CSC373

Week 6: Network Flow (contd)

Nisarg Shah

Recap

- Some more DP
  - Traveling salesman problem (TSP)

- Start of network flow
  - Problem statement
  - Ford-Fulkerson algorithm
  - Running time
  - Correctness using max-flow, min-cut

This Lecture

- Network flow in polynomial time
  - Edmonds-Karp algorithm (shortest augmenting path)

- Applications of network flow
  - Bipartite matching & Hall's theorem
  - Edge-disjoint paths & Menger's theorem
  - Multiple sources/sinks
  - Circulation networks
  - Lower bounds on flows
  - Survey design
  - Image segmentation

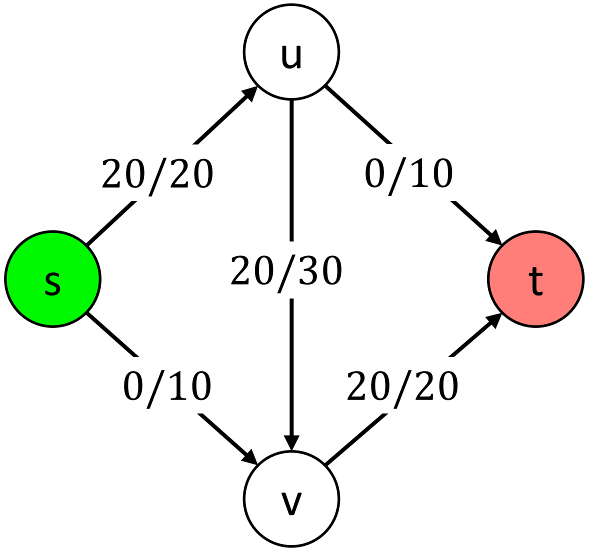
Ford-Fulkerson Recap

- Define the residual graph G_f of flow f
  - G_f has the same vertices as G
  - For each edge e $= (u, v)$ in G, G_f has at most two edges
    - Forward edge $e = (u, v)$ with capacity $c(e) - f(e)$
      - We can send this much additional flow on e
    - Reverse edge $e^{rev} = (v, u)$ with capacity $f(e)$
      - The maximum "reverse" flow we can send is the maximum amount by which we can reduce flow on e, which is $f(e)$
    - We only add each edge if its capacity > 0

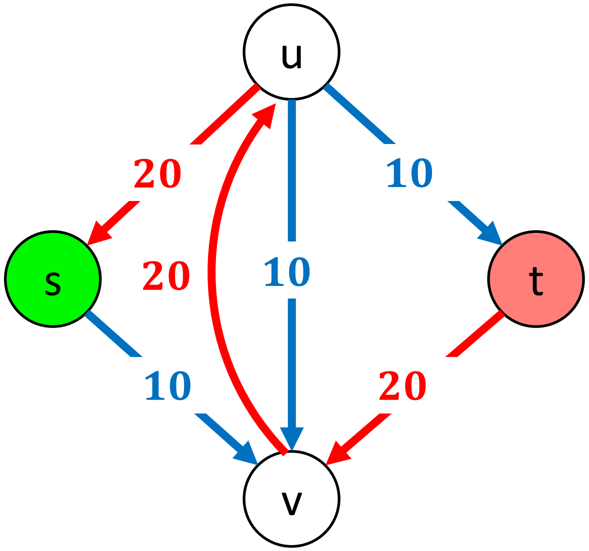
Ford-Fulkerson Recap

- Example!

Flow f



Residual graph G_f



Ford-Fulkerson Recap

```
MaxFlow(G):
  // initialize:
  Set f(e) = 0 for all e in G

  // while there is an s-t path in G_f:
  While P = FindPath(s,t,Residual(G,f))!=None:
    f = Augment(f,P)
    UpdateResidual(G,f)
  EndWhile

  Return f
```

Ford-Fulkerson Recap

- Running time:
  - #Augmentations:
    - At every step, flow and capacities remain integers
    - For path P in G_f, $\text{bottleneck}(P, f) > 0$ implies $\text{bottleneck}(P, f) \geq 1$
    - Each augmentation increases flow by at least 1
    - At most $C = \sum_{e \text{ leaving } s} c(e)$ augmentations
  - Time for an augmentation:
    - G_f has n vertices and at most $2m$ edges
    - Finding an s-t path in G_f takes $O(m+n)$ time
  - Total time: $O((m+n) \cdot C)$

Edmonds-Karp Algorithm

- At every step, find the shortest path from s to t in G_f, and augment.

Minimum number of edges

```
MaxFlow(G):
  // initialize:
  Set f(e) = 0 for all e in G

  // Find shortest s-t path in G_f & augment:
  While P = BFS(s,t,Residual(G,f))!=None:
    f = Augment(f,P)
    UpdateResidual(G,f)
  EndWhile
  Return f
```

Proof

- $d(v)$ = shortest distance of v from s in residual graph G_f
- Lemma 1: During the execution of the algorithm, $d(v)$ does not decrease for any v.
- Proof:
  - Suppose augmentation $f \to f'$ decreases $d(v)$ for some v
  - Choose the v with the smallest $d(v)$ in $G_{f'}$
    - Say $d(v) = k$ in $G_{f'}$, so $d(v) \geq k + 1$ in G_f
  - Look at node u just before v on a shortest path $s \to v$ in $G_{f'}$
    - $d(u) = k - 1$ in $G_{f'}$
    - $d(u)$ didn't decrease, so $d(u) \leq k - 1$ in G_f

Proof

- $d(v)$ = shortest distance of v from s in residual graph G_f

- Lemma 1: During the execution of the algorithm, $d(v)$ does not decrease for any v.
- Proof:

	$d(u)$	$d(v)$
G_f	$\leq k-1$	$\geq k+1$
	$\downarrow$	$\downarrow$
$G_{f'}$	$k-1$	k

- In G_f, (u, v) must be missing
- We must have added (u, v) by selecting (v, u) in augmenting path P
- But P is a shortest path, so it cannot have edge (v, u) with $d(v) > d(u)$

Proof

- Call edge (u, v) critical in an augmentation step if
  - It's part of the augmenting path P and its capacity is equal to $\text{bottleneck}(P, f)$
  - Augmentation step removes e and adds e^{rev} (if missing)
- Lemma 2: Between any two steps in which (u, v) is critical, $d(u)$ increases by at least 2
- Proof of Edmonds-Karp running time
  - Each $d(u)$ can go from 0 to n (Lemma 1)
  - So, each edge (u, v) can be critical at most $n/2$ times (Lemma 2)
  - So, there can be at most $m \cdot n/2$ augmentation steps
  - Each augmentation takes $O(m)$ time to perform
  - Hence, $O(m^2 n)$ operations in total!

Proof

- Lemma 2: Between any two steps in which (u, v) is critical, $d(u)$ increases by at least 2
- Proof:
  - Suppose (u, v) was critical in G_f
    - So, the augmentation step must have removed it
  - Let $k = d(u)$ in G_f
    - Because (u, v) is part of a shortest path, $d(v) = k + 1$ in G_f
  - For (u, v) to be critical again, it must be added back at some point
    - Suppose $f' \to f''$ steps adds it back
    - Augmenting path in f' must have selected (v, u)
    - In $G_{f'}$: $d(u) = d(v) + 1 \geq (k + 1) + 1 = k + 2$

Lemma 1 on v

Edmonds-Karp Proof Overview

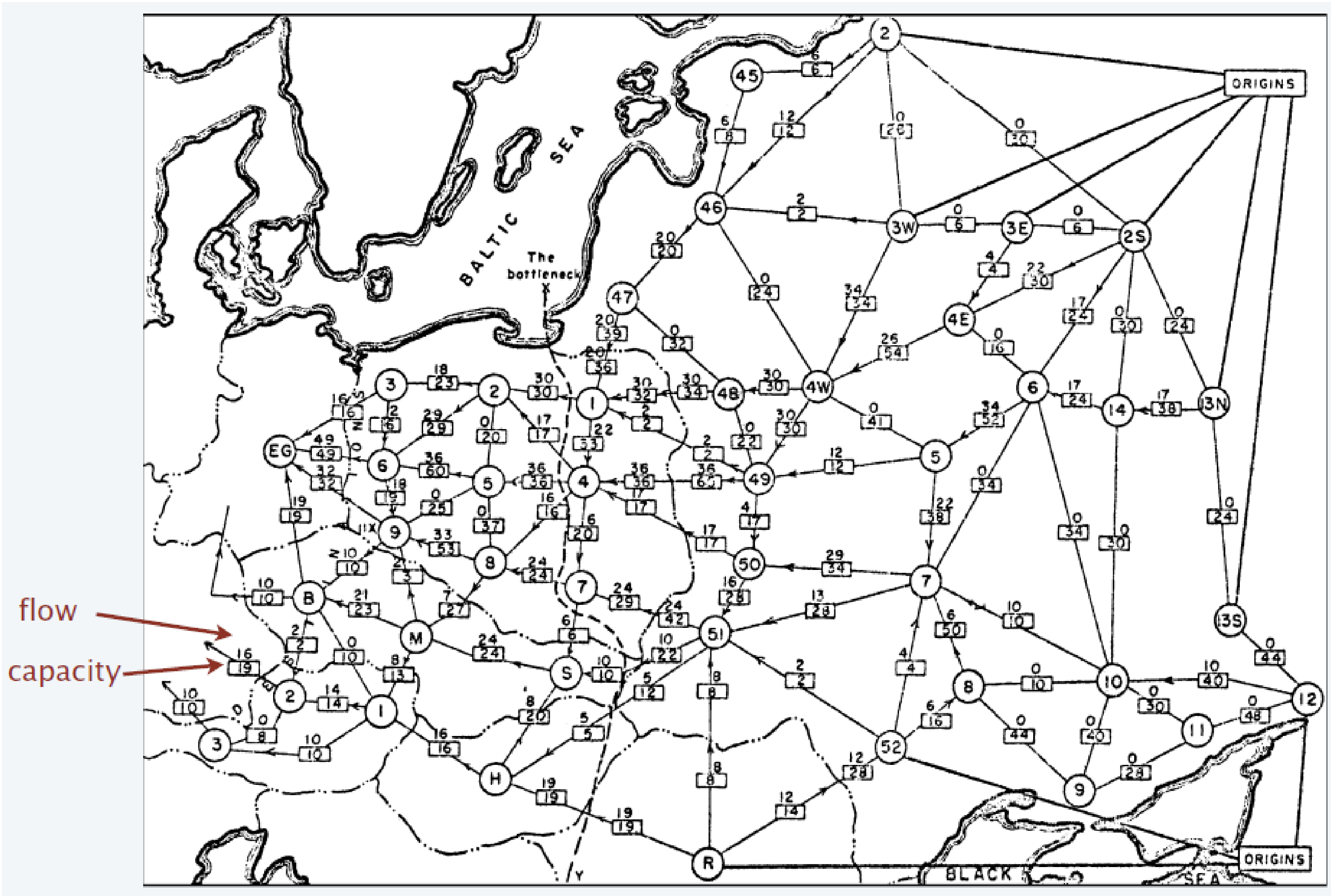
- Note:
  - Some graphs require $\Omega(mn)$ augmentation steps
  - But we may be able to reduce the time to run each augmentation step

- Two algorithms use this idea to reduce run time
  - Dinitz's algorithm [1970] $\Rightarrow O(mn^2)$
  - Sleator–Tarjan algorithm [1983] $\Rightarrow O(m\, n \log n)$
    - Using the dynamic trees data structure

Network Flow Applications

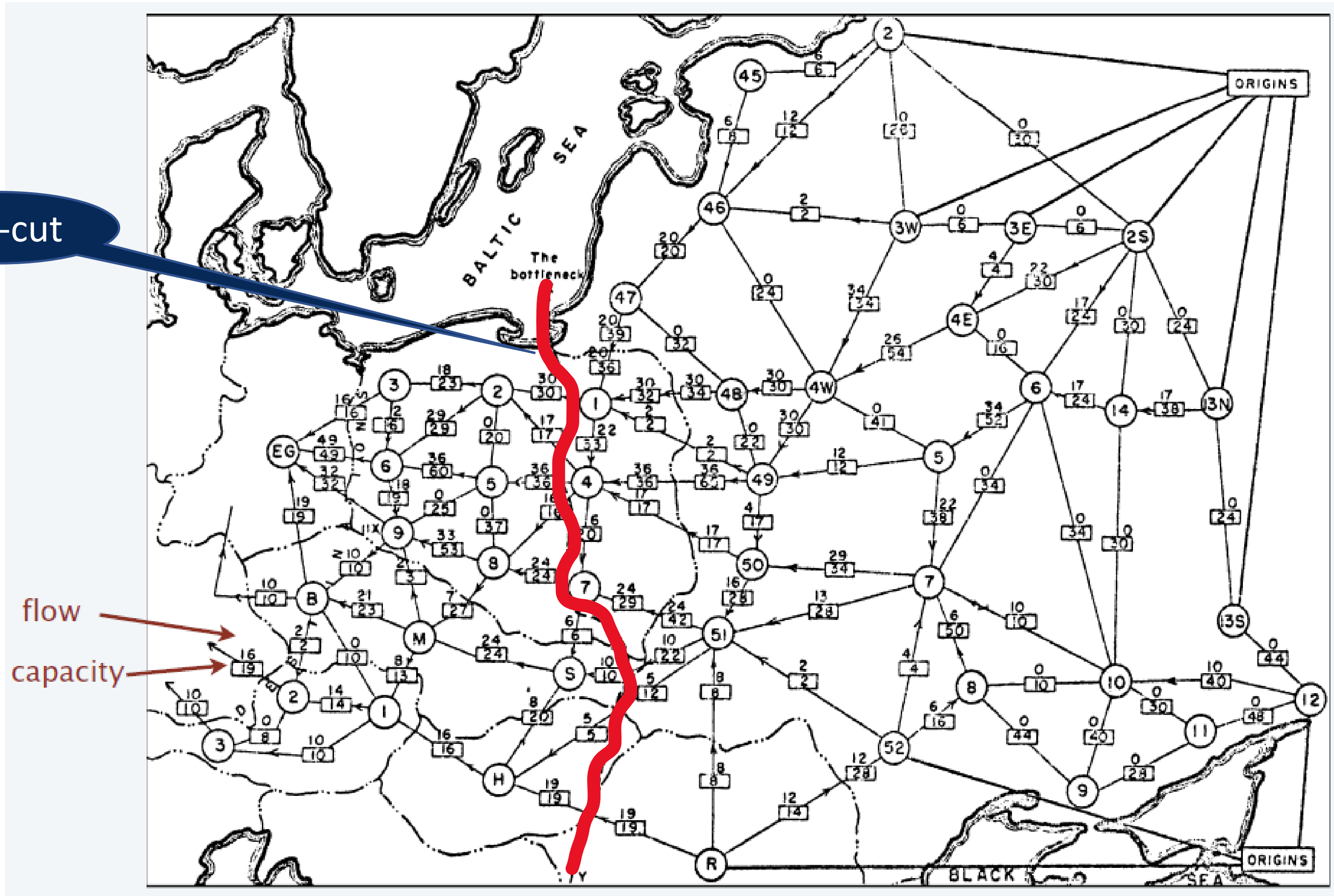
Rail network connecting Soviet Union with Eastern European countries

(Tolstoǐ 1930s)



Rail network connecting Soviet Union with Eastern European countries

(Tolstoǐ 1930s)



Min-cut

flow

capacity

Integrality Theorem

- Before we look at applications, we need the following special property of the max-flow computed by Ford-Fulkerson and its variants

- Observation:
  - If edge capacities are integers, then the max-flow computed by Ford-Fulkerson and its variants are also integral (i.e., the flow on each edge is an integer).
  - Easy to check that each augmentation step preserves integral flow

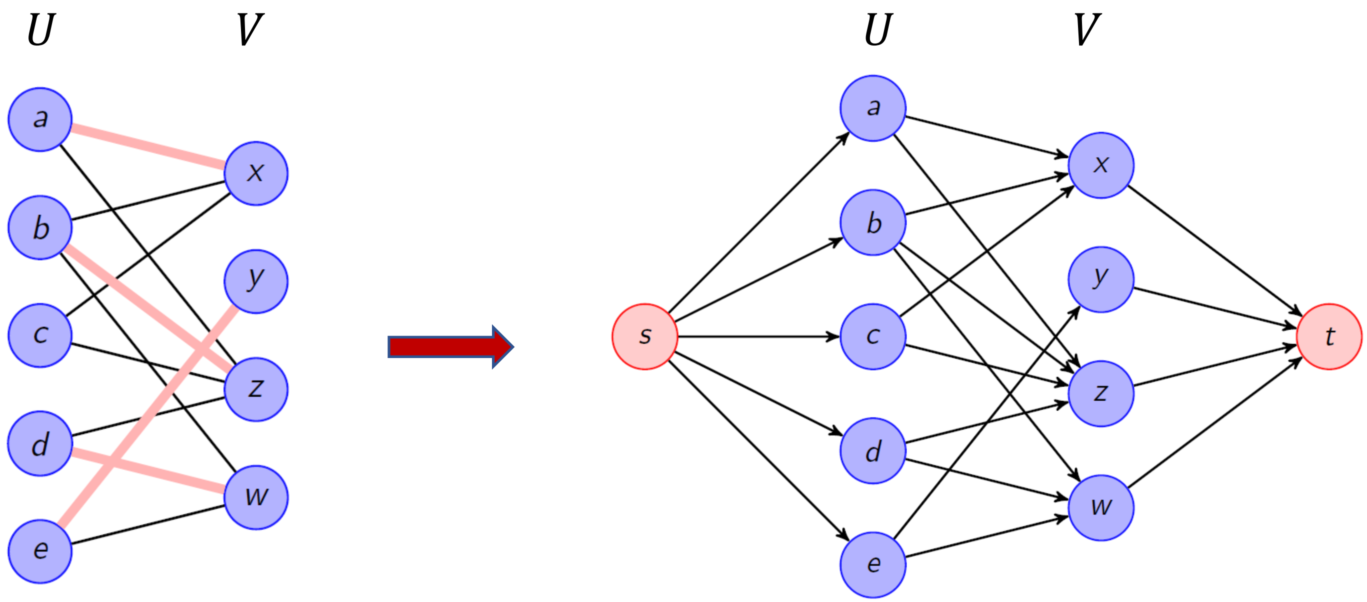
Bipartite Matching

- Problem
  - Given a bipartite graph $G = (U \cup V, E)$, find a maximum cardinality matching

- We do not know any efficient greedy or dynamic programming algorithm for this problem.

- But it can be reduced to max-flow.

Bipartite Matching



- Create a directed flow graph where we…
  - Add a source node s and target node t
  - Add edges, all of capacity 1:
    - $s \rightarrow u$ for each $u \in U$, $v \rightarrow t$ for each $v \in V$
    - $u \rightarrow v$ for each $(u, v) \in E$

Bipartite Matching

- Observation
  - There is a 1-1 correspondence between matchings of size k in the original graph and flows with value k in the corresponding flow network.

- Proof: (matching $\Rightarrow$ integral flow)
  - Take a matching $M = \{(u_1, v_1), \dots, (u_k, v_k)\}$ of size k
  - Construct the corresponding unique flow f_M where…
    - Edges $s \to u_i$, $u_i \to v_i$, and $v_i \to t$ have flow 1, for all $i = 1, \dots, k$
    - The rest of the edges have flow 0
  - This flow has value k

Bipartite Matching

- Observation
  - There is a 1-1 correspondence between matchings of size k in the original graph and flows with value k in the corresponding flow network.

- Proof: (integral flow $\Rightarrow$ matching)
  - Take any flow f with value k
  - The corresponding unique matching M_f = set of edges from U to V with a flow of 1
    - Since flow of k comes out of s, unit flow must go to k distinct vertices in U
    - From each such vertex in U, unit flow goes to a distinct vertex in V
    - Uses integrality theorem

Bipartite Matching

- Perfect matching = flow with value n
  - where $n = |U| = |V|$

- Recall naïve Ford-Fulkerson running time:
  - $O((m+n) \cdot C)$, where $C =$ sum of capacities of edges leaving s
  - Q: What's the runtime when used for bipartite matching?

- Some variants are faster…
  - Dinitz's algorithm runs in time $O(m\sqrt{n})$ when all edge capacities are 1

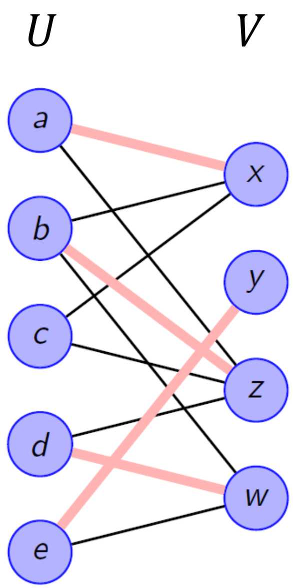
Hall's Marriage Theorem

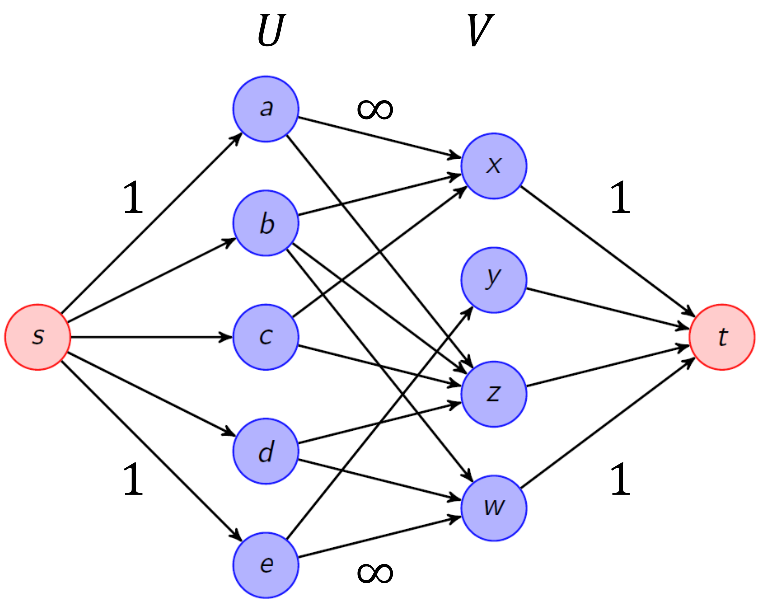
- When does a bipartite graph have a perfect matching?
  - Well, when the corresponding flow network has value n
  - But can we interpret this condition in terms of edges of the original bipartite graph?
  - For $S \subseteq U$, let $N(S) \subseteq V$ be the set of all nodes in V adjacent to some node in S

- Observation:
  - If G has a perfect matching, $|N(S)| \geq |S|$ for each $S \subseteq U$
  - Because each node in S must be matched to a distinct node in $N(S)$

Hall's Marriage Theorem

- We'll consider a slightly different flow network, which is still equivalent to bipartite matching
  - All $U \rightarrow V$ edges now have ∞ capacity
  - $s \rightarrow U$ and $V \rightarrow t$ edges are still unit capacity





Hall's Marriage Theorem

- Hall's Theorem:
  - G has a perfect matching iff $|N(S)| \geq |S|$ for each $S \subseteq V$

- Proof (reverse direction, via network flow):
  - Suppose G doesn't have a perfect matching
  - Hence, max-flow = min-cut $< n$
  - Let (A, B) be the min-cut
    - Can't have any $U \to V$ (∞ capacity edges)
    - Has unit capacity edges $s \to U \cap B$ and $V \cap A \to t$

Hall's Marriage Theorem

- Hall's Theorem:
  - G has a perfect matching iff $|N(S)| \geq |S|$ for each $S \subseteq V$

- Proof (reverse direction, via network flow):
  - $cap(A, B) = |U \cap B| + |V \cap A| < n = |U|$
  - So $|V \cap A| < |U \cap A|$
  - But $N(U \cap A) \subseteq V \cap A$ because the cut doesn't include any ∞ edges
  - So $|N(U \cap A)| \leq |V \cap A| < |U \cap A|$. ■

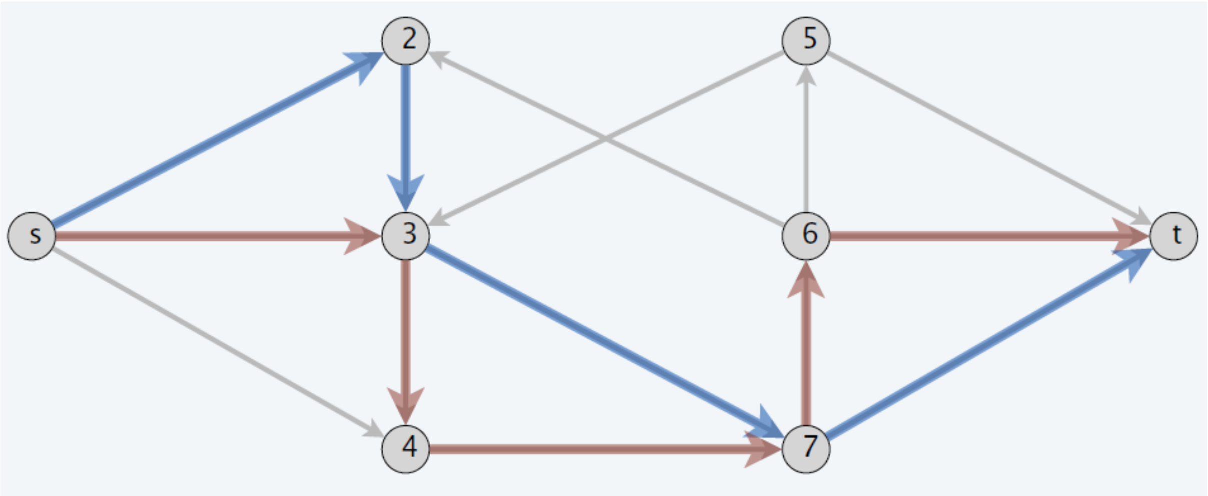
Some Notes

- Runtime for bipartite perfect matching
  - 1955: $O(mn) \rightarrow$ Ford-Fulkerson
  - 1973: $O(m\sqrt{n}) \rightarrow$ blocking flow (Hopcroft-Karp, Karzanov)
  - 2004: $O(n^{2.378}) \rightarrow$ fast matrix multiplication (Mucha–Sankowsi)
  - 2013: $\tilde{O}\left(m^{10/7}\right) \rightarrow$ electrical flow (Mądry)
  - Best running time is still an open question
- Nonbipartite graphs
  - Hall's theorem $\rightarrow$ Tutte's theorem
  - 1965: $O(n^4) \rightarrow$ Blossom algorithm (Edmonds)
  - 1980/1994: $O(m\sqrt{n}) \rightarrow$ Micali-Vazirani

Edge-Disjoint Paths

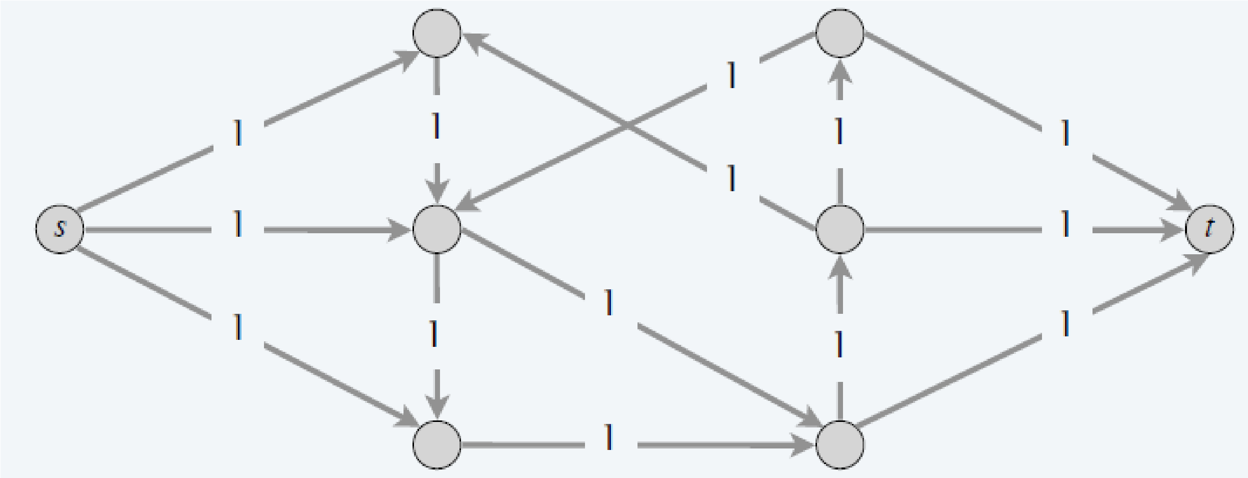
- Problem
  - Given a directed graph $G = (V, E)$, two nodes s and t, find the maximum number of edge-disjoint $s \to t$ paths

- Two $s \to t$ paths P and P' are edge-disjoint if they don't share an edge



Edge-Disjoint Paths

- Application:
  - Communication networks
- Max-flow formulation
  - Assign unit capacity on all edges



Edge-Disjoint Paths

- Theorem:
  - There is 1-1 correspondence between sets of k edge-disjoint $s \to t$ paths and integral flows of value k
- Proof (paths → flow)
  - Let $\{P_1, \dots, P_k\}$ be a set of k edge-disjoint $s \to t$ paths
  - Define flow f where $f(e) = 1$ whenever $e \in P_i$ for some i, and 0 otherwise
  - Since paths are edge-disjoint, flow conservation and capacity constraints are satisfied
  - Unique integral flow of value k

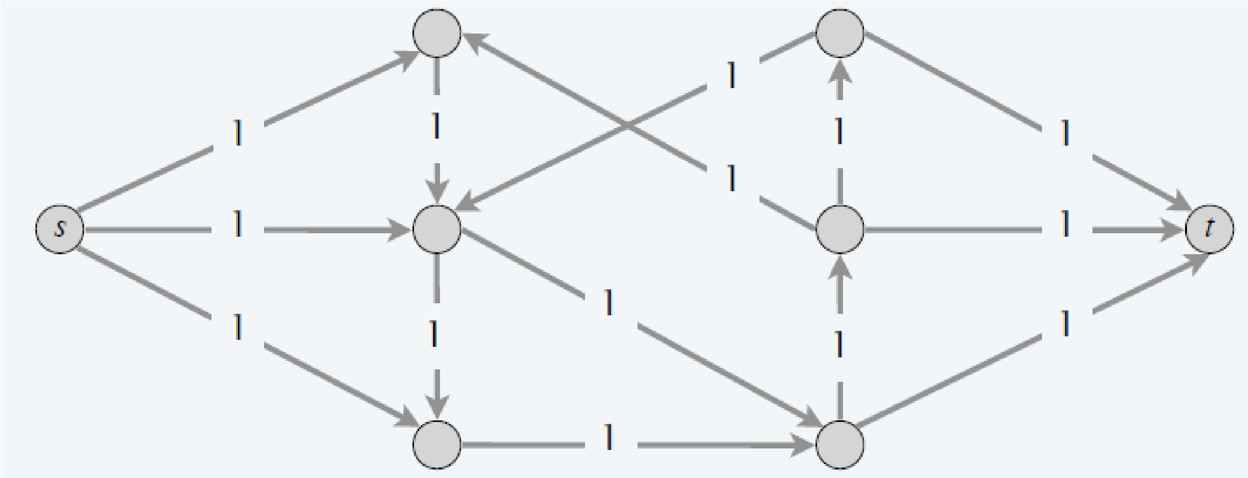
Edge-Disjoint Paths

- Theorem:
  - There is 1-1 correspondence between k edge-disjoint $s \to t$ paths and integral flows of value k

- Proof (flow → paths)
  - Let f be an integral flow of value k
  - k outgoing edges from s have unit flow
  - Pick one such edge (s, u_1)
    - By flow conservation, u_1 must have unit outgoing flow (which we haven't used up yet).
    - Pick such an edge and continue building a path until you hit t
  - Repeat this for the other $k - 1$ edges from s with unit flow ■

Edge-Disjoint Paths

- Maximum number of edge-disjoint $s \to t$ paths
  - Equals max flow in this network
  - By max-flow min-cut theorem, also equals minimum cut
  - Exercise: minimum cut = minimum number of edges we need to delete to disconnect s from t
    - Hint: Show each direction separately ($\leq$ and $\geq$)



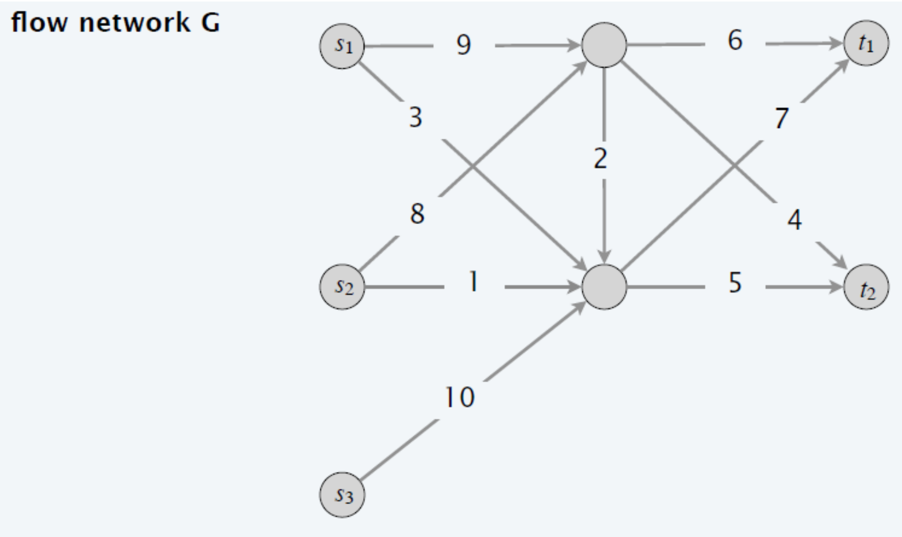
Edge-Disjoint Paths

- Exercise!
  - Show that to compute the maximum number of edge-disjoint s-t paths in an undirected graph, you can create a directed flow network by adding each undirected edge in both directions and setting all capacities to 1

- Menger's Theorem
  - In any directed/undirected graph, the maximum number of edge-disjoint (resp. vertex-disjoint) $s \to t$ paths equals the minimum number of edges (resp. vertices) whose removal disconnects s and t

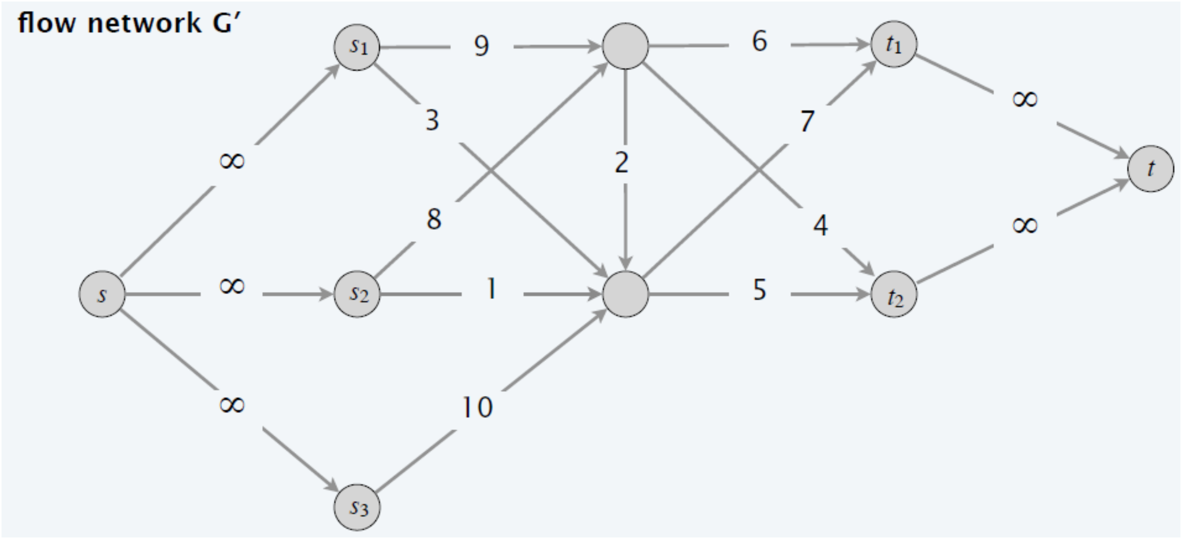
Multiple Sources/Sinks

- Problem
  - Given a directed graph $G = (V, E)$ with edge capacities $c: E \to \mathbb{N}$, sources $s_1, \dots, s_k$ and sinks $t_1, \dots, t_\ell$, find the maximum total flow from sources to sinks.



Multiple Sources/Sinks

- Network flow formulation
  - Add a new source s, edges from s to each s_i with ∞ capacity
  - Add a new sink t, edges from each t_j to t with ∞ capacity
  - Find max-flow from s to t
  - Claim: $1 - 1$ correspondence between flows in two networks



Circulation

- Input
  - Directed graph $G = (V, E)$
  - Edge capacities $c : E \to \mathbb{N}$
  - Node demands $d : V \to \mathbb{Z}$
- Output
  - Some circulation $f : E \to \mathbb{N}$ satisfying
    - For each $e \in E : 0 \leq f(e) \leq c(e)$
    - For each $v \in V : \sum_{e \text{ entering } v} f(v) - \sum_{e \text{ leaving } v} f(v) = d(v)$

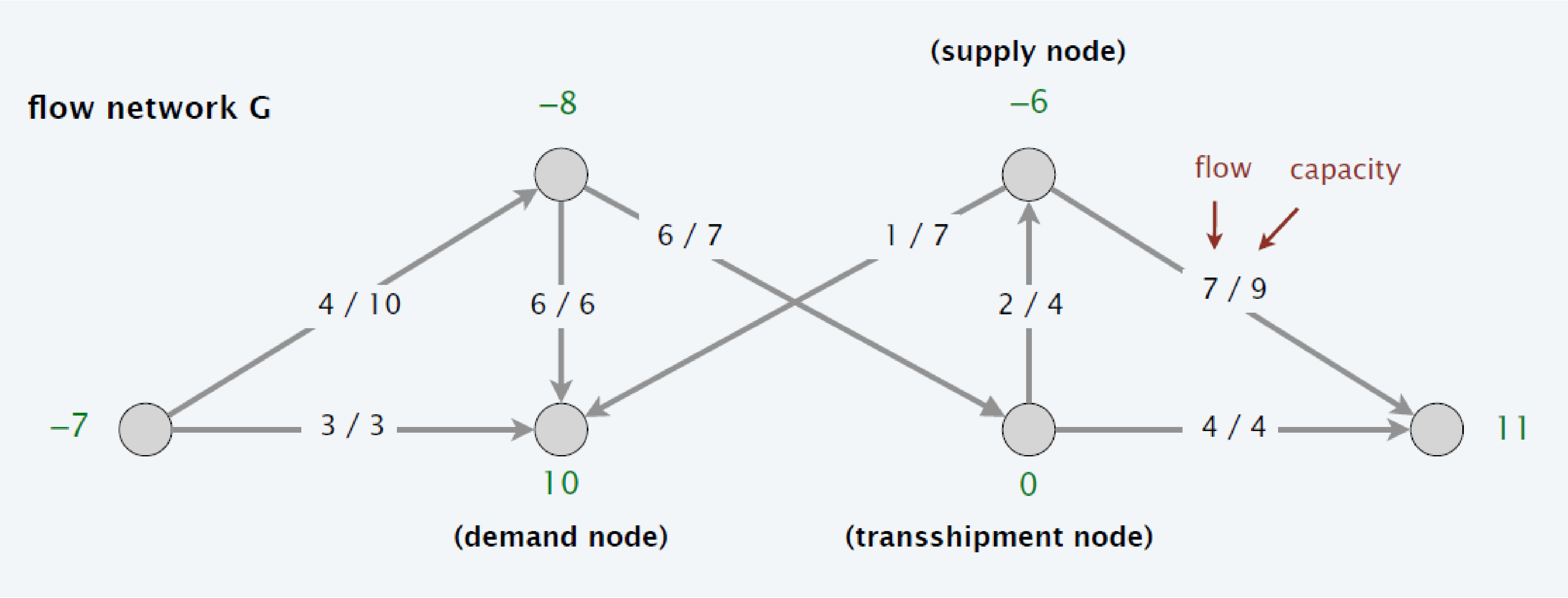
- Note that you need $\sum_{v:d(v)>0} d(v) = \sum_{v:d(v)<0} -d(v)$
- What are demands?

Circulation

- Demand at v = amount of flow you need to take out at node v
  - $d(v) > 0$: You need to take some flow out at v
    - So, there should be $d(v)$ *more* incoming flow than outgoing flow
    - "Demand node"
  - $d(v) < 0$: You need to put some flow in at v
    - So, there should be $|d(v)|$ *more* outgoing flow than incoming flow
    - "Supply node"
  - $d(v) = 0$: Node has flow conservation
    - Equal incoming and outgoing flows
    - "Transshipment node"

Circulation

- Example



Circulation

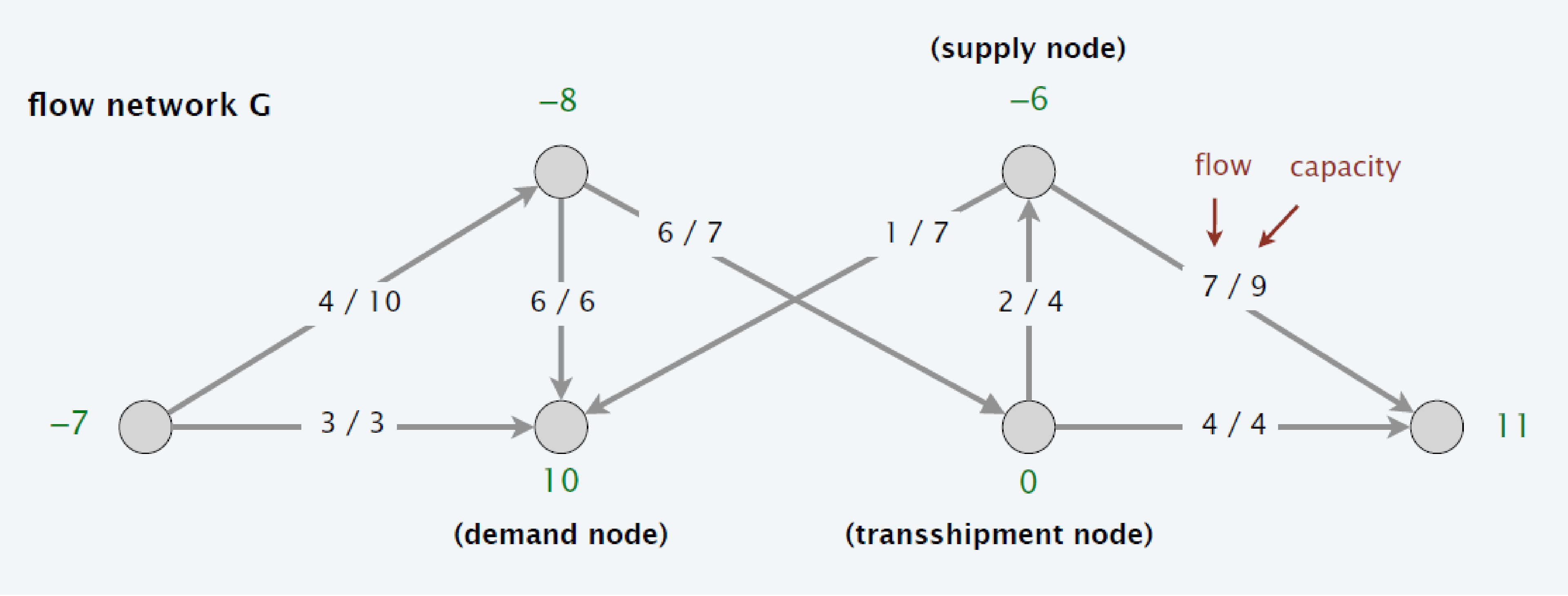
- Network-flow formulation G'
  - Add a new source s and a new sink t
  - For each "supply" node v with $d(v) < 0$, add edge (s, v) with capacity $-d(v)$
  - For each "demand" node v with $d(v) > 0$, add edge (v, t) with capacity $d(v)$

- Claim:
  - G has a circulation iff G' has max flow of value

$$\sum_{v:d(v)>0} d(v) = \sum_{v:d(v)<0} -d(v)$$

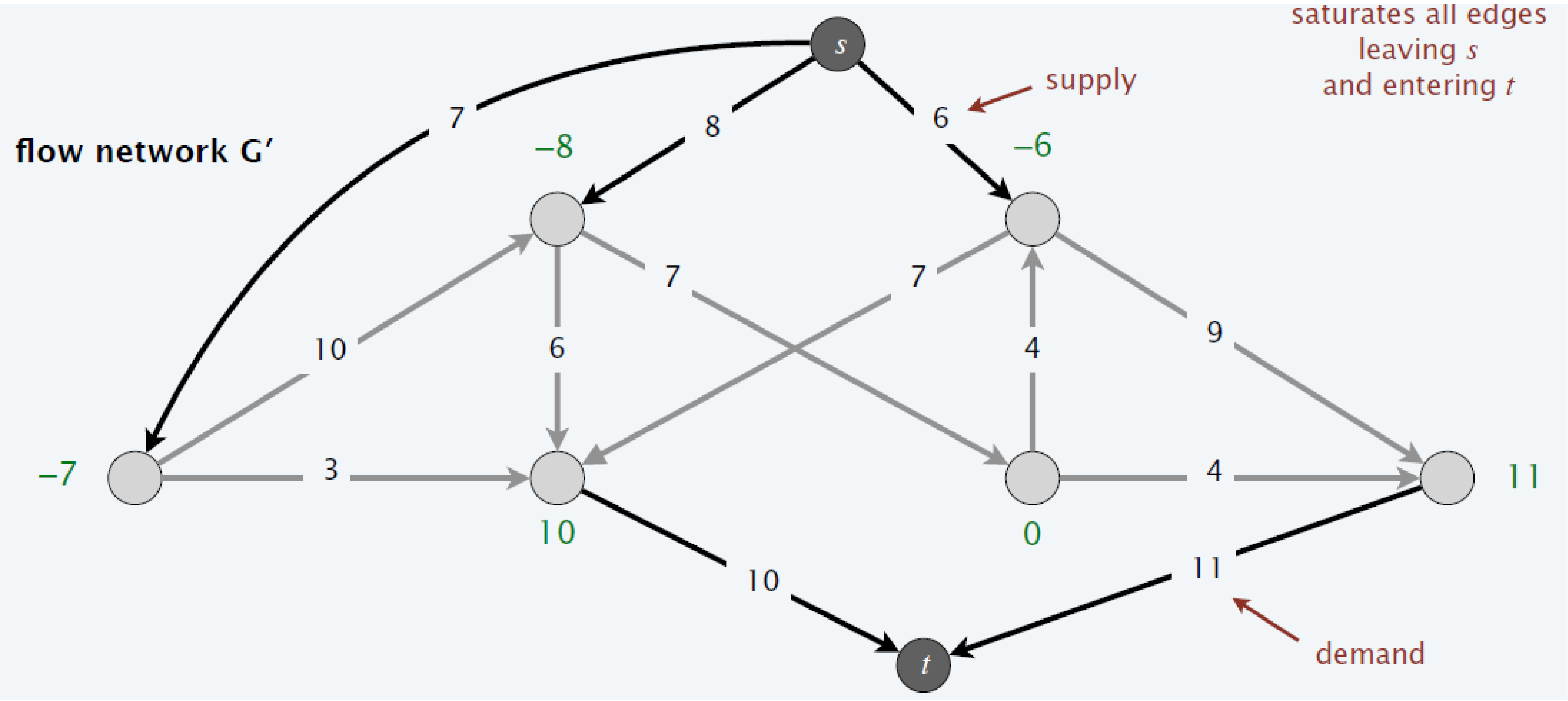
Circulation

- Example



Circulation

- Example



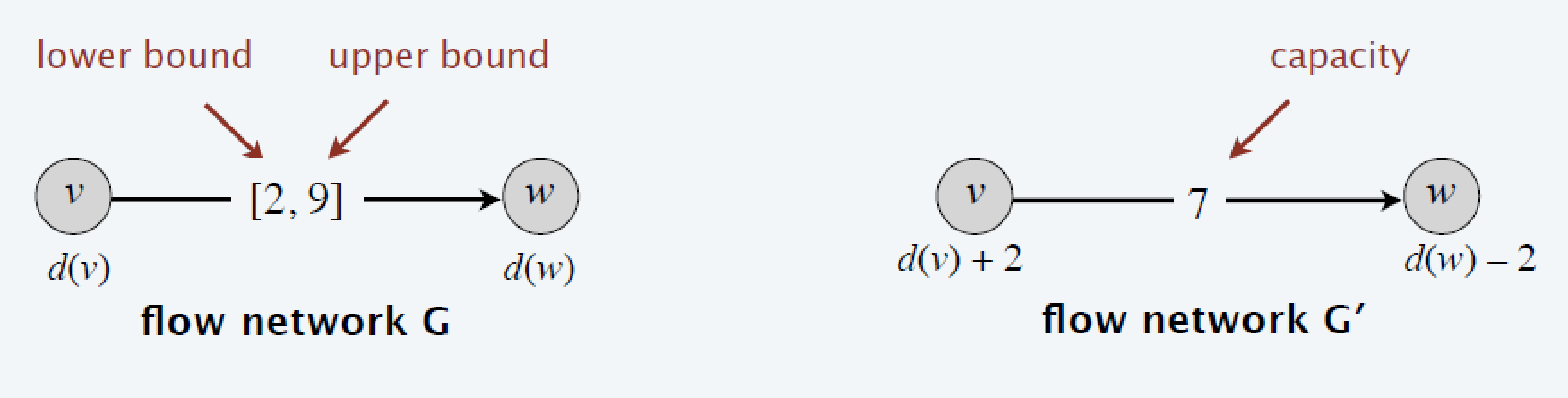
Circulation with Lower Bounds

- Input
  - Directed graph $G = (V, E)$
  - Edge capacities $c : E \to \mathbb{N}$ and lower bounds $\ell : E \to \mathbb{N}$
  - Node demands $d : V \to \mathbb{Z}$
- Output
  - Some circulation $f : E \to \mathbb{N}$ satisfying
    - For each $e \in E$: $\ell(e) \le f(e) \le c(e)$
    - For each $v \in V$: $\sum_{e \text{ entering } v} f(v) - \sum_{e \text{ leaving } v} f(v) = d(v)$

- Note that you still need $\sum_{v:d(v)>0} d(v) = \sum_{v:d(v)<0} -d(v)$

Circulation with Lower Bounds

- Transform to circulation without lower bounds
  - Do the following operation to each edge



- Claim: Circulation in G iff circulation in G'
  - Proof sketch: $f(e)$ gives a valid circulation in G iff $f(e) - \ell(e)$ gives a valid circulation in G'

Survey Design

- Problem
  - We want to design a survey about m products
    - We have one question in mind for each product
    - Need to ask product j's question to between p_j and p_j' consumers
  - There are a total of n consumers
    - Consumer i owns a subset of products O_i
    - We can ask consumer i questions about only these products
    - We want to ask consumer i between c_i and c_i' questions
  - Is there a survey meeting all these requirements?

Survey Design

- Bipartite matching is a special case
  - $c_i = c_i' = p_j = p_j' = 1$ for all i and j

- Formulate as circulation with lower bounds
  - Create a network with special nodes s and t
  - Edge from s to each consumer i with flow $\in [c_i, c_i']$
  - Edge from each consumer i to each product $j \in O_i$ with flow $\in [0,1]$
  - Edge from each product j to t with flow $\in [p_j, p_j']$
  - Edge from t to s with flow in $[0, \infty]$
  - All demands and supplies are 0

Survey Design

- Max-flow formulation:
  - Feasible survey iff feasible circulation in this network

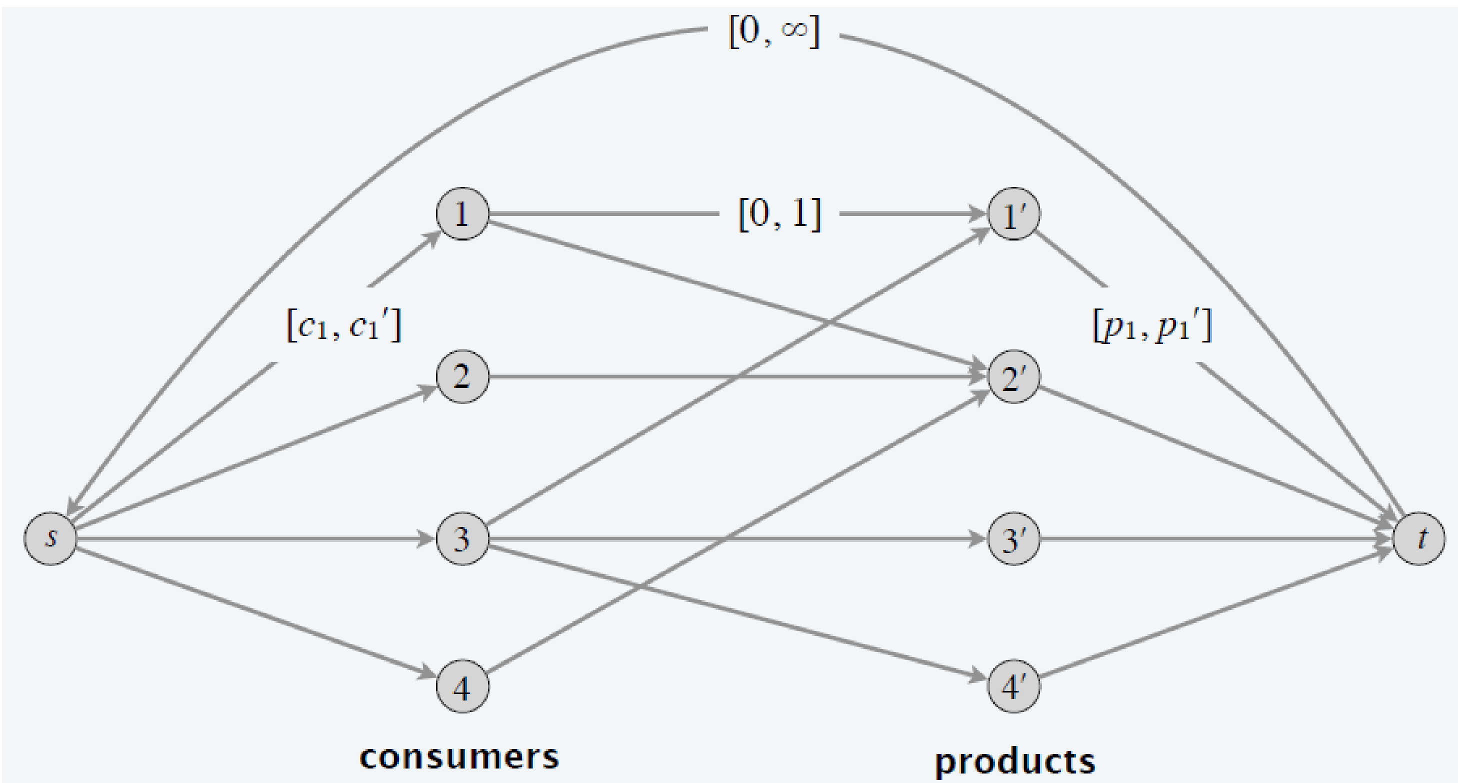


Image Segmentation

- Foreground/background segmentation
  - Given an image, separate "foreground" from "background"
- Here's the power of PowerPoint (or the lack thereof)

Remove background

Image Segmentation

- Foreground/background segmentation
  - Given an image, separate "foreground" from "background"
- Here's what remove.bg gets using AI

Remove background

Image Segmentation

- Informal problem
  - Given an image (2D array of pixels), and likelihood estimates of different pixels being foreground/background, label each pixel as foreground or background
  - Want to prevent having too many neighboring pixels where one is labeled foreground but the other is labeled background

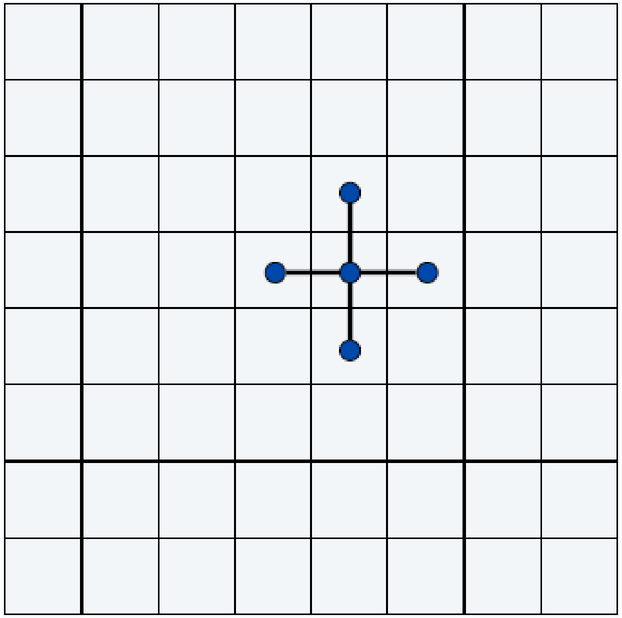


Image Segmentation

- Input
  - An image (2D array of pixels)
  - a_i = likelihood of pixel i being in foreground
  - b_i = likelihood of pixel i being in background
  - $p_{i,j}$ = penalty for “separating” pixels i and j (i.e. labeling one of them as foreground and the other as background)
- Output
  - Label each pixel as “foreground” or “background”
  - Minimize “total penalty”
    - Want it to be high if a_i is high but i is labeled background, b_i is high but i is labeled foreground, or $p_{i,j}$ is high but i and j are separated

Image Segmentation

- Recall
  - a_i = likelihood of pixels i being in foreground
  - b_i = likelihood of pixels i being in background
  - $p_{i,j}$ = penalty for separating pixels i and j
  - Let $E =$ pairs of neighboring pixels
- Output
  - Minimize total penalty
    - $A =$ set of pixels labeled foreground
    - $B =$ set of pixels labeled background
    - Penalty =

$$\sum_{i \in A} b_i + \sum_{j \in B} a_j + \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{i,j}$$

Image Segmentation

- Formulate as a min-cut problem
  - Want to divide the set of pixels V into (A, B) to minimize

$$\sum_{i \in A} b_i + \sum_{j \in B} a_j + \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{i,j}$$

  - Nodes:
    - source s, target t, and v_i for each pixel i
  - Edges:
    - (s, v_i) with capacity a_i for all i
    - (v_i, t) with capacity b_i for all i
    - (v_i, v_j) and (v_j, v_i) with capacity $p_{i,j}$ each for all neighboring (i, j)

Image Segmentation

- Formulate as min-cut problem
  - Here's what the network looks like

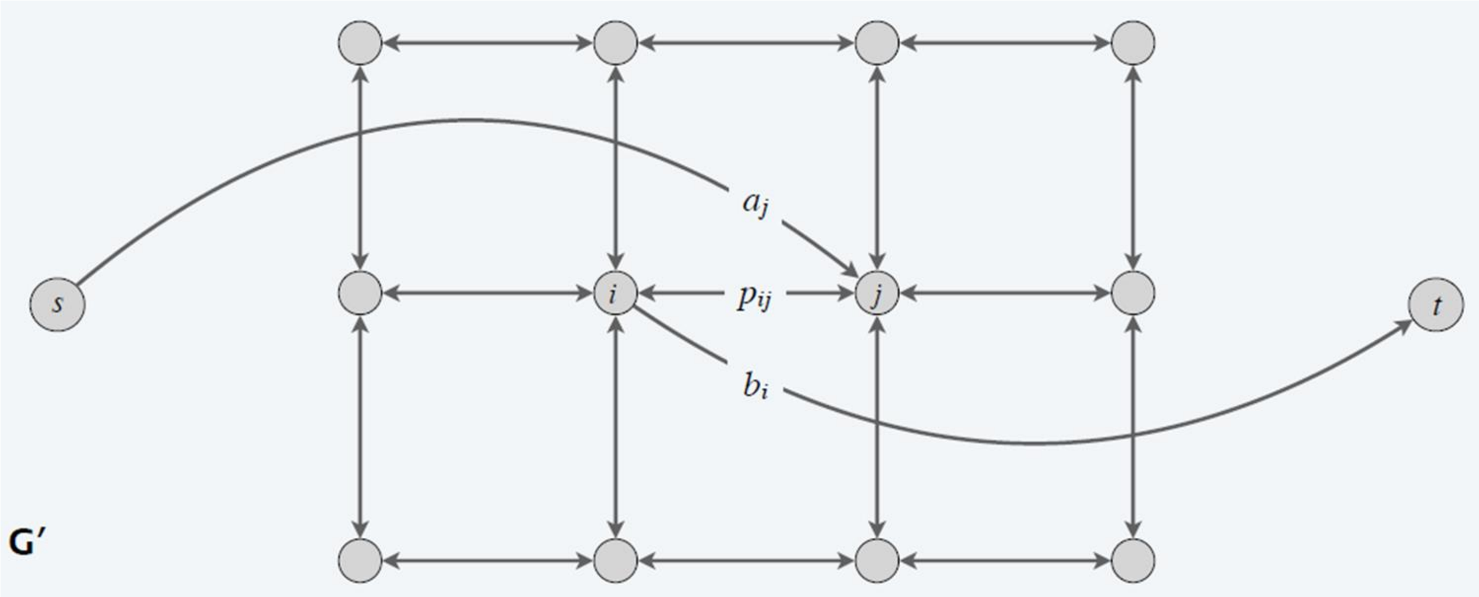


Image Segmentation

If i and j are labeled differently, it will add $p_{i,j}$ exactly once

- Consider the min-cut (A, B)

$$cap(A,B) = \sum_{i \in A} b_i + \sum_{j \in B} a_j + \sum_{\substack{(i,j) \in E \\ i \in A, j \in B}} p_{i,j}$$

- Exactly what we want to minimize!

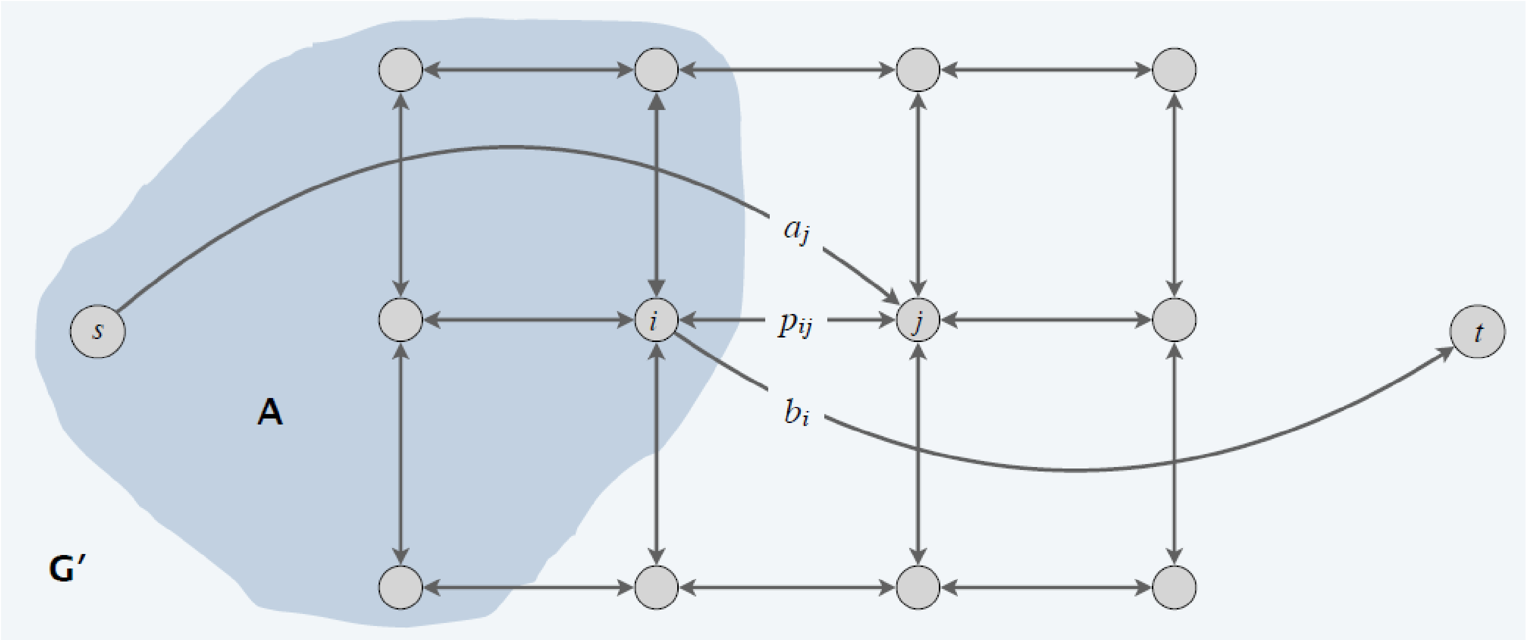


Image Segmentation

- GrabCut [Rother-Kolmogorov-Blake 2004]

"GrabCut" — Interactive Foreground Extraction using Iterated Graph Cuts

Carsten Rother* Vladimir Kolmogorov† Andrew Blake‡
Microsoft Research Cambridge, UK

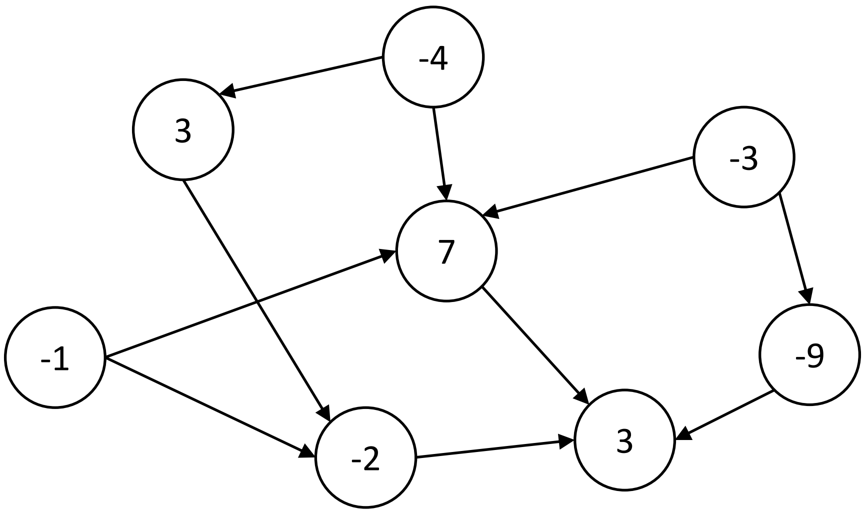
Figure 1: **Three examples of GrabCut .** The user drags a rectangle loosely around an object. The object is then extracted automatically.

Profit Maximization (Yeaa…!)

- **Problem**
  - There are n tasks
  - Performing task i generates a profit of p_i
    - We allow $p_i < 0$ (i.e., performing task i may be costly)
  - There is a set E of precedence relations
    - $(i,j) \in E$ indicates that if we perform i, we must also perform j
- **Goal**
  - Find a subset of tasks S which, subject to the precedence constraints, maximizes $profit(S) = \sum_{i \in S} p_i$

Profit Maximization

- We can represent the input as a graph
  - Nodes = tasks, node weights = profits,
  - Edges = precedence constraints
  - Goal: find a subset of nodes S with highest total weight s.t. if $i \in S$ and $(i, j) \in E$, then $j \in S$ as well

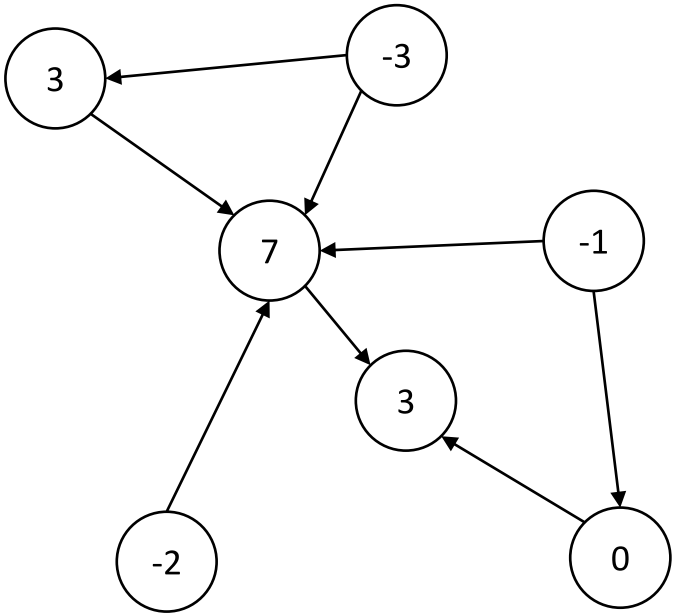


Profit Maximization

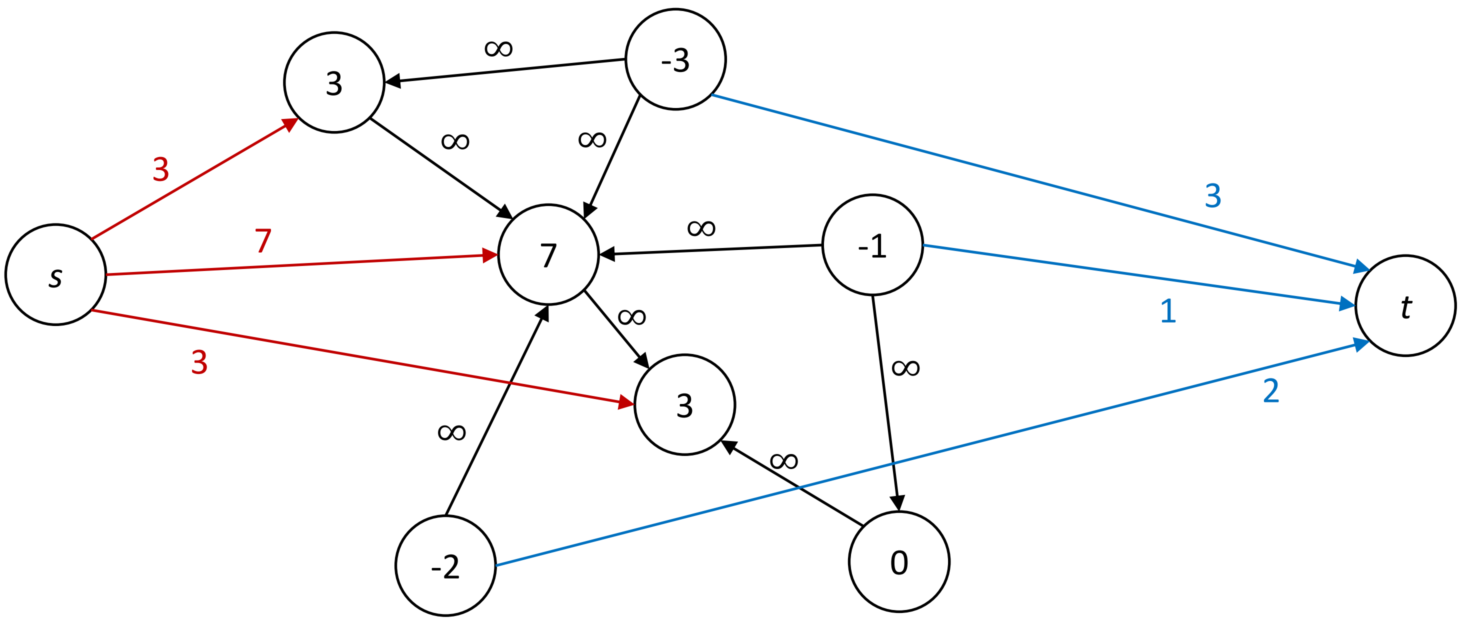
- Want to formulate as a min-cut
  - Add source s and target t
  - min-cut $(A, B) \Rightarrow$ want desired solution to be $S = A \setminus \{s\}$
  - Goals:
    - $cap(A, B)$ should nicely relate to $profit(S)$
    - Precedence constraints *must be* respected
      - “Hard” constraints are usually enforced using infinite capacity edges

- Construction:
  - Add each $(i, j) \in E$ with *infinite* capacity
  - For each i:
    - If $p_i > 0$, add (s, i) with capacity p_i
    - If $p_i < 0$, add (i, t) with capacity $-p_i$

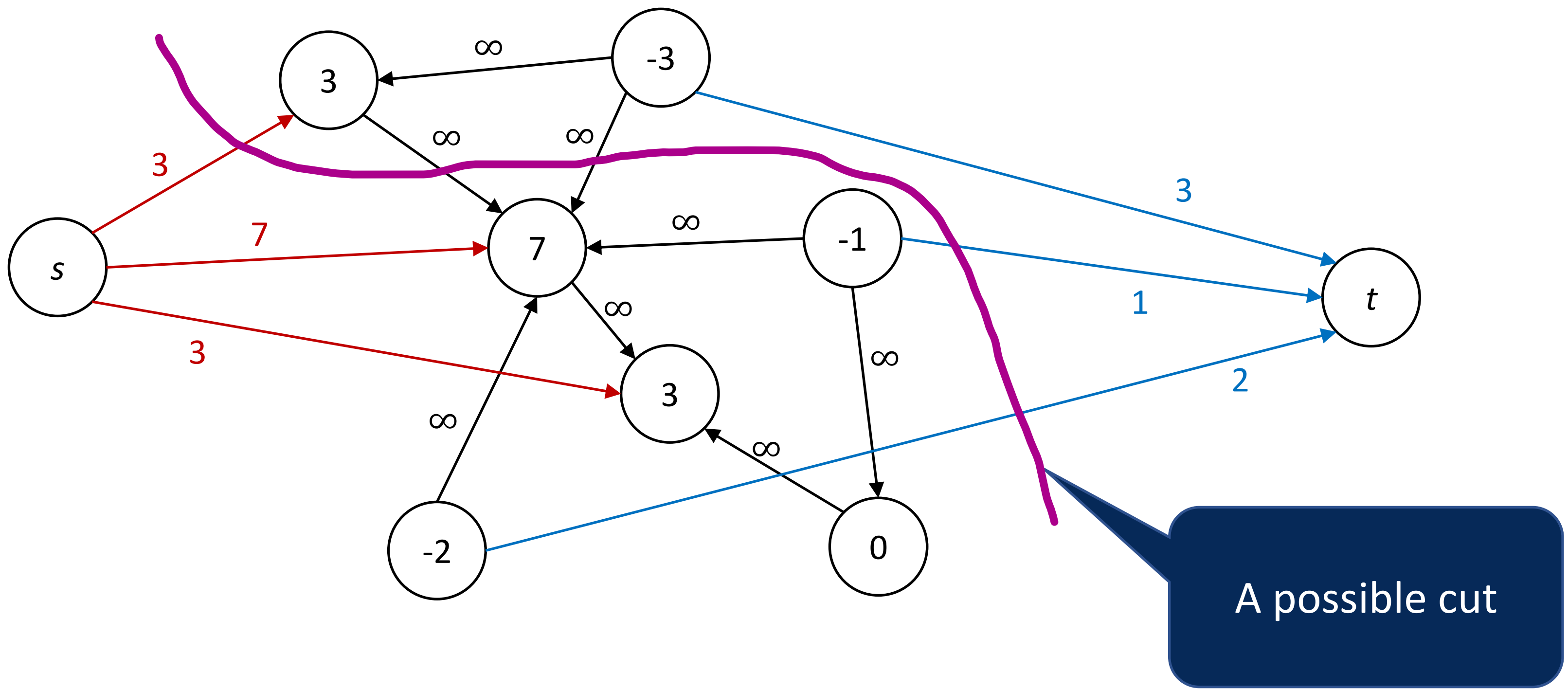
Profit Maximization



Profit Maximization



Profit Maximization



QUESTION: What is the capacity of this cut?

Profit Maximization

Exercise: Show that…

1. A finite capacity cut exists.
2. If $cap(A, B)$ is finite, then $A \setminus \{s\}$ is a valid solution;
3. Minimizing $cap(A, B)$ maximizes $profit(A \setminus \{s\})$
   - Show that $cap(A, B) = \text{constant} - profit(A \setminus \{s\})$, where the constant is independent of the choice of (A, B)