2 + \alpha - \frac{\alpha}{m}$ while PluralityMatching has distortion $2 + \alpha$.
  • Randomized: No rule has distortion better than $\sim \frac{(3+\alpha)}{2} - \frac{(1-\alpha)}{m}$ while there exists a randomized rule (using only plurality votes) with distortion $2 + \alpha - \frac{\alpha}{m}$.

• Other types of extra information
  • “Voter passion” [Abramowitz, Anshelevich, Zhu, 2019]
  • Locations of alternatives known [Chen, Li, Wang, 2020; Anshelevich, Zhu, 2021]
Committee Selection

• Voter costs for committees:
  • Additive costs: \( c_i(S) = \sum_{a \in S} d(i, a) \)
  • \( q \)-costs: \( c_i(S) = \min_{a \in S} d(i, a) \)

• Theorem [Goel, Hulett, Krishnaswamy, 2018]:
  • Under additive costs, applying a single-winner rule with distortion \( d \) recursively to choose a committee of size \( k \) achieves distortion at most \( d \).

• Theorem [Caragiannis, Shah, Voudouris, 2022]:
  • Under \( q \)-costs, the optimal distortion of deterministic rules follows a trichotomy:
    • \( q \in [1, \frac{k}{3}] \) : \( \infty \)
    • \( q \in (\frac{k}{3}, \frac{k}{2}] \) : \( \Theta(n) \)
    • \( q \in (\frac{k}{2}, k] \) : 3
  • Open question: For \( q > \frac{k}{2} \), what distortion can be achieved in polynomial time?
    • Current best is 9
Many, Many Open Questions

• Extensions for metric distortion less-studied than for utilitarian distortion
  • Participatory budgeting?
  • Strategyproofness?
  • Ranked ballots + additional queries?
  • Information-distortion tradeoff? [Kempe 2020a]
  • ...

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• Metric distortion framework
  • Model
  • Known results

• Applications beyond voting
Actually, More Voting First!

• Distributed elections
  • Voters partitioned into groups that conduct separate elections [Borodin, Lev, Shah, Strangway, 2019; Filos-Ratsikas, Micha, Voudouris, 2020; Filos-Ratsikas, Voudouris, 2021; Anshelevich, Filos-Ratsikas, Voudouris, 2022]

• Representative candidates
  • Alternatives sampled from the pool of voters [Cheng, Dughmi, Kempe, 2017; Cheng, Dughmi, Kempe, 2018]

• Voter abstentions
  • What if only a fraction of the voters vote? [Borodin, Lev, Shah, Strangway, 2019; Seddighin, Latifian, Ghodsi, 2021; Anagnostides, Fotakis, Patsilinakos, 2021]

• Approval-based cost functions for metric distortion [Pierczynski, Skowron, 2019]
Beyond Voting

• One-Sided Matching
  • Match $m$ agents to $m$ items, where agents have cardinal utilities for the items but only provide ordinal rankings

• Theorem [Filos-Ratsikas, Frederiksen, Zhang, 2014]:
  • The best distortion of any randomized rule is $\Theta(\sqrt{m})$.

• Theorem [Amanatidis, Birmpas, Filos-Ratsikas, Voudouris, 2021]:
  • The best distortion of any deterministic rule is $\Theta(m^2)$.
  • They also analyze the information-distortion tradeoff via queries.

• Surprisingly, identical bounds as single-winner voting!

• Other work [Ma, Menon, Larson, 2021; Bishop, Chan, Mandal, Tran-Thanh, 2022]
Beyond Voting

• Resource allocation
  • Allocate \( m \) goods to \( n \) agents
  • [Halpern, Shah, 2021]: When every agent ranks the goods
  • [Ebadian, Freeman, Shah, 2022]: When \( k \) agents provide no information while the rest provide cardinal utilities

• Secretary problem [Hoefer, Kodric, 2017]

• Graph-theoretic problems
  • Maximum-weight matching [Anshelevich, Sekar, 2016a]
  • Max \( k \)-sum, densest \( k \)-subgraph, maximum traveling salesman [Anshelevich, Sekar, 2016b]
  • Min-weight and max-min bipartite matching, facility location, \( k \)-center, \( k \)-median [Filos-Ratsikas, Voudouris, 2021; Anshelevich, Zhu, 2021]
Future Work: Ballot Design

• Common ballot designs
  • Pairwise comparisons, “Do you like candidate \( a \) at least twice as much as candidate \( b \)?”, ...

• Better models of cognitive burden
  • Psychology, HCI, ...

• Voter errors in answering ballots
  • Expressive ballots can also induce errors

• Intangible aspects of ballot design
  • Barcelona PB team: “Knapsack votes are good because they help voters understand the limitations of the budget.”
Future Work: Distortion vs Other Desiderata

• Distortion & Truthfulness
  • With ranked ballots, near-optimal distortion can be achieved via truthful aggregation
  • What happens with other ballot formats?

• Distortion & Axioms
  • Can we achieve low distortion together with popular axioms?
  • Especially, proportional representation for committee selection

• Distortion & Explainability
  • Explaining the voting rule vs explaining what it does
Future Work: More Complex Voting Paradigms

- Design optimal voting rules for more complex voting paradigms
  - Participatory budgeting
  - Districting

- Model end-to-end voting
  - In participatory budgeting, voting is but the final step of a year-long process

- Compare models of democracy
  - E.g., direct democracy, representative democracy, and liquid democracy
AI-Driven Decisions

RoboVote is a free service that helps users combine their preferences or opinions into optimal decisions. To do so, RoboVote employs state-of-the-art voting methods developed in artificial intelligence research.

Learn More

Poll Types

RoboVote offers two types of polls, which are tailored to different scenarios; it is up to users to indicate to RoboVote which scenario best fits the problem at hand.

Objective Opinions

In this scenario, some alternatives are objectively better than others, and the opinion of a participant reflects an attempt to estimate the correct order. RoboVote’s proposed outcome is guaranteed to be as close as possible — based on the available information — to the best outcome. Examples include deciding which product prototype to develop, or which company to invest in, based on a metric such as projected revenue or market share.

Subjective Preferences

In this scenario participants’ preferences reflect their subjective taste; RoboVote proposes an outcome that mathematically makes participants as happy as possible overall. Common examples include deciding which restaurant or movie to go to as a group.

Ready to get started?

CREATE A POLL
• Abramowitz, B., Anshelevich, E. and Zhu, W. Awareness of voter passion greatly improves the distortion of metric social choice. WINE, pp. 3-16, 2019.


• Anshelevich, E. and Sekar, S. Truthful mechanisms for matching and clustering in an ordinal world. WINE, pp. 265-278, 2016b.
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References


• Caragiannis, I. and Procaccia, A. D. Voting almost maximizes social welfare despite limited communication. AIJ, 175(9-10), pp. 1655-1671, 2011.


References


References

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References


Thank you!

Questions?