CSC373

Week 2: Greedy Algorithms

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Recap

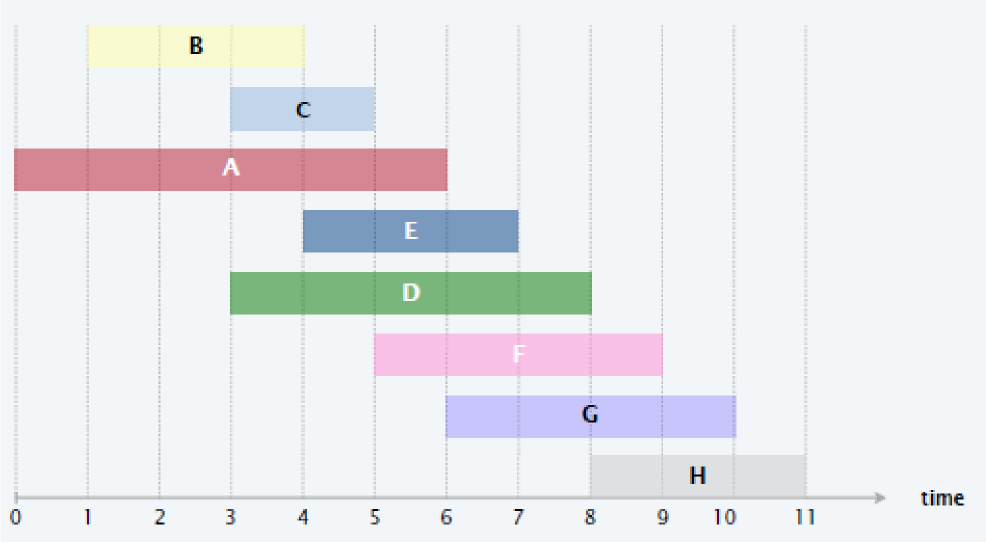
- Divide & Conquer
  - Master theorem
  - Counting inversions in $O(n \log n)$
  - Finding closest pair of points in $\mathbb{R}^2$ in $O(n \log n)$
  - Fast integer multiplication in $O\left(n^{\log_2 3}\right)$
  - Fast matrix multiplication in $O\left(n^{\log_2 7}\right)$
  - Finding k^{th} smallest element (in particular, median) in $O(n)$

Greedy Algorithms

- Greedy/myopic algorithm outline
  - Goal: find a solution x maximizing/minimizing objective function f
  - Challenge: space of possible solutions x is too large
  - Insight: x is composed of several parts (e.g., x is a set or a sequence)
  - Approach: Instead of computing x directly…
    - Compute it one part at a time
    - Select the next part "greedily" to get the most immediate "benefit" (this needs to be defined carefully for each problem)
    - Polynomial running time is typically guaranteed
    - Need to prove that this will always return an optimal solution despite having no foresight

Interval Scheduling

- Problem
  - Job j starts at time s_j and finishes at time f_j
  - Two jobs i and j are compatible if $[s_i, f_i)$ and $[s_j, f_j)$ don't overlap
    - Note: we allow a job to start right when another finishes
  - Goal: find maximum-size subset of mutually compatible jobs



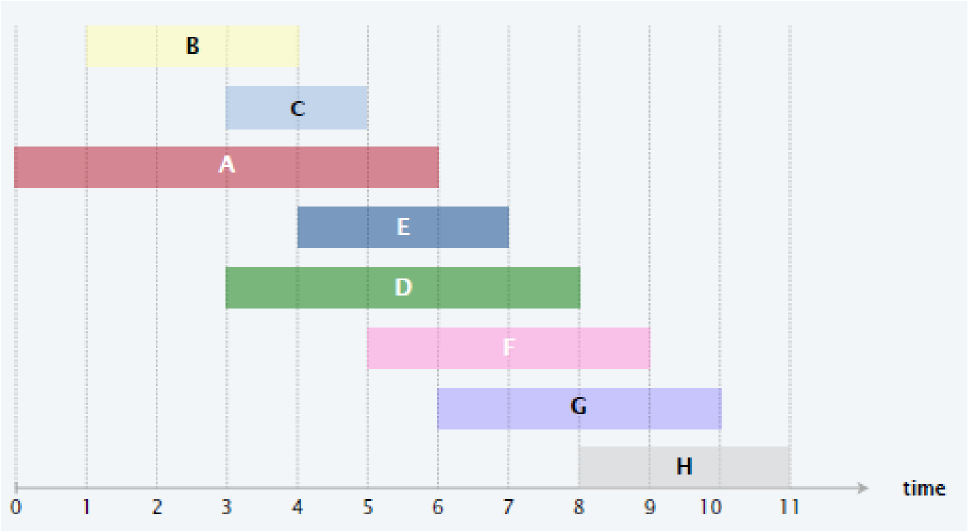
Interval Scheduling

- Greedy template
  - Consider jobs in some "natural" order
  - Take a job if it's compatible with the ones already chosen

- What order?
  - Earliest start time: ascending order of s_j
  - Earliest finish time: ascending order of f_j
  - Shortest interval: ascending order of $f_j - s_j$
  - Fewest conflicts: ascending order of c_j, where c_j is the number of remaining jobs that conflict with j

Example

- Earliest start time: ascending order of s_j
- Earliest finish time: ascending order of f_j
- Shortest interval: ascending order of $f_j - s_j$
- Fewest conflicts: ascending order of c_j, where c_j is the number of remaining jobs that conflict with j



Interval Scheduling

- Does it work?

Counterexamples for

earliest start time

shortest interval

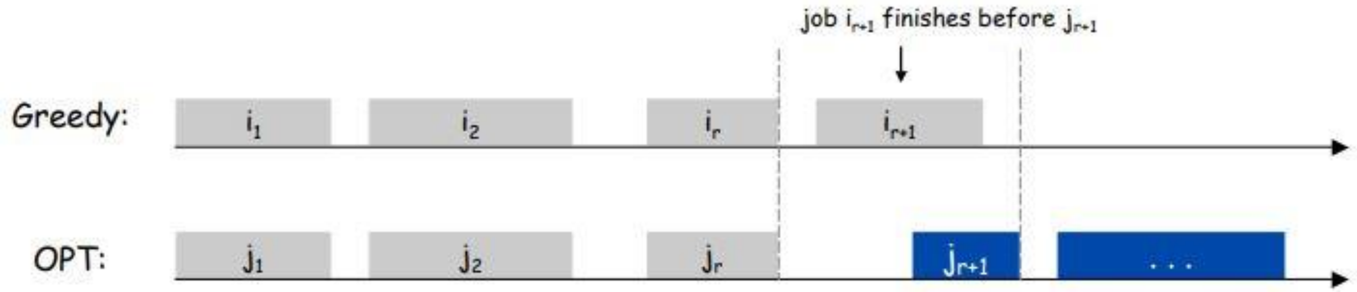
fewest conflicts

Interval Scheduling

- Implementing greedy with earliest finish time (EFT)
  - Sort jobs by finish time, say $f_1 \leq f_2 \leq \cdots \leq f_n$
    - $O(n \log n)$
  - For each job j, we need to check if it's compatible with *all* previously added jobs
    - Naively, this can take $O(n)$ time per job j, so $O(n^2)$ total time
    - We only need to check if $s_j \geq f_{i^*}$, where i^* is the *last added job*
      - For any jobs i added before i^*, $f_i \leq f_{i^*}$
      - By keeping track of f_{i^*}, we can check job j in $O(1)$ time
  - Running time: $O(n \log n)$

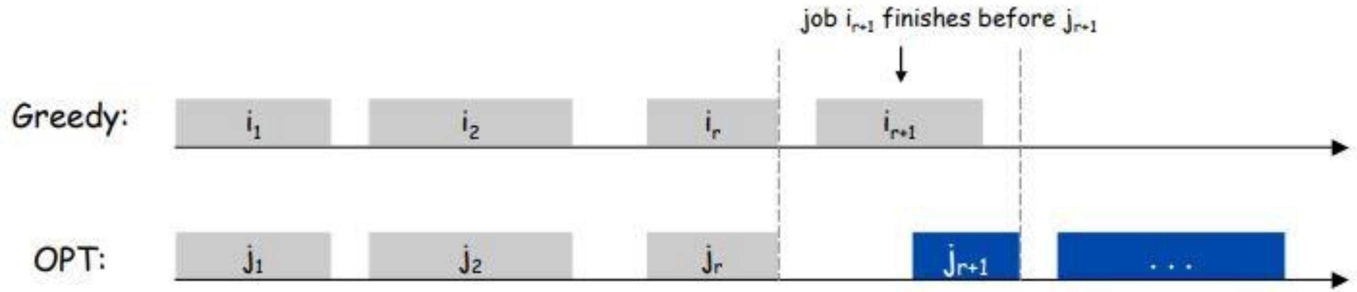
Interval Scheduling

- Proof of optimality by contradiction
  - Suppose for contradiction that greedy is not optimal
  - Say greedy selects jobs $i_1, i_2, \ldots, i_k$ sorted by finish time
  - Consider an optimal solution $j_1, j_2, \ldots, j_m$ (also sorted by finish time) which matches greedy for as many indices as possible
    - That is, we want $j_1 = i_1, \ldots, j_r = i_r$ for the greatest possible r
  - Both i_{r+1} and j_{r+1} must be compatible with the previous selection ($i_1 = j_1, \ldots, i_r = j_r$)



Interval Scheduling

- Proof of optimality by contradiction
  - Consider a new solution $i_1, i_2, \ldots, i_r, i_{r+1}, j_{r+2}, \ldots, j_m$
    - We have replaced j_{r+1} by i_{r+1} in our reference optimal solution
    - <u>This is still feasible</u> because $f_{i_{r+1}} \leq f_{j_{r+1}} \leq s_{j_t}$ for $t \geq r + 2$
    - <u>This is still optimal</u> because m jobs are selected
    - But it matches the greedy solution in $r + 1$ indices
      - This is the desired contradiction



Interval Scheduling

- Proof of optimality by induction
  - Let S_j be the subset of jobs picked by greedy after considering the first j jobs in the increasing order of finish time
    - Define $S_0 = \emptyset$
  - We call this partial solution *promising* if there is a way to extend it to an optimal solution by picking some subset of jobs $j+1, \dots, n$
    - $\exists T \subseteq \{j+1, \dots, n\}$ such that $O_j = S_j \cup T$ is optimal
  - Inductive claim: For all $t \in \{0,1, \dots, n\}$, S_t is promising
  - If we prove this, then we are done!
    - For $t = n$, if S_n is promising, then it must be optimal (Why?)
    - We chose $t = 0$ as our base case since it is "trivial"

Interval Scheduling

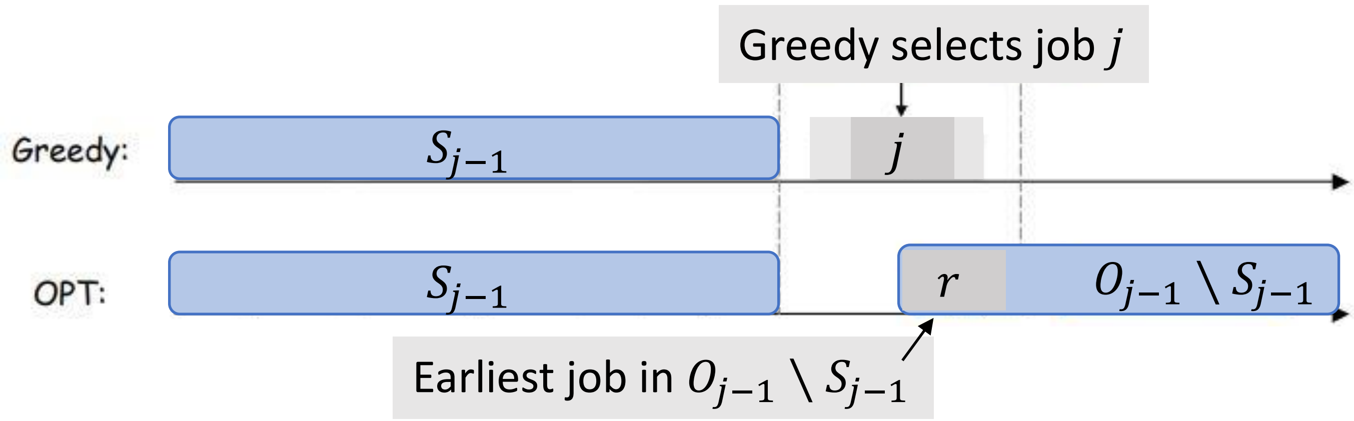
- Proof of optimality by induction
  - S_j is *promising* if $\exists T \subseteq \{j+1, \dots, n\}$ such that $O_j = S_j \cup T$ is optimal
  - Inductive claim: For all $t \in \{0,1,\dots,n\}$, S_t is promising
  - Base case: For $t = 0$, $S_0 = \emptyset$ is clearly promising
    - Any optimal solution extends it
  - Induction hypothesis: Suppose the claim holds for $t = j-1$ and optimal solution O_{j-1} extends S_{j-1}
  - Induction step: At $t = j$, we have two possibilities:

    1) Greedy did not select job j, so $S_j = S_{j-1}$
       - Job j must conflict with some job in S_{j-1}
       - Since $S_{j-1} \subseteq O_{j-1}$, O_{j-1} also cannot include job j
       - $O_j = O_{j-1}$ also extends $S_j = S_{j-1}$

Interval Scheduling

- Proof of optimality by induction
  - Induction step: At $t = j$, we have two possibilities:

    2) Greedy selected job j, so $S_j = S_{j-1} \cup \{ j \}$
    - Consider the earliest job r in $O_{j-1} \setminus S_{j-1}$
    - Consider O_j obtained by replacing r with j in O_{j-1}
    - Prove that O_j is still feasible
    - O_j extends S_j, as desired!



Contradiction vs Induction

- Both methods make the same claim
  - "The greedy solution after j iterations can be extended to an optimal solution, $\forall j$"

- They also use the same key argument
  - "If the greedy solution after j iterations can be extended to an optimal solution, then the greedy solution after $j+1$ iterations can be extended to an optimal solution as well"

  - For proof by induction, this is the key induction step
  - For proof by contradiction, we take the greatest j for which the greedy solution can be extended to an optimal solution, and derive a contradiction by extending the greedy solution after $j+1$ iterations

Interval Partitioning

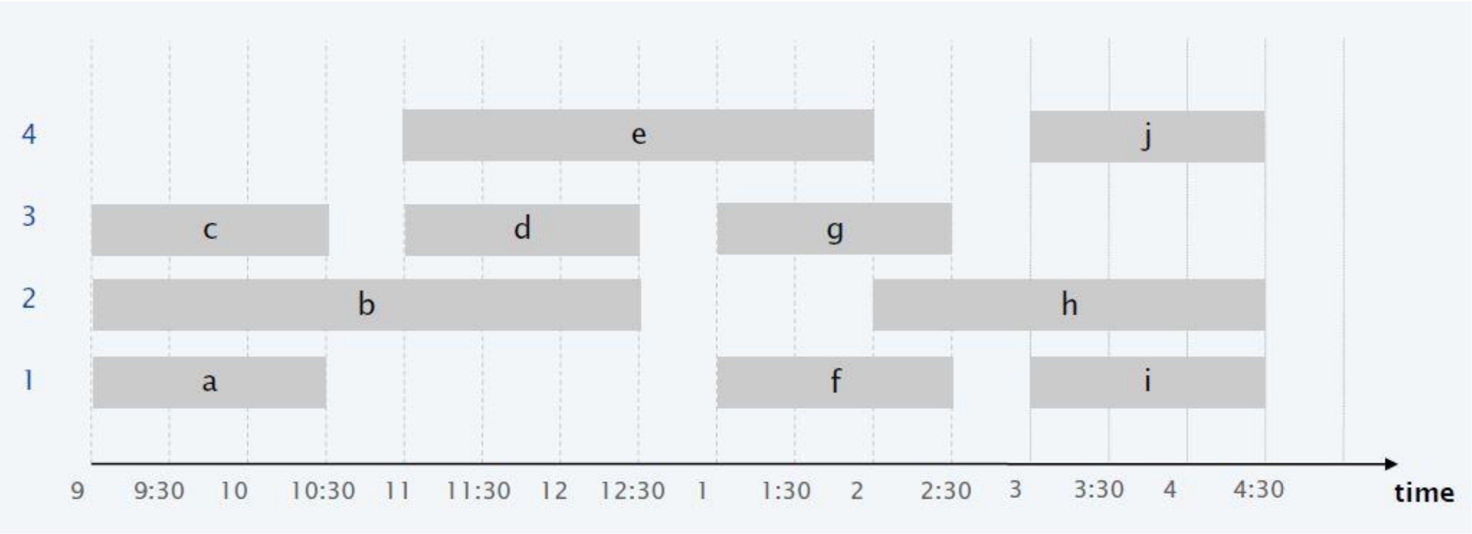
- Problem
  - Job j starts at time s_j and finishes at time f_j
  - Two jobs are compatible if they don't overlap
  - Goal: group jobs into fewest partitions such that jobs in the same partition are compatible

- One idea
  - Find the maximum compatible set using the previous greedy EFT algorithm, call it one partition, recurse on the remaining jobs.
  - Doesn't work (check by yourselves)

Interval Partitioning

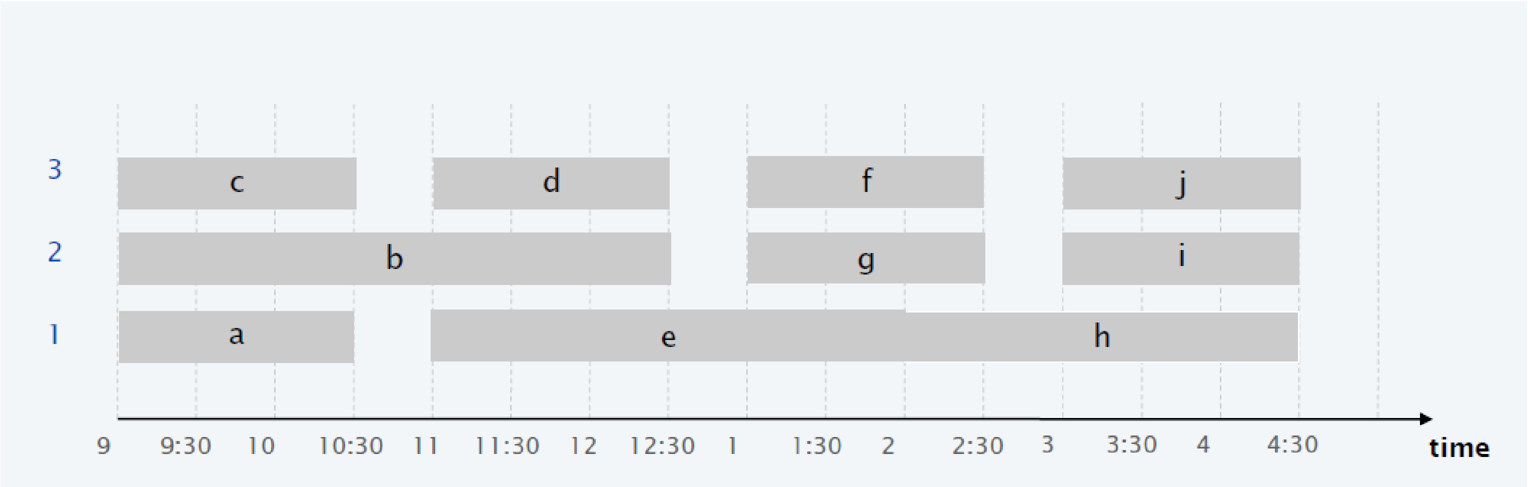
- Think of scheduling lectures for various courses into as few classrooms as possible

- This schedule uses 4 classrooms for scheduling 10 lectures



Interval Partitioning

- Think of scheduling lectures for various courses into as few classrooms as possible
- This schedule uses 3 classrooms for scheduling 10 lectures

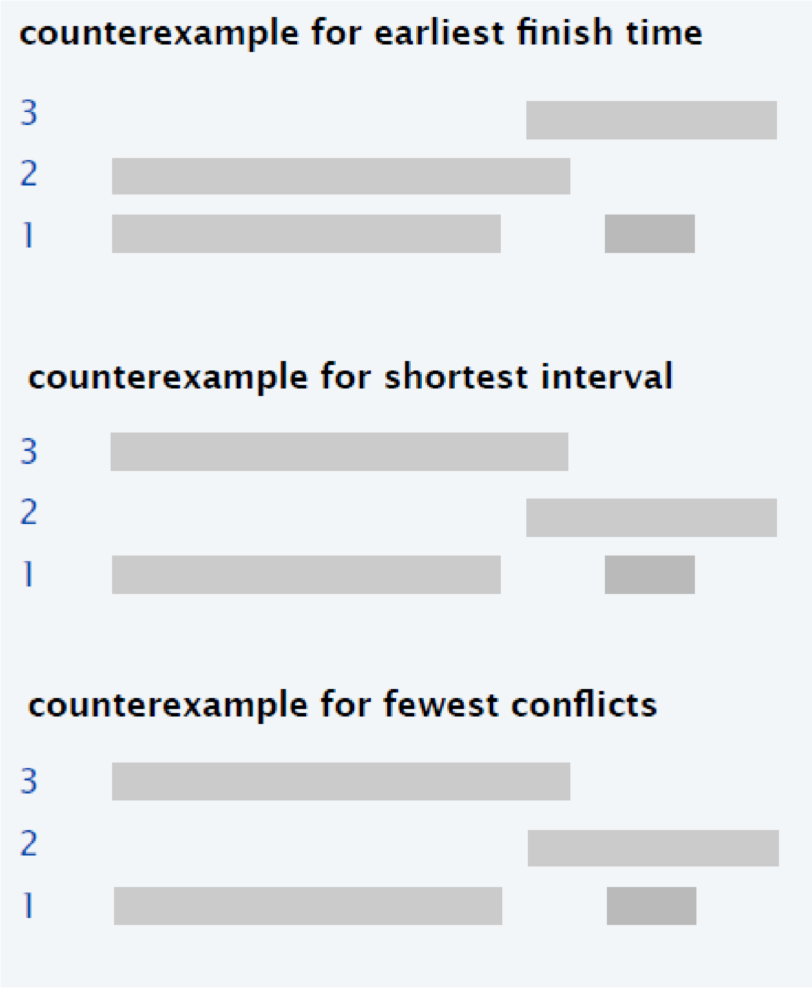


Interval Partitioning

- Let's go back to the greedy template!
  - Go through lectures in some "natural" order
  - Assign each lecture to an (arbitrary?) compatible classroom, and create a new classroom if the lecture conflicts with every existing classroom

- Order of lectures?
  - Earliest start time: ascending order of s_j
  - Earliest finish time: ascending order of f_j
  - Shortest interval: ascending order of $f_j - s_j$
  - Fewest conflicts: ascending order of c_j, where c_j is the number of remaining jobs that conflict with j

Interval Partitioning



- At least when you assign each lecture to an arbitrary compatible classroom, three of these heuristics do not work.
- The fourth one works! (next slide)

Interval Partitioning

EARLIESTSTARTTIMEFIRST(n, s_1, s_2, ..., s_n, f_1, f_2, ..., f_n)

SORT lectures by start time so that $s_1 \leq s_2 \leq \ldots \leq s_n$.

$d \leftarrow 0$ ⟵ number of allocated classrooms

FOR $j = 1$ TO n

  IF lecture j is compatible with some classroom

    Schedule lecture j in any such classroom k.

  ELSE

    Allocate a new classroom $d + 1$.

    Schedule lecture j in classroom $d + 1$.

    $d \leftarrow d + 1$

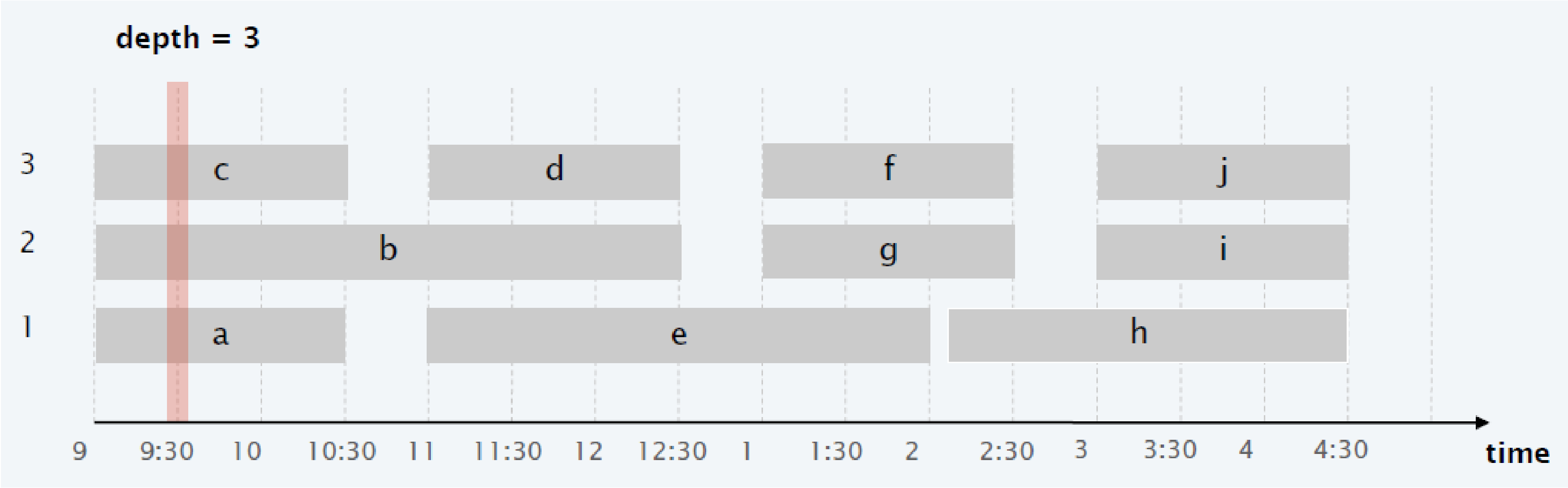
RETURN schedule.

Interval Partitioning

- Running time
  - Key step: check if the next lecture can be scheduled at some classroom
  - Store classrooms in a priority queue
    - key = latest finish time of any lecture in the classroom
  - Is lecture j compatible with some classroom?
    - Same as “Is s_j at least as large as the minimum key?”
    - If yes: add lecture j to classroom k with minimum key, and increase its key to f_j
    - Otherwise: create a new classroom, add lecture j, set key to f_j
  - $O(n)$ priority queue operations, $O(n \log n)$ time

Interval Partitioning

- Proof of optimality (lower bound)
  - # classrooms needed ≥ "depth"
    - depth = maximum number of lectures running at any time
    - Recall, as before, that job i runs in $[s_i, f_i)$
  - Claim: our greedy algorithm uses only these many classrooms!



Interval Partitioning

- Proof of optimality (upper bound)
  - Let d = # classrooms used by greedy
  - Classroom d was opened because there was a lecture j which was incompatible with some lectures already scheduled in each of $d - 1$ other classrooms
  - All these d lectures end after s_j
  - *Since we sorted by start time*, they all start at/before s_j
  - So, at time s_j, we have d mutually overlapping lectures
  - Hence, depth $\geq d =$ #classrooms used by greedy ∎
  - Note: before we proved that #classrooms used by any algorithm (including greedy) $\geq$ depth, so greedy uses exactly as many classrooms as the depth.

Interval Graphs

NOT IN SYLLABUS

- Interval scheduling and interval partitioning can be seen as graph problems

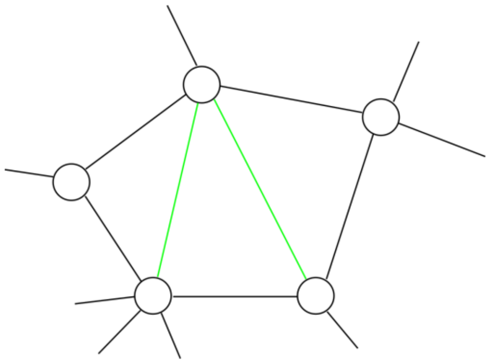
- Input
  - Graph $G = (V, E)$
  - Vertices V = jobs/lectures
  - Edge $(i, j) \in E$ if jobs i and j are incompatible

- Interval scheduling = maximum independent set (MIS)
- Interval partitioning = graph coloring

Interval Graphs

NOT IN SYLLABUS

- MIS and graph coloring are NP-hard for general graphs
- But they're efficiently solvable for "interval graphs"
  - ➢ Graphs which can be obtained from incompatibility of intervals
  - ➢ In fact, this holds even when we are not given an interval representation of the graph
- Can we extend this result further?
  - ➢ Yes! Chordal graphs
    - o Every cycle with 4 or more vertices has a chord



Minimizing Lateness

- Problem
  - We have a single machine
  - Each job j requires t_j units of time and is due by time d_j
  - If it's scheduled to start at s_j, it will finish at $f_j = s_j + t_j$
  - Lateness: $\ell_j = \max\{0, f_j - d_j\}$
  - Goal: minimize the maximum lateness, $L = \max_j \ell_j$

- Contrast with interval scheduling
  - We can decide the start time
  - There are soft deadlines

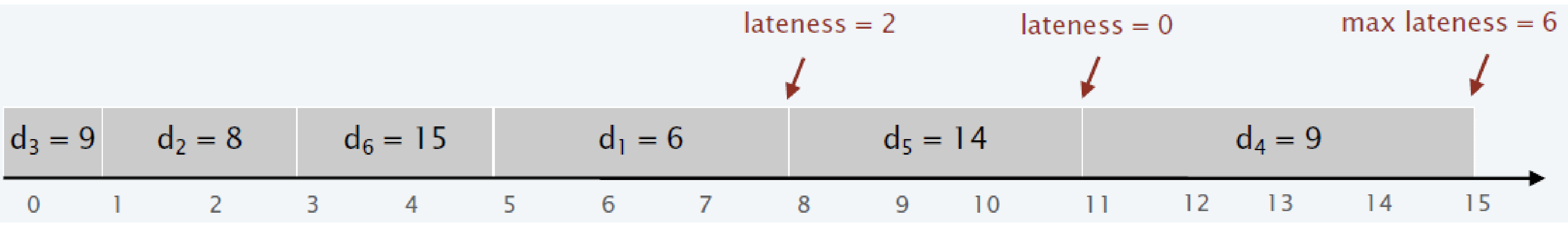
Minimizing Lateness

- Example

Input

	1	2	3	4	5	6
t_j	3	2	1	4	3	2
d_j	6	8	9	9	14	15

An example schedule



Minimizing Lateness

- Let's go back to greedy template
  - Consider jobs one-by-one in some "natural" order
  - Schedule jobs in this order (nothing special to do here, since we have to schedule all jobs and there is only one machine available)
- Natural orders?
  - Shortest processing time first: ascending order of processing time t_j
  - Earliest deadline first: ascending order of due time d_j
  - Smallest slack first: ascending order of $d_j - t_j$

Minimizing Lateness

- Counterexamples

  - Shortest processing time first
    - Ascending order of processing time t_j

	1	2
t_j	1	10
d_j	100	10

  - Smallest slack first
    - Ascending order of $d_j - t_j$

	1	2
t_j	1	10
d_j	2	10

Minimizing Lateness

- By now, you should know what's coming…
- We'll prove that earliest deadline first works!

$\text{EarliestDeadlineFirst}(n, t_1, t_2, \ldots, t_n, d_1, d_2, \ldots, d_n)$

Sort n jobs so that $d_1 \le d_2 \le \ldots \le d_n$.

$t \leftarrow 0$

For $j = 1$ to n

  Assign job j to interval $[t, t + t_j]$.

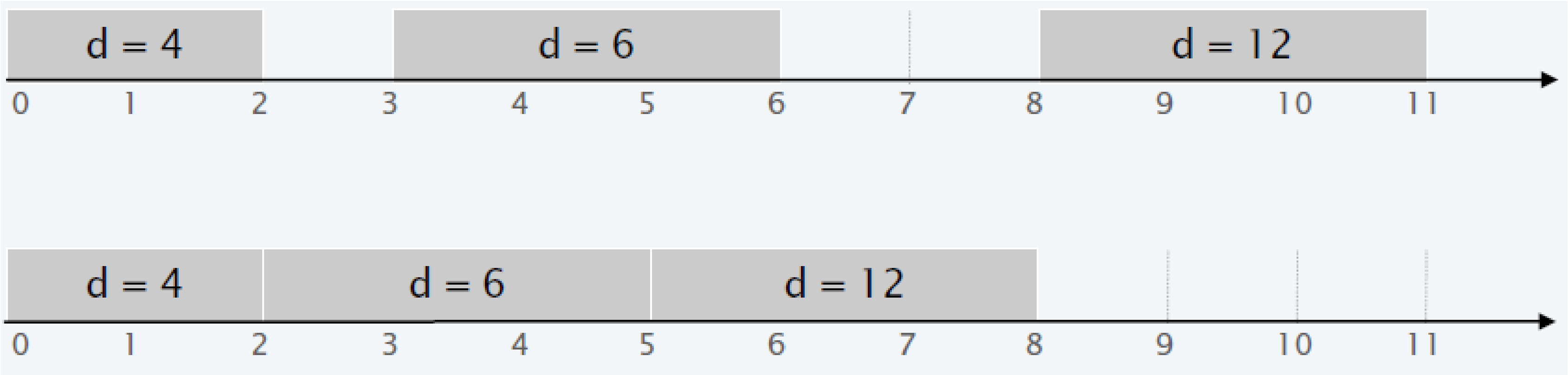
  $s_j \leftarrow t$; $f_j \leftarrow t + t_j$

  $t \leftarrow t + t_j$

Return intervals $[s_1, f_1], [s_2, f_2], \ldots, [s_n, f_n]$.

Minimizing Lateness

- Observation 1
  - ➢ There is an optimal schedule with no idle time



Minimizing Lateness

- Observation 2
  - Earliest deadline first has no idle time
- Let us define an “inversion”
  - (i, j) such that $d_i < d_j$ but j is scheduled before i
- Observation 3
  - By definition, earliest deadline first has no inversions
- Observation 4
  - If a schedule with no idle time has at least one inversion, it has a pair of inverted jobs scheduled consecutively

Minimizing Lateness

- Observation 5
  - Swapping adjacently scheduled inverted jobs doesn't increase lateness but reduces #inversions by one
- Proof
  - Check that swapping an adjacent inverted pair reduces the total #inversions by one

Minimizing Lateness

- Observation 5
  - Swapping adjacently scheduled inverted jobs doesn't increase lateness but reduces #inversions by one
- Proof
  - Let ℓ_k and ℓ'_k denote the lateness of job k before & after swap
  - Let $L = \max_k \ell_k$ and $L' = \max_k \ell'_k$
  - 1) $\ell_k = \ell'_k$ for all $k \neq i, j$ (no change in their finish time)
  - 2) $\ell'_i \leq \ell_i$ (i is moved early)

Minimizing Lateness

- Observation 5
  - Swapping adjacently scheduled inverted jobs doesn't increase lateness but reduces #inversions by one
- Proof
  - 3) $\ell'_j = f'_j - d_j = f_i - d_j \le f_i - d_i = \ell_i$
    - This uses the fact that, due to the inversion, $d_j \ge d_i$
  - $L' = \max\left\{\ell'_i, \ell'_j, \max_{k \ne i,j} \ell'_k\right\} \le \max\left\{\ell_i, \ell_i, \max_{k \ne i,j} \ell_k\right\} \le L$

Minimizing Lateness

- Observations 4+5 are the key!

- Recall the proof of optimality of the greedy algorithm for interval scheduling:
  - Took an optimal solution matching greedy for r steps, and produced another optimal solution matching greedy for $r + 1$ steps
  - "Wrapped" this in a proof by contradiction or a proof by induction
  - Observations 4+5 provide something similar
    - If optimal solution doesn't fully match greedy (#inversions ≥ 1), we can swap an adjacent inverted pair and reduce #inversions by one

Minimizing Lateness

- Proof of optimality by contradiction
  - Suppose for contradiction that the greedy EDF solution is not optimal
  - Consider an optimal schedule S^* with the fewest inversions
    - Without loss of generality, suppose it has no idle time
  - Because EDF is not optimal, S^* has at least one inversion
  - By Observation 4, it has an adjacent inversion (i, j)
  - By Observation 5, swapping the adjacent pair keeps the schedule optimal but reduces the #inversions by 1
  - Contradiction! ■

Minimizing Lateness

- Proof of optimality by (reverse) induction
  - Claim: For each $r \in \left\{0,1,\ldots,\binom{n}{2}\right\}$, there is an optimal schedule with *at most* r inversions
  - Base case of $r = \binom{n}{2}$: trivial, any optimal schedule works
  - Induction hypothesis: Suppose the claim holds for $r = t + 1$
  - Induction step: Take an optimal schedule with at most $t + 1$ inversions
    - If it has at most t inversions, we're done!
    - If it has exactly $t + 1 \geq 1$ inversions…
      - Assume no idle time WLOG
      - Find and swap an adjacent inverted pair (Observations 4 & 5)
      - #inversions reduces by one to t, so we're done!
  - QED!
  - Claim for $r = 0$ shows optimality of EDF

Contradiction vs Induction

- Choose the method that feels natural to you

- It may be the case that…
  - For some problems, a proof by contradiction feels more natural
  - But for other problems, a proof by induction feels more natural
  - No need to stick to one method

- As we saw for interval partitioning, sometimes you may require an entirely different kind of proof

Lossless Compression

- Problem
  - We have a document that is written using n distinct labels
  - Naïve encoding: represent each label using $\log n$ bits
  - If the document has length m, this uses $m \log n$ bits

- English document with no punctuations etc.
- $n = 26$, so we can use 5 bits
  - $a = 00000$
  - $b = 00001$
  - $c = 00010$
  - $d = 00011$
  - …

Lossless Compression

- Is this optimal?
  - What if a, e, r, s are much more frequent in the document than x, q, z?
  - Can we assign shorter codes to more frequent letters?

- Say we assign…
  - $a = 0, b = 1, c = 01, \ldots$
  - See a problem?
    - What if we observe the encoding '01'?
    - Is it 'ab'? Or is it 'c'?

Lossless Compression

- To avoid conflicts, we need a *prefix-free encoding*
  - Map each label x to a bit-string $c(x)$ such that for all distinct labels x and y, $c(x)$ is not a prefix of $c(y)$
  - Then it's impossible to have a scenario like this

    $$\underbrace{\underbrace{\ldots\ldots\ldots\ldots\ldots}_{c(x)}\ldots\ldots\ldots\ldots}_{c(y)}$$

  - Now, we can read left to right
    - Whenever the part to the left becomes a valid encoding, greedily decode it, and continue with the rest

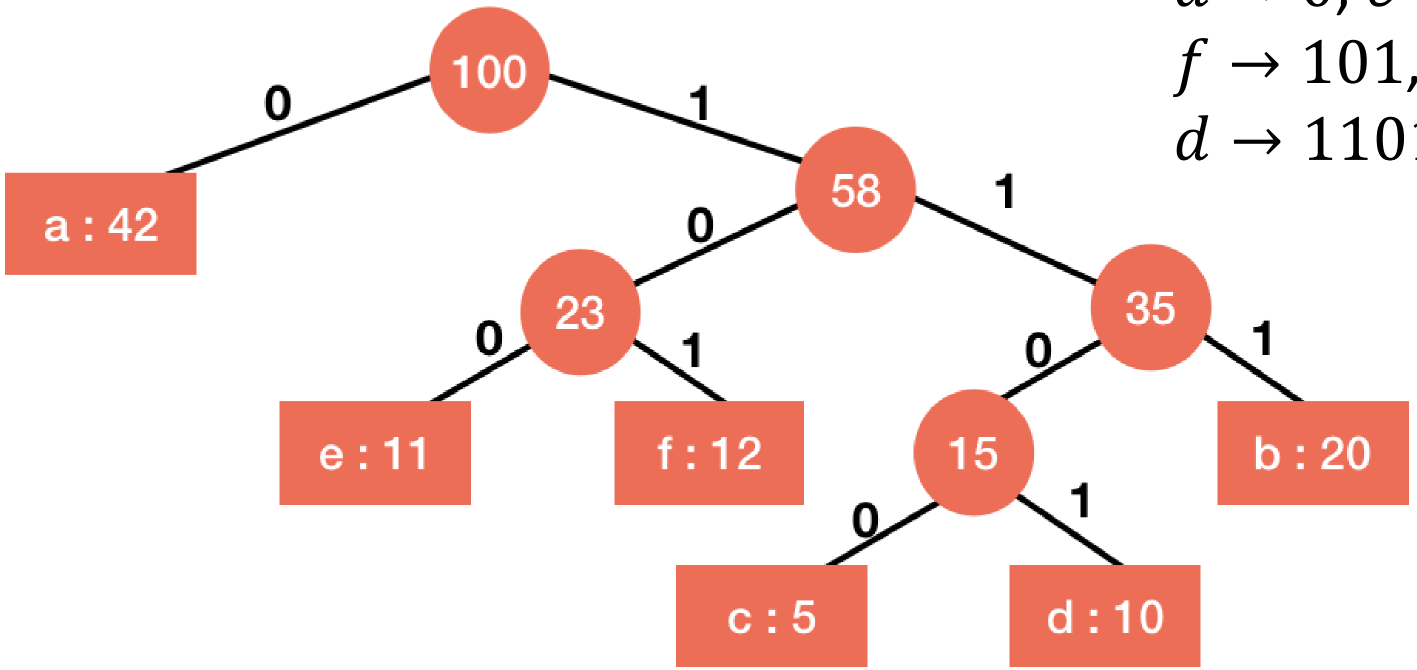
Lossless Compression

- Formal problem
  - Given n symbols and their frequencies $(w_1, \dots, w_n)$, find a prefix-free encoding with lengths $(\ell_1, \dots, \ell_n)$ assigned to the symbols which minimizes $\sum_{i=1}^{n} w_i \cdot \ell_i$
    - Note that $\sum_{i=1}^{n} w_i \cdot \ell_i$ is the length of the compressed document

- Example
  - $(w_a, w_b, w_c, w_d, w_e, w_f) = (42,20,5,10,11,12)$
  - No need to remember the numbers ☺

Lossless Compression

- Observation: prefix-free encoding = tree



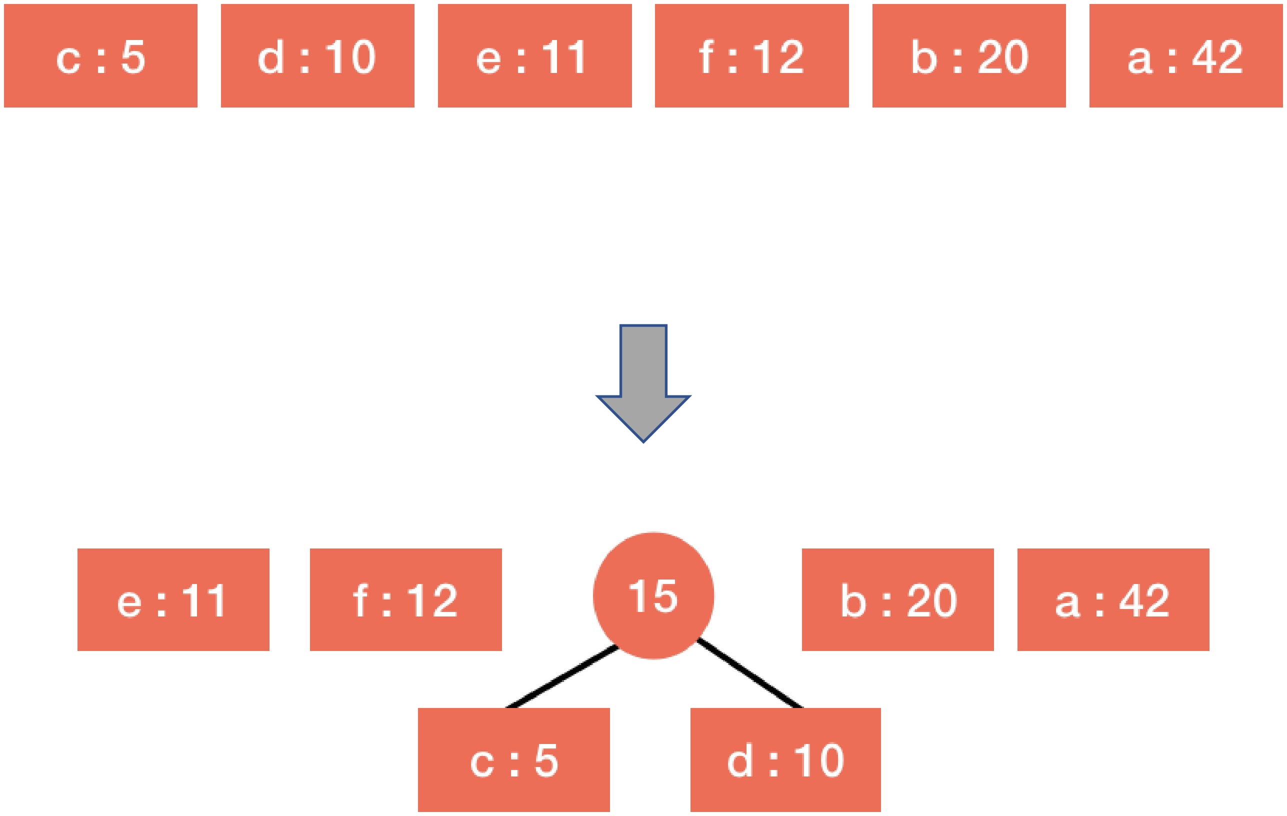
$a \to 0, e \to 100,$
$f \to 101, c \to 1100,$
$d \to 1101, b \to 111$

Lossless Compression

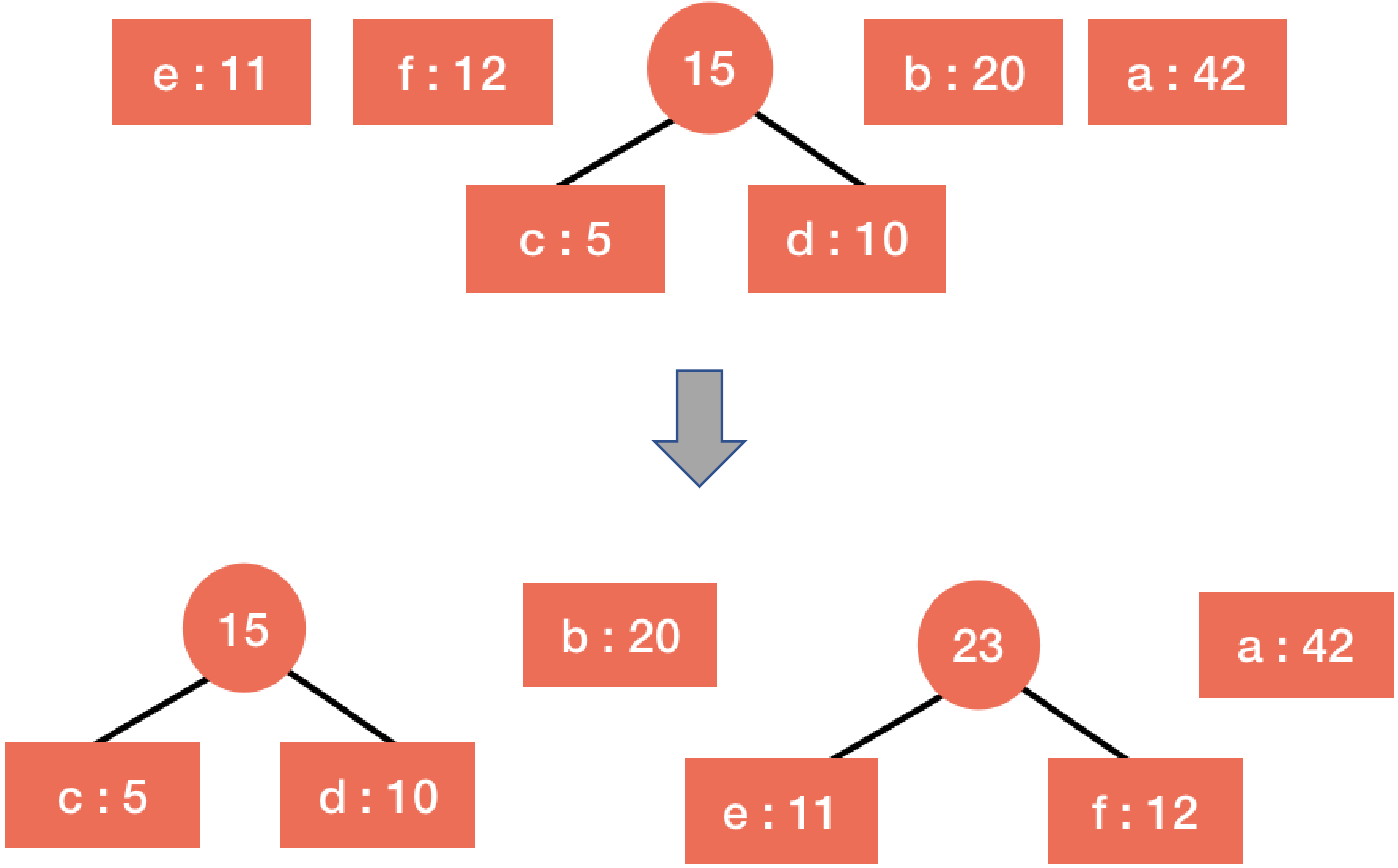
- Huffman Coding
  - Build a priority queue by adding (x, w_x) for each symbol x
  - While |queue|≥ 2
    - Take the two symbols with the lowest weight (x, w_x) and (y, w_y)
    - Merge them into one symbol with weight $w_x + w_y$

- Let's see this on the previous example

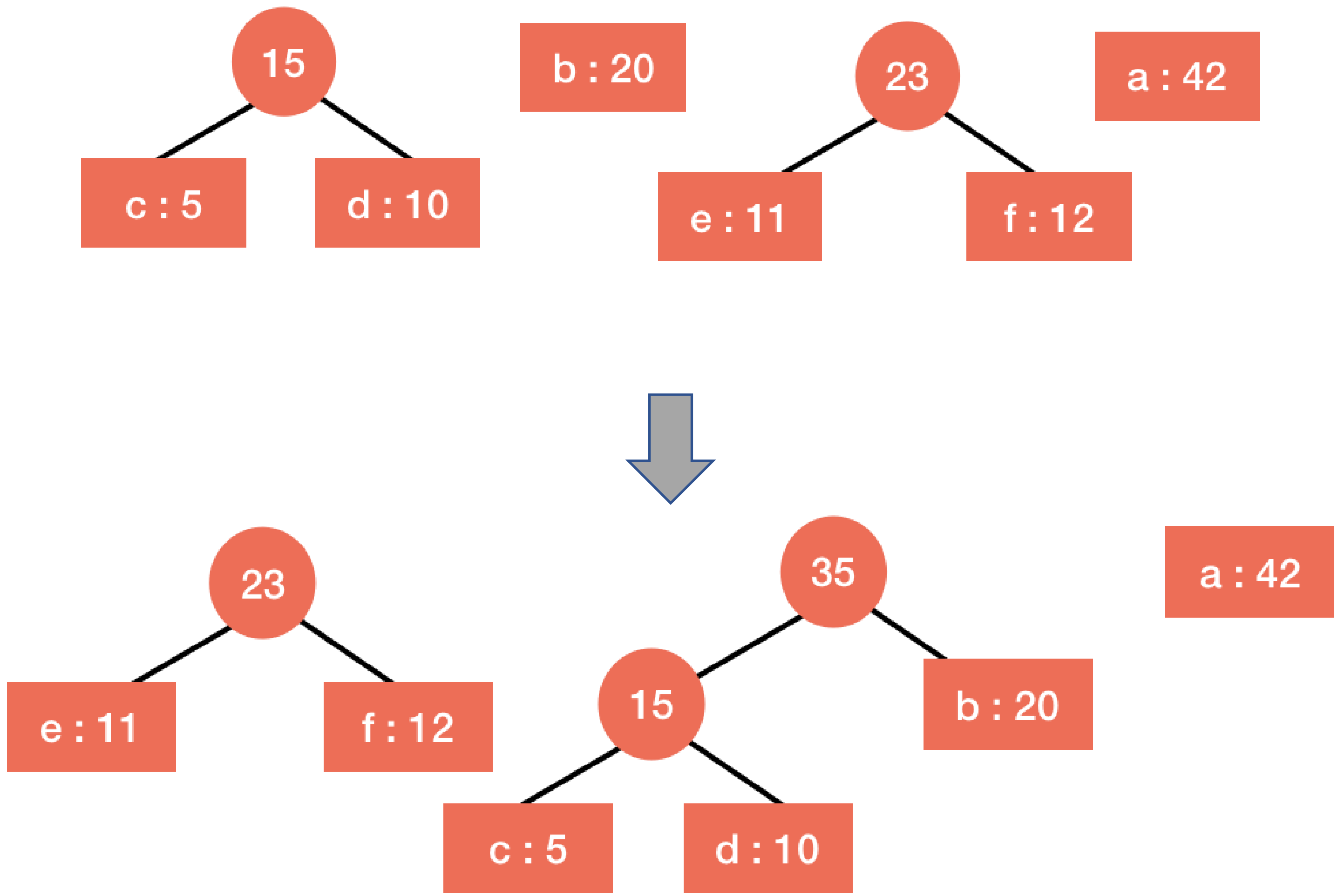
Lossless Compression



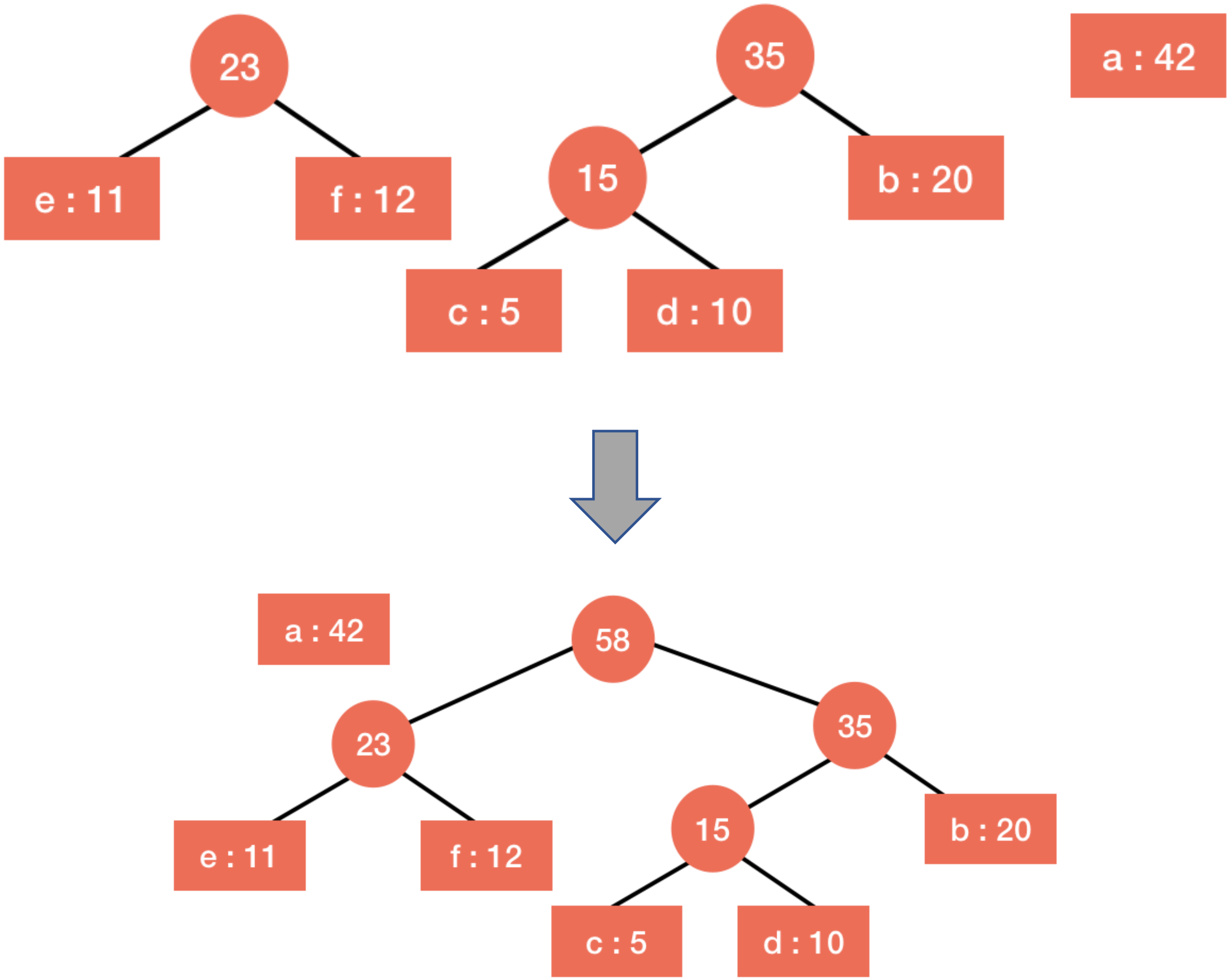
Lossless Compression



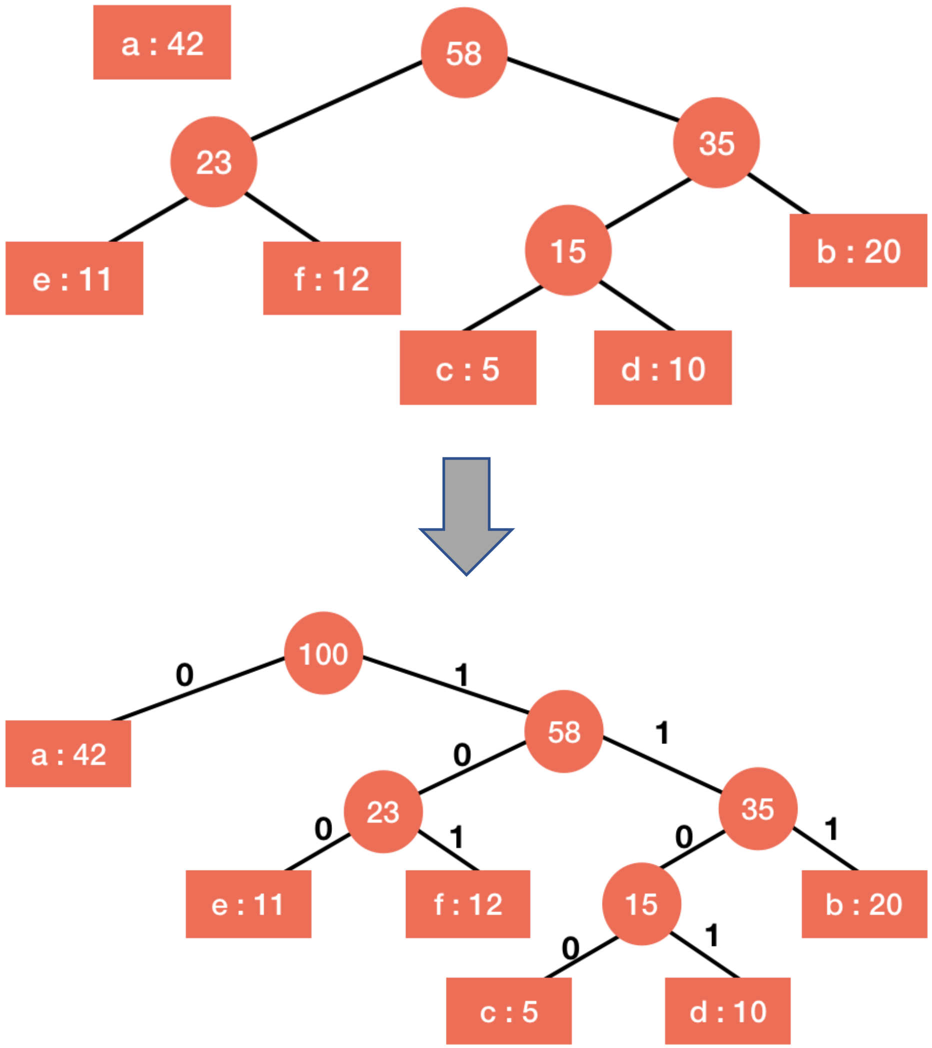
Lossless Compression



Lossless Compression



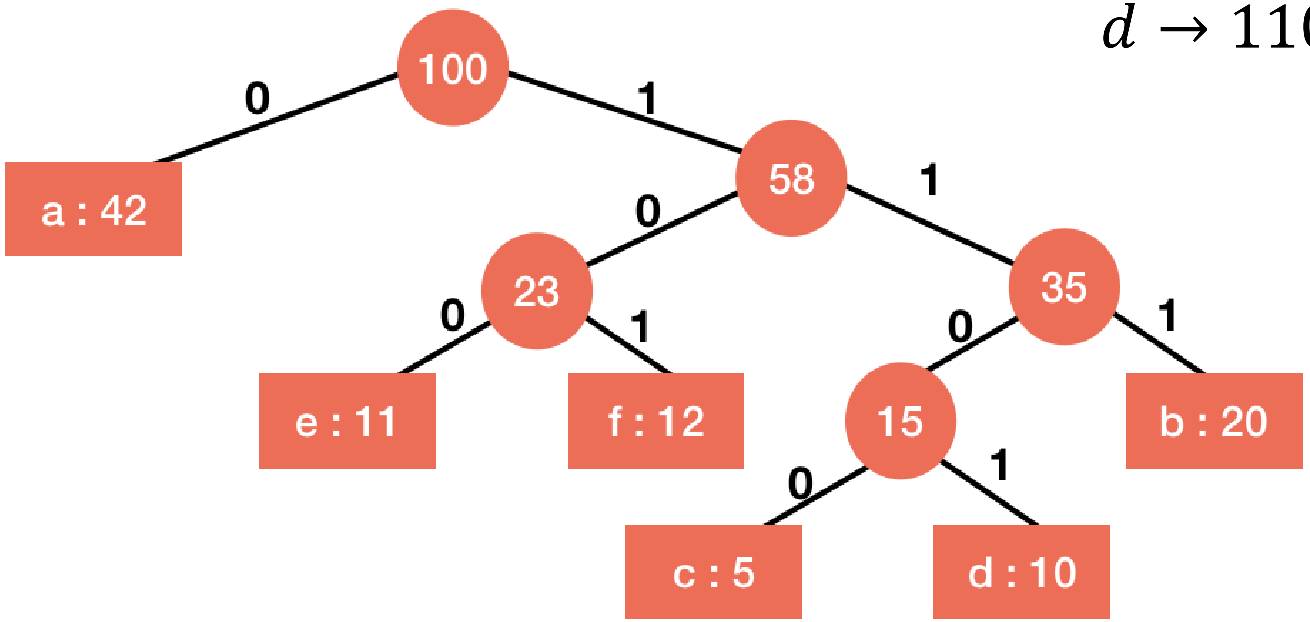
Lossless Compression



Lossless Compression

- Final Outcome

$a \to 0, e \to 100,$
$f \to 101, c \to 1100,$
$d \to 1101, b \to 111$



Lossless Compression

- Running time
  - $O(n \log n)$
  - Can be made $O(n)$ if the labels are given to you sorted by their frequencies
    - Exercise! Think of using two queues…

- Proof of optimality
  - Induction on the number of symbols n
  - Base case: For $n = 2$, both encodings which assign 1 bit to each symbol are optimal
  - Hypothesis: Assume it returns an optimal encoding with $n - 1$ symbols

Lossless Compression

- Proof of optimality
  - Consider the case of n symbols
  - Lemma 1: If $w_x < w_y$, then $\ell_x \geq \ell_y$ in any optimal tree.
  - Proof:
    - Suppose for contradiction that $w_x < w_y$ and $\ell_x < \ell_y$.
    - Swapping x and y strictly reduces the overall length as $w_x \cdot \ell_y + w_y \cdot \ell_x < w_x \cdot \ell_x + w_y \cdot \ell_y$ (check!)
    - QED!

Lossless Compression

- Proof of optimality
  - Consider the two symbols x and y with lowest frequency which Huffman combines in the first step
  - Lemma 2: ∃ optimal tree T in which x and y are siblings (i.e., for some p, they are assigned encodings $p0$ and $p1$).
  - Proof:
    1. Take any optimal tree
    2. Let x be the label with the lowest frequency.
    3. If x doesn't have the longest encoding, swap it with one that has
    4. Due to optimality, x must have a sibling (check!)
    5. If it's not y, swap it with y
    6. Check that Steps 3 and 5 do not change the overall length. ■

Lossless Compression

- Proof of optimality
  - Let x and y be the two least frequency symbols that Huffman combines in the first step into "xy"
  - Let H be the Huffman tree produced
  - Let T be an optimal tree in which x and y are siblings
  - Let H' and T' be obtained from H and T by treating xy as one symbol with frequency $w_x + w_y$
  - Induction hypothesis: $Length(H') \leq Length(T')$
  - $Length(H) = Length(H') + \left(w_x + w_y\right) \cdot 1$
  - $Length(T) = Length(T') + \left(w_x + w_y\right) \cdot 1$
  - So $Length(H) \leq Length(T)$ ■

Other Greedy Algorithms

- If you aren't familiar with the following algorithms, spend some time checking them out!
  - Dijkstra's shortest path algorithm
  - Kruskal and Prim's minimum spanning tree algorithms