CSC373

Week 7:
Linear Programming

Illustration Courtesy:
Kevin Wayne & Denis Pankratov
Recap

• **Network flow**
  - Ford-Fulkerson algorithm
    - Ways to make the running time polynomial
  - Correctness using max-flow, min-cut
  - Applications:
    - Edge-disjoint paths
    - Multiple sources/sinks
    - Circulation
    - Circulation with lower bounds
    - Survey design
    - Image segmentation
    - Profit maximization
Brewery Example

• A brewery can invest its inventory of corn, hops and malt into producing some amount of ale and some amount of beer
  ➢ Per unit resource requirement and profit of the two items are as given below

<table>
<thead>
<tr>
<th>Beverage</th>
<th>Corn (pounds)</th>
<th>Hops (ounces)</th>
<th>Malt (pounds)</th>
<th>Profit ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ale (barrel)</td>
<td>5</td>
<td>4</td>
<td>35</td>
<td>13</td>
</tr>
<tr>
<td>Beer (barrel)</td>
<td>15</td>
<td>4</td>
<td>20</td>
<td>23</td>
</tr>
<tr>
<td>constraint</td>
<td>480</td>
<td>160</td>
<td>1190</td>
<td></td>
</tr>
</tbody>
</table>

Example Courtesy: Kevin Wayne
Brewery Example

- Suppose it produces $A$ units of ale and $B$ units of beer.
- Then we want to solve this program:
Linear Function

- \( f: \mathbb{R}^n \rightarrow \mathbb{R} \) is a linear function if \( f(x) = a^T x \) for some \( a \in \mathbb{R}^n \)
  - Example: \( f(x_1, x_2) = 3x_1 - 5x_2 = (3 \ -5)^T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \)

- Linear objective: \( f \)

- Linear constraints:
  - \( g(x) = c \), where \( g: \mathbb{R}^n \rightarrow \mathbb{R} \) is a linear function and \( c \in \mathbb{R} \)
  - Line in the plane (or a hyperplane in \( \mathbb{R}^n \))
  - Example: \( 5x_1 + 7x_2 = 10 \)
Linear Function

• Geometrically, $a$ is the normal vector of the line (or hyperplane) represented by $a^T x = c$
Linear Inequality

- $a^T x \leq c$ represents a “half-space”
Linear Programming

• Maximize/minimize a linear function subject to linear equality/inequality constraints

Objective function

\[ \text{max } x_1 + 6x_2 \]

Constraints

\[ x_1 \leq 200 \]
\[ x_2 \leq 300 \]
\[ x_1 + x_2 \leq 400 \]
\[ x_1, x_2 \geq 0 \]

Could be min

Linear objective!

Linear constraints: inequalities/equalities
Geometrically...

Objective function: \( \max x_1 + 6x_2 \)

Constraints:
\[
\begin{align*}
x_1 & \leq 200 \\
x_2 & \leq 300 \\
x_1 + x_2 & \leq 400 \\
x_1, x_2 & \geq 0
\end{align*}
\]

Any point here is a feasible solution.

Feasible region – polytope, aka intersection of half-spaces!
Back to Brewery Example

Hops
4A + 4B ≤ 160

Malt
35A + 20B ≤ 1190

(0, 32)

(12, 28)

(26, 14)

(34, 0)

Corn
5A + 15B ≤ 480
Back to Brewery Example

\[ 13A + 23B = $1600 \]
\[ 13A + 23B = $800 \]
\[ 13A + 23B = $442 \]
Optimal Solution At A Vertex

• **Claim:** Regardless of the objective function, there must be a vertex that is an optimal solution.
Convexity

- **Convex set**: $S$ is convex if $x, y \in S, \lambda \in [0,1] \Rightarrow \lambda x + (1 - \lambda)y \in S$

- **Vertex**: A point which cannot be written as a strict convex combination of any two points in the set

- **Observation**: Feasible region of an LP is a convex set
Optimal Solution At A Vertex

- Intuitive proof of the claim:
  - Start at some point $x$ in the feasible region
  - If $x$ is not a vertex:
    - Find a direction $d$ such that points within a positive distance of $\epsilon$ from $x$ in both $d$ and $-d$ directions are within the feasible region
    - Objective must *not decrease* in at least one of the two directions
    - Follow that direction until you reach a new point $x'$ for which at least one more constraint is “tight”
  - Repeat until we are at a vertex
LP, Standard Formulation

• Input: $c, a_1, a_2, \ldots, a_m \in \mathbb{R}^n, b \in \mathbb{R}^m$
  ➢ There are $n$ variables and $m$ constraints

• Goal:

Maximize $c^T x$
Subject to $a_1^T x \leq b_1$
$\quad a_2^T x \leq b_2$
$\quad \vdots$
$\quad a_m^T x \leq b_m$

$x \geq 0$

$n$ variables

$m$ constraints

$n$ more constraints
LP, Standard Matrix Form

- **Input:** $c, a_1, a_2, \ldots, a_m \in \mathbb{R}^n, b \in \mathbb{R}^m$
  - There are $n$ variables and $m$ constraints
- **Goal:**

  - Maximize $c^T x$
  - Subject to $Ax \leq b$
  - $x \geq 0$

  - $n$ variables
  - $m$ constraints
  - $n$ more constraints
Convert to Standard Form

• What if the LP is not in standard form?
  ➢ Constraints that use $\geq$
    o $a^T x \geq b \iff -a^T x \leq -b$
  ➢ Constraints that use equality
    o $a^T x = b \iff a^T x \leq b, a^T x \geq b$
  ➢ Objective function is a minimization
    o Minimize $c^T x \iff$ Maximize $-c^T x$
  ➢ Variable is unconstrained
    o $x$ with no constraint $\iff$ Replace $x$ by two variables $x'$ and $x''$, replace every occurrence of $x$ with $x' - x''$, and add constraints $x' \geq 0, x'' \geq 0$
LP Transformation Example

minimize $-2x_1 + 3x_2$
subject to
\[ x_1 + x_2 = 7 \]
\[ x_1 - 2x_2 \leq 4 \]
\[ x_1 \geq 0 . \]

maximize $2x_1 - 3x_2$
subject to
\[ x_1 + x_2 = 7 \]
\[ x_1 - 2x_2 \leq 4 \]
\[ x_1 \geq 0 . \]

maximize $2x_1 - 3x'_2 + 3x''_2$
subject to
\[ x_1 + x'_2 - x''_2 = 7 \]
\[ x_1 - 2x'_2 + 2x''_2 \leq 4 \]
\[ x_1, x'_2, x''_2 \geq 0 . \]
Optimal Solution

• Does an LP always have an optimal solution?

• No! The LP can “fail” for two reasons:

  1. It is *infeasible*, i.e., \( \{ x \mid Ax \leq b \} = \emptyset \)
    
    o E.g., the set of constraints is \( \{ x_1 \leq 1, -x_1 \leq -2 \} \)

  2. It is *unbounded*, i.e., the objective function can be made arbitrarily large (for maximization) or small (for minimization)
    
    o E.g., “maximize \( x_1 \) subject to \( x_1 \geq 0 \)”

• But if the LP has an optimal solution, we know that there must be a vertex which is optimal
Simplex Algorithm

let \( v \) be any vertex of the feasible region
while there is a neighbor \( v' \) of \( v \) with better objective value:
set \( v = v' \)

- Simple algorithm, easy to specify geometrically
- Worst-case running time is exponential
- Excellent performance in practice
Simplex: Geometric View

let $v$ be any vertex of the feasible region
while there is a neighbor $v'$ of $v$ with better objective value:
set $v = v'$

max $x_1 + 6x_2$
$x_1 \leq 200$
$x_2 \leq 300$
$x_1 + x_2 \leq 400$
$x_1, x_2 \geq 0$
Algorithmic Implementation

Start at a vertex of feasible polytope

Is there a neighbor vertex with better objective value?

Move to a neighbor vertex with better objective value

Terminate, declare the current solution and value as optimal
How Do We Implement This?

• We’ll work with the slack form of LP
  ➢ Convenient for implementing simplex operations
  ➢ We want to maximize $z$ in the slack form, but for now, forget about the maximization objective

Standard form:

Maximize $c^T x$
Subject to $Ax \leq b$
$x \geq 0$

Slack form:

$z = c^T x$
$s = b - Ax$
$s, x \geq 0$
Slack Form

maximize \[ 2x_1 - 3x_2 + 3x_3 \]
subject to
\[
\begin{align*}
x_1 + x_2 - x_3 & \leq 7 \\
-x_1 - x_2 + x_3 & \leq -7 \\
x_1 - 2x_2 + 2x_3 & \leq 4 \\
x_1, x_2, x_3 & \geq 0.
\end{align*}
\]

Nonbasic Variables

Basic Variables

\[
\begin{align*}
x_4 &= 7 - x_1 - x_2 + x_3 \\
x_5 &= -7 + x_1 + x_2 - x_3 \\
x_6 &= 4 - x_1 + 2x_2 - 2x_3 \\
x_1, x_2, x_3, x_4, x_5, x_6 & \geq 0.
\end{align*}
\]
Slack Form

\[
\begin{align*}
    z &= 2x_1 - 3x_2 + 3x_3 \\
    x_4 &= 7 - x_1 - x_2 + x_3 \\
    x_5 &= -7 + x_1 + x_2 - x_3 \\
    x_6 &= 4 - x_1 + 2x_2 - 2x_3 \\
    x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{align*}
\]
Simplex: Step 1

• Start at a feasible vertex
  ➢ How do we find a feasible vertex?
  ➢ For now, assume $b \geq 0$ (that is, each $b_i \geq 0$)
    o In this case, $x = 0$ is a feasible vertex.
    o In the slack form, this means setting the nonbasic variables to 0
  ➢ We’ll later see what to do in the general case

Standard form:  Slack form:

Maximize $c^T x$  $z = c^T x$
Subject to $Ax \leq b$  $s = b - Ax$
$\quad \quad \quad x \geq 0$  $\quad \quad \quad s, x \geq 0$
Simple: Step 2

• What next? Let’s look at an example

\[
\begin{align*}
  z &= 3x_1 + x_2 + 2x_3 \\
  x_4 &= 30 - x_1 - x_2 - 3x_3 \\
  x_5 &= 24 - 2x_1 - 2x_2 - 5x_3 \\
  x_6 &= 36 - 4x_1 - x_2 - 2x_3 \\
  x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{align*}
\]

• To increase the value of \( z \):
  - Find a nonbasic variable with a positive coefficient
    - This is called an \textit{entering variable}
  - See how much you can increase its value without violating any constraints
This is because the current values of $x_2$ and $x_3$ are 0, and we need $x_4, x_5, x_6 \geq 0$.
Simple: Step 2

Tightest obstacle

➢ Solve the tightest obstacle for the nonbasic variable

\[
x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}
\]

○ Substitute the entering variable (called pivot) in other equations
○ Now \(x_1\) becomes basic and \(x_6\) becomes non-basic
○ \(x_6\) is called the **leaving variable**
Simplex: Step 2

After one iteration of this step:

- The basic feasible solution (i.e., substituting 0 for all nonbasic variables) improves from $z = 0$ to $z = 27$

- Repeat!
Simplex: Step 2

Entropy variable
Try to increase!

Leaving variable
Tightest obstacle!

\[
\begin{align*}
    z &= 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \\
    x_1 &= 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \\
    x_4 &= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \\
    x_5 &= 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2} \\

    x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{align*}
\]
Simplex: Step 2

Entering variable
Try to increase!

Leaving variable
Tightest obstacle!

Pivot!
Simplex: Step 2

- There is no entering variable (nonbasic variable with positive coefficient).
- What now? Nothing! We are done.
- Take the basic feasible solution ($x_3 = x_5 = x_6 = 0$).
- Gives the optimal value $z = 28$
- In the optimal solution, $x_1 = 8, x_2 = 4, x_3 = 0$
Simplex Overview

Start at a vertex of feasible polytope

Is there a neighbor vertex with better objective value?

Move to a neighbor vertex with better objective value

Terminate, declare the current solution and value as optimal
Simplex Overview

Assuming $b \geq 0$, start with a basic feasible solution

Is there a neighbor vertex with better objective value?

Move to a neighbor vertex with better objective value

Terminate, declare the current solution and value as optimal
**Simplex Overview**

- Assuming $b \geq 0$, start with a basic feasible solution.
- Is there an entering variable with positive coefficient?
- Move to a neighbor vertex with better objective value.
- Terminate, declare the current solution and value as optimal.
Simplex Overview

Assuming $b \geq 0$, start with a basic feasible solution

Is there an entering variable with positive coefficient?

Pivot on a leaving variable

Terminate, declare the current solution and value as optimal

373F21 - Nisarg Shah
Simplex Overview

Assuming \( b \geq 0 \), start with a basic feasible solution.

Is there an entering variable with positive coefficient?

Pivot on a leaving variable.

Terminate, declare optimal value.
Some Outstanding Issues

• What if the entering variable has no upper bound?
  - If it doesn’t appear in any constraints, or only appears in constraints where it can go to $\infty$
  - Then $z$ can also go to $\infty$, so declare that LP is unbounded

• What if pivoting doesn’t change the constant in $z$?
  - Known as degeneracy, and can lead to infinite loops
  - Can be prevented by “perturbing” $b$ by a small random amount in each coordinate
  - Or by carefully breaking ties among entering and leaving variables, e.g., by smallest index (known as Bland’s rule)
Some Outstanding Issues

• We assumed $b \geq 0$, and then started with the vertex $x = 0$
• What if this assumption does not hold?

$$
\begin{align*}
LP_1 & : \text{Max } c^T x \\
\text{s.t. } a_1^T x & \leq b_1 \\
& a_2^T x \leq b_2 \\
& \vdots \\
& a_m^T x \leq b_m \\
& x \geq 0
\end{align*}
$$

$$
\begin{align*}
LP_2 & : \text{Max } c^T x \\
\text{s.t. } a_1^T x + s_1 & = b_1 \\
& a_2^T x + s_2 = b_2 \\
& \vdots \\
& a_m^T x + s_m = b_m \\
& x, s \geq 0
\end{align*}
$$

$$
\begin{align*}
LP_3 & : \text{Max } c^T x \\
\text{s.t. } a_1^T x + s_1 & = b_1 \\
& -a_2^T x - s_2 = -b_2 \\
& \vdots \\
& -a_m^T x - s_m = -b_m \\
& x, s \geq 0
\end{align*}
$$

Multiply every constraint with negative $b_i$ by $-1$ so RHS is now positive.
Some Outstanding Issues

- We assumed $b \geq 0$, and then started with the vertex $x = 0$
- What if this assumption does not hold?

\[
\begin{align*}
LP_3 & : \quad \text{Max } c^T x \\
\text{s.t. } & \quad a_1^T x + s_1 = b_1 \\
& \quad -a_2^T x - s_2 = -b_2 \\
& \quad \vdots \\
& \quad -a_m^T x - s_m = -b_m \\
x, s \geq 0
\end{align*}
\]

\[
\begin{align*}
LP_4 & : \quad \text{Min } \sum_i z_i \\
\text{s.t. } & \quad a_1^T x + s_1 + z_1 = b_1 \\
& \quad -a_2^T x - s_2 + z_2 = -b_2 \\
& \quad \vdots \\
& \quad -a_m^T x - s_m + z_m = -b_m \\
x, s, z \geq 0
\end{align*}
\]

Remember: we only want to find a basic feasible solution to $LP_1$
Some Outstanding Issues

• We assumed \( b \geq 0 \), and then started with the vertex \( x = 0 \)
• What if this assumption does not hold?

What now?
• Solve \( LP_4 \) using simplex with the initial basic solution being \( x = s = 0, z = |b| \)
• If its optimum value is 0, extract a basic feasible solution \( x^* \) from it, use it to solve \( LP_1 \) using simplex
• If optimum value for \( LP_4 \) is greater than 0, then \( LP_1 \) is infeasible

\[
LP_4
\]
Min \( \sum_i z_i \)
s.t. \( a_1^T x + s_1 + z_1 = b_1 \)
\(-a_2^T x - s_2 + z_2 = -b_2 \)
\vdots
\(-a_m^T x - s_m + z_m = -b_m \)
\( x, s, z \geq 0 \)

Remember: the RHS is now positive
Some Outstanding Issues

• Curious about pseudocode? Proof of correctness? Running time analysis?

• See textbook for details, but this is NOT in syllabus!
• Notes
  ➢ #vertices of a polytope can be exponential in the #constraints
    o There are examples where simplex takes exponential time if you choose your pivots arbitrarily
    o No pivot rule known which guarantees polynomial running time
  ➢ Other algorithms known which run in polynomial time
    o Ellipsoid method, interior point method, ...
    o Ellipsoid uses $O(mn^3L)$ arithmetic operations
      • $L = \text{length of input in binary}$
    o But no known strongly polynomial time algorithm
      • Number of arithmetic operations = poly(m,n)
      • We know how to avoid dependence on length(b), but not for length(A)
Certificate of Optimality

• Suppose you design a state-of-the-art LP solver that can solve very large problem instances

• You want to convince someone that you have this new technology without showing them the code
  ➢ **Idea:** They can give you very large LPs and you can quickly return the optimal solutions
  ➢ **Question:** But how would they know that your solutions are optimal, if they don’t have the technology to solve those LPs?
Certificate of Optimality

Suppose I tell you that \((x_1, x_2) = (100, 300)\) is optimal with objective value 1900

How can you check this?

- **Note:** Can easily substitute \((x_1, x_2)\), and verify that it is feasible, and its objective value is indeed 1900
Certificate of Optimality

\[
\begin{align*}
\text{max} & \quad x_1 + 6x_2 \\
\text{s.t.} & \quad x_1 \leq 200 \\
& \quad x_2 \leq 300 \\
& \quad x_1 + x_2 \leq 400 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

• Claim: \((x_1, x_2) = (100, 300)\) is optimal with objective value 1900

• Any solution that satisfies these inequalities also satisfies their positive combinations
  ➢ E.g. \(2*\text{first\_constraint} + 5*\text{second\_constraint} + 3*\text{third\_constraint}\)
  ➢ Try to take combinations which give you \(x_1 + 6x_2\) on LHS
Certificate of Optimality

\[ \begin{align*}
\text{max } & \quad x_1 + 6x_2 \\
& x_1 \leq 200 \\
& x_2 \leq 300 \\
& x_1 + x_2 \leq 400 \\
& x_1, x_2 \geq 0
\end{align*} \]

• Claim: \((x_1, x_2) = (100,300)\) is optimal with objective value 1900

• first_constraint + 6*second_constraint
  - \(x_1 + 6x_2 \leq 200 + 6 \times 300 = 2000\)
  - This shows that no feasible solution can beat 2000
Certificate of Optimality

\[
\begin{align*}
\text{max} & \quad x_1 + 6x_2 \\
x_1 & \leq 200 \\
x_2 & \leq 300 \\
x_1 + x_2 & \leq 400 \\
x_1, x_2 & \geq 0
\end{align*}
\]

• Claim: \((x_1, x_2) = (100, 300)\) is optimal with objective value 1900

• \(5\ast\text{second\_constraint} + \text{third\_constraint}\)
  \[
  5x_2 + (x_1 + x_2) \leq 5 \ast 300 + 400 = 1900
  \]
  This shows that no feasible solution can beat 1900
    o No need to proceed further
    o We already know one solution that achieves 1900, so it must be optimal!
Is There a General Algorithm?

• Introduce variables $y_1, y_2, y_3$ by which we will be multiplying the three constraints
  ➢ Note: These need not be integers. They can be reals.

<table>
<thead>
<tr>
<th>Multiplier</th>
<th>Inequality</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_1$</td>
<td>$x_1 \leq 200$</td>
</tr>
<tr>
<td>$y_2$</td>
<td>$x_2 \leq 300$</td>
</tr>
<tr>
<td>$y_3$</td>
<td>$x_1 + x_2 \leq 400$</td>
</tr>
</tbody>
</table>

• After multiplying and adding constraints, we get:

$$(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400y_3$$
Is There a General Algorithm?

We have:

\[(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400y_3\]

What do we want?

- \(y_1, y_2, y_3 \geq 0\) because otherwise direction of inequality flips
- LHS to look like objective \(x_1 + 6x_2\)
  - In fact, it is sufficient for LHS to be an upper bound on objective
  - So, we want \(y_1 + y_3 \geq 1\) and \(y_2 + y_3 \geq 6\)
Is There a General Algorithm?

➢ We have:

\[ (y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400y_3 \]

➢ What do we want?

- \( y_1, y_2, y_3 \geq 0 \)
- \( y_1 + y_3 \geq 1, \ y_2 + y_3 \geq 6 \)
- Subject to these, we want to minimize the upper bound \( 200y_1 + 300y_2 + 400y_3 \)
Is There a General Algorithm?

We have:
\[(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400y_3\]

What do we want?
- This is just another LP!
- Called the dual
- Original LP is called the primal
Is There a General Algorithm?

The problem of verifying optimality is another LP.

- For any \((y_1, y_2, y_3)\) that you can find, the objective value of the dual is an upper bound on the objective value of the primal.
- If you found a specific \((y_1, y_2, y_3)\) for which this dual objective becomes equal to the primal objective for the \((x_1, x_2)\) given to you, then you would know that the given \((x_1, x_2)\) is optimal for primal (and your \((y_1, y_2, y_3)\) is optimal for dual).
Is There a General Algorithm?

The problem of verifying optimality is another LP

- **Issue 1:** But...but...if I can’t solve large LPs, how will I solve the dual to verify if optimality of \((x_1, x_2)\) given to me?
  - You don’t. Ask the other party to give you both \((x_1, x_2)\) and the corresponding \((y_1, y_2, y_3)\) for proof of optimality

- **Issue 2:** What if there are no \((y_1, y_2, y_3)\) for which dual objective matches primal objective under optimal solution \((x_1, x_2)\)?
  - As we will see, this can’t happen!
Is There a General Algorithm?

Primal LP

\[ \max \ c^T x \]
\[ A x \leq b \]
\[ x \geq 0 \]

Dual LP

\[ \min \ y^T b \]
\[ y^T A \geq c^T \]
\[ y \geq 0 \]

- General version, in our standard form for LPs
Is There a General Algorithm?

**Primal LP**

\[
\begin{align*}
\text{max} & \quad c^T x \\
\text{Ax} & \leq b \\
x & \geq 0
\end{align*}
\]

**Dual LP**

\[
\begin{align*}
\text{min} & \quad y^T b \\
y^T A & \geq c^T \\
y & \geq 0
\end{align*}
\]

- \(c^T x\) for any feasible \(x\) \(\leq\) \(y^T b\) for any feasible \(y\)
- \(\max_{\text{primal feasible } x} c^T x \leq \min_{\text{dual feasible } y} y^T b\)
- If there is \((x^*, y^*)\) with \(c^T x^* = (y^*)^T b\), then both must be optimal
- In fact, for optimal \((x^*, y^*)\), we claim that this must happen!
  - Does this remind you of something? Max-flow, min-cut...
# Weak Duality

- From here on, assume primal LP is feasible and bounded
- **Weak duality theorem:**
  - For any primal feasible $x$ and dual feasible $y$, $c^T x \leq y^T b$
- **Proof:**
  
  $$c^T x \leq (y^T A)x = y^T (Ax) \leq y^T b$$

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\max \ c^T x$</td>
<td>$\min \ y^T b$</td>
</tr>
<tr>
<td>$Ax \leq b$</td>
<td>$y^T A \geq c^T$</td>
</tr>
<tr>
<td>$x \geq 0$</td>
<td>$y \geq 0$</td>
</tr>
</tbody>
</table>
**Strong Duality**

- **Strong duality theorem:**
  - For any primal optimal $x^*$ and dual optimal $y^*$, $c^T x^* = (y^*)^T b$
Strong Duality Proof

• **Farkas’ lemma** (one of many, many versions):
  - Exactly one of the following holds:
    1) There exists $x$ such that $Ax \leq b$
    2) There exists $y$ such that $y^T A = 0$, $y \geq 0$, $y^T b < 0$

• **Geometric intuition:**
  - Define image of $A = \text{set of all possible values of } Ax$
  - It is known that this is a “linear subspace” (e.g., a line in a plane, a line or plane in 3D, etc)

This slide is not in the scope of the course
• **Farkas’ lemma**: Exactly one of the following holds:
  1) There exists $x$ such that $Ax \leq b$
  2) There exists $y$ such that $y^T A = 0$, $y \geq 0$, $y^T b < 0$

1) Image of $A$ contains a point “below” $b$

2) The region “below” $b$ doesn’t intersect image of $A$
   this is witnessed by normal vector to the image of $A$
Strong Duality

<table>
<thead>
<tr>
<th>Primal LP</th>
<th>Dual LP</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \max c^T x ]</td>
<td>[ \min y^T b ]</td>
</tr>
<tr>
<td>[ Ax \leq b ]</td>
<td>[ y^T A \geq c^T ]</td>
</tr>
<tr>
<td>[ x \geq 0 ]</td>
<td>[ y \geq 0 ]</td>
</tr>
</tbody>
</table>

- **Strong duality theorem:**
  - For any primal optimal \( x^* \) and dual optimal \( y^* \), \( c^T x^* = (y^*)^T b \)
  - **Proof (by contradiction):**
    - Let \( z^* = c^T x^* \) be the optimal primal value.
    - Suppose optimal dual objective value \( > z^* \)
    - So, there is no \( y \) such that \( y^T A \geq c^T \) and \( y^T b \leq z^* \), i.e.,
      \[
      \begin{pmatrix} -A^T \\ b^T \end{pmatrix} y \leq \begin{pmatrix} c \\ z^* \end{pmatrix}
      \]
Strong Duality

- There is no $y$ such that \( \begin{pmatrix} -A^T \\ b^T \end{pmatrix} y \leq \begin{pmatrix} c \\ z^* \end{pmatrix} \)
- By Farkas’ lemma, there is $x$ and $\lambda$ such that
  \[
  (x^T \quad \lambda) \begin{pmatrix} -A^T \\ b^T \end{pmatrix} = 0, x \geq 0, \lambda \geq 0, -x^T c + \lambda z^* < 0
  \]

- **Case 1: $\lambda > 0$**
  - Note: $c^T x > \lambda z^*$ and $Ax = 0 = \lambda b$.
  - Divide both by $\lambda$ to get $A \begin{pmatrix} x \\ \lambda \end{pmatrix} = b$ and $c^T \begin{pmatrix} x \\ \lambda \end{pmatrix} > z^*$
    - Contradicts optimality of $z^*$

- **Case 2: $\lambda = 0$**
  - We have $Ax = 0$ and $c^T x > 0$
  - Adding $x$ to optimal $x^*$ of primal improves objective value beyond $z^*$ \( \Rightarrow \) contradiction
Exercise: Formulating LPs

- A canning company operates two canning plants (A and B).

- Three suppliers of fresh fruits:

- Shipping costs in $/tonne: 

- Plant capacities and labour costs:

- Selling price: $50/tonne, no limit

- Objective: Find which plant should get how much supply from each grower to maximize profit
Exercise: Formulating LPs

• Similarly to the brewery example from earlier:
  ➢ A brewery can invest its inventory of corn, hops and malt into producing three types of beer
  ➢ Per unit resource requirement and profit are as given below
  ➢ The brewery cannot produce positive amounts of both A and B
  ➢ Goal: maximize profit

<table>
<thead>
<tr>
<th>Beverage</th>
<th>Corn (kg)</th>
<th>Hops (kg)</th>
<th>Malt (kg)</th>
<th>Profit ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>5</td>
<td>4</td>
<td>35</td>
<td>13</td>
</tr>
<tr>
<td>B</td>
<td>15</td>
<td>4</td>
<td>20</td>
<td>23</td>
</tr>
<tr>
<td>C</td>
<td>10</td>
<td>7</td>
<td>25</td>
<td>15</td>
</tr>
<tr>
<td>Limit</td>
<td>500</td>
<td>300</td>
<td>1000</td>
<td></td>
</tr>
</tbody>
</table>
Exercise: Formulating LPs

• Similarly to the brewery example from the beginning:
  ➢ A brewery can invest its inventory of corn, hops and malt into producing three types of beer
  ➢ Per unit resource requirement and profit are as given below
  ➢ The brewery can only produce $C$ in integral quantities up to 100
  ➢ Goal: maximize profit

<table>
<thead>
<tr>
<th>Beverage</th>
<th>Corn (kg)</th>
<th>Hops (kg)</th>
<th>Malt (kg)</th>
<th>Profit ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>5</td>
<td>4</td>
<td>35</td>
<td>13</td>
</tr>
<tr>
<td>B</td>
<td>15</td>
<td>4</td>
<td>20</td>
<td>23</td>
</tr>
<tr>
<td>C</td>
<td>10</td>
<td>7</td>
<td>25</td>
<td>15</td>
</tr>
<tr>
<td>Limit</td>
<td>500</td>
<td>300</td>
<td>1000</td>
<td></td>
</tr>
</tbody>
</table>
Exercise: Formulating LPs

• Similarly to the brewery example from the beginning:
  ➢ A brewery can invest its inventory of corn, hops and malt into producing three types of beer
  ➢ Per unit resource requirement and profit are as given below
  ➢ Goal: maximize profit, but if there are multiple profit-maximizing solutions, then...
    o Break ties to choose those with the largest quantity of $A$
    o Break any further ties to choose those with the largest quantity of $B$

<table>
<thead>
<tr>
<th>Beverage</th>
<th>Corn (kg)</th>
<th>Hops (kg)</th>
<th>Malt (kg)</th>
<th>Profit ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>5</td>
<td>4</td>
<td>35</td>
<td>13</td>
</tr>
<tr>
<td>B</td>
<td>15</td>
<td>4</td>
<td>20</td>
<td>23</td>
</tr>
<tr>
<td>C</td>
<td>10</td>
<td>7</td>
<td>25</td>
<td>15</td>
</tr>
<tr>
<td>Limit</td>
<td>500</td>
<td>300</td>
<td>1000</td>
<td></td>
</tr>
</tbody>
</table>
More Tricks

• Constraint: $|x| \leq 3$
  - Replace with constraints $x \leq 3$ and $-x \leq 3$
  - What if the constraint is $|x| \geq 3$?

• Objective: minimize $3|x| + y$
  - Add a variable $t$
  - Add the constraints $t \geq x$ and $t \geq -x$ (so $t \geq |x|$)
  - Change the objective to minimize $3t + y$
  - What if the objective is to maximize $3|x| + y$?

• Objective: minimize $\max(3x + y, x + 2y)$
  - Hint: minimizing $3|x| + y$ in the earlier bullet was equivalent to minimizing $\max(3x + y, -3x + y)$

• ...
NOW I KNOW
LINEAR PROGRAMMING