CSC373

Week 6:
Linear Programming

Illustration Courtesy:
Kevin Wayne & Denis Pankratov

Announcement

- ACM ICPC Qualification Round
- Oct 24, 3-8pm EST

- Sign up at: https://www.teach.cs.toronto.edu/~acm/

- Top 9 participants will be chosen to represent U of T at the regional contest (broken into three teams of 3 each)

Recap

- Network flow
  - Ford-Fulkerson algorithm
    - Ways to make the running time polynomial
  - Correctness using max-flow, min-cut
  - Applications:
    - Edge-disjoint paths
    - Multiple sources/sinks
    - Circulation
    - Circulation with lower bounds
    - Survey design
    - Image segmentation
    - Profit maximization

Brewery Example

- A brewery can invest its inventory of corn, hops and malt into producing some amount of ale and some amount of beer
  - Per unit resource requirement and profit of the two items are as given below

Beverage	Corn (pounds)	Hops (ounces)	Malt (pounds)	Profit ($)
Ale (barrel)	5	4	35	13
Beer (barrel)	15	4	20	23
constraint	480	160	1190	

Example Courtesy: Kevin Wayne

Brewery Example

Beverage	Corn (pounds)	Hops (ounces)	Malt (pounds)	Profit ($)
Ale (barrel)	5	4	35	13
Beer (barrel)	15	4	20	23
constraint	480	160	1190	

- Suppose it produces A units of ale and B units of beer
- Then we want to solve this program:

objective function

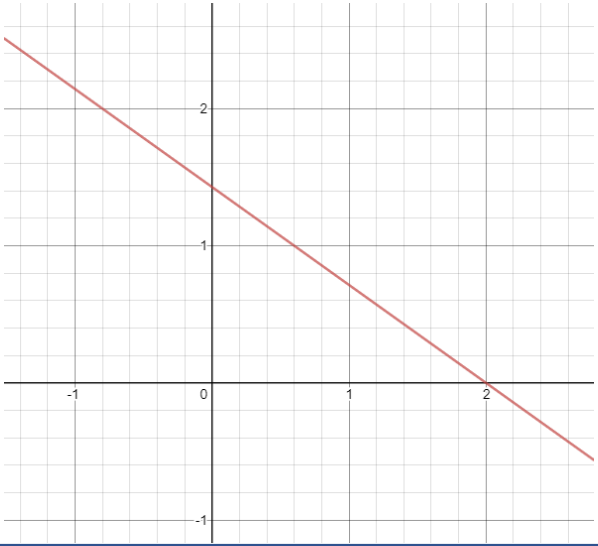
$$\begin{array}{lcccccl}
 & \text{Ale} & & \text{Beer} & & & \\
\max & 13A & + & 23B & & & \text{Profit} \\
\text{s. t.} & 5A & + & 15B & \leq & 480 & \text{Corn} \\
 & 4A & + & 4B & \leq & 160 & \text{Hops} \\
 & 35A & + & 20B & \leq & 1190 & \text{Malt} \\
 & A & , & B & \geq & 0 &
\end{array}$$

constraint

decision variable

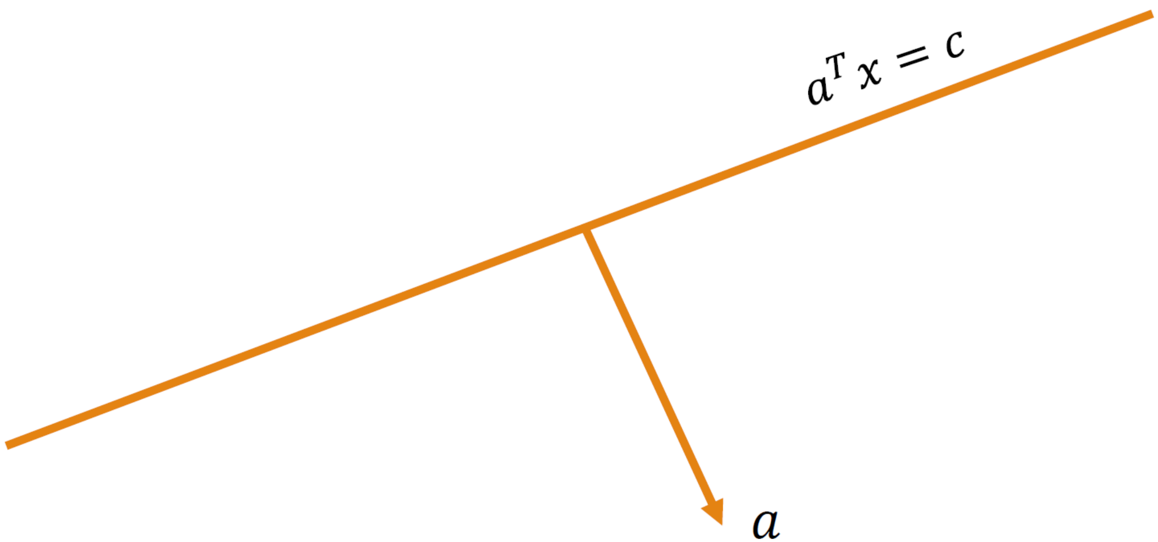
Linear Function

- $f: \mathbb{R}^n \to \mathbb{R}$ is a linear function if $f(x) = a^T x$ for some $a \in \mathbb{R}^n$
  - Example: $f(x_1, x_2) = 3x_1 - 5x_2 = \binom{3}{-5}^T \binom{x_1}{x_2}$
- Linear objective: f
- Linear constraints:
  - $g(x) = c$, where $g: \mathbb{R}^n \to \mathbb{R}$ is a linear function and $c \in \mathbb{R}$
  - Line in the plane (or a hyperplane in $\mathbb{R}^n$)
  - Example: $5x_1 + 7x_2 = 10$



Linear Function

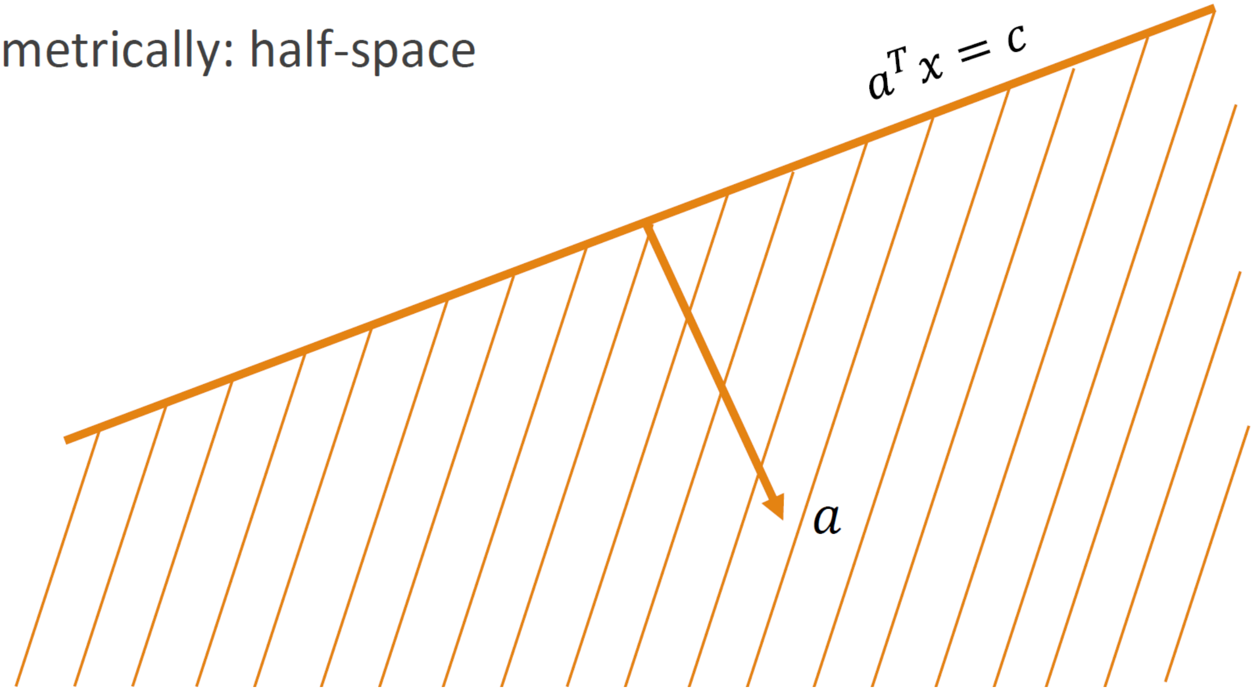
- Geometrically, a is the normal vector of the line(or hyperplane) represented by $a^T x = c$



Linear Inequality

- $a^T x \leq c$ represents a "half-space"

Geometrically: half-space



Linear Programming

- Maximize/minimize a linear function subject to linear equality/inequality constraints

Could be min

Linear objective!

Objective function $\max x_1 + 6x_2$

Constraints
$$x_1 \leq 200$$
$$x_2 \leq 300$$
$$x_1 + x_2 \leq 400$$
$$x_1, x_2 \geq 0$$

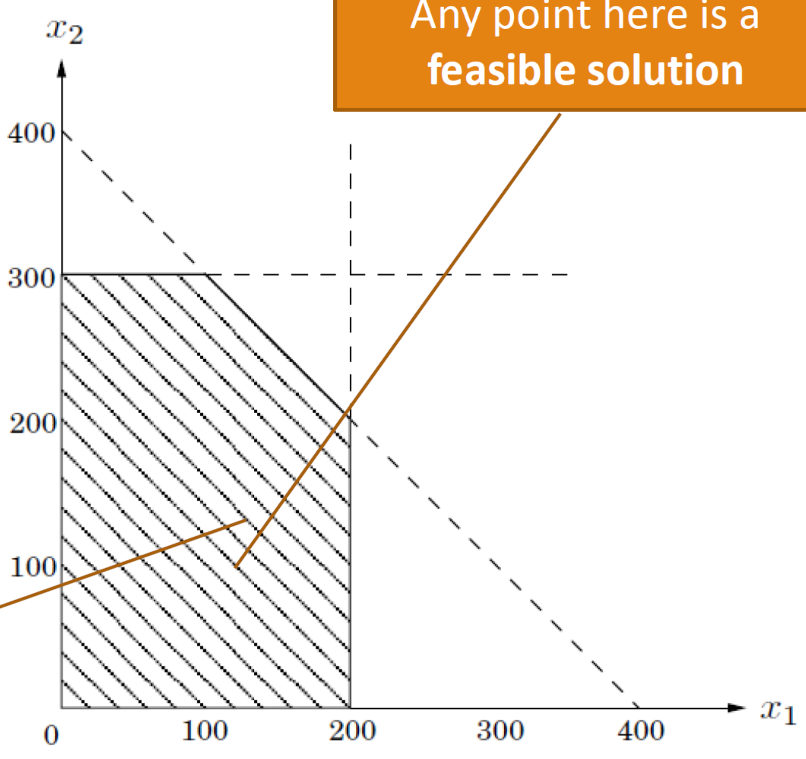
Linear constraints: inequalities/equalities

Geometrically…

Objective function $\max x_1 + 6x_2$

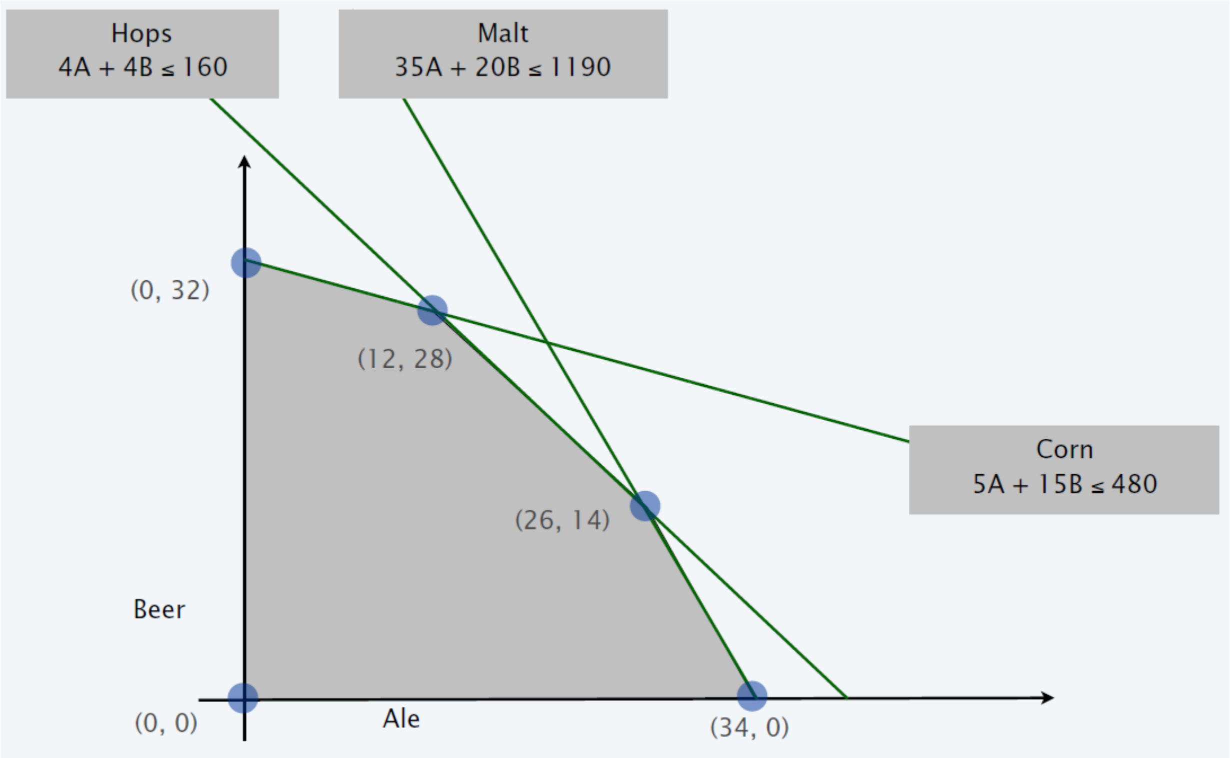
Constraints
$$x_1 \leq 200$$
$$x_2 \leq 300$$
$$x_1 + x_2 \leq 400$$
$$x_1, x_2 \geq 0$$

Any point here is a **feasible solution**

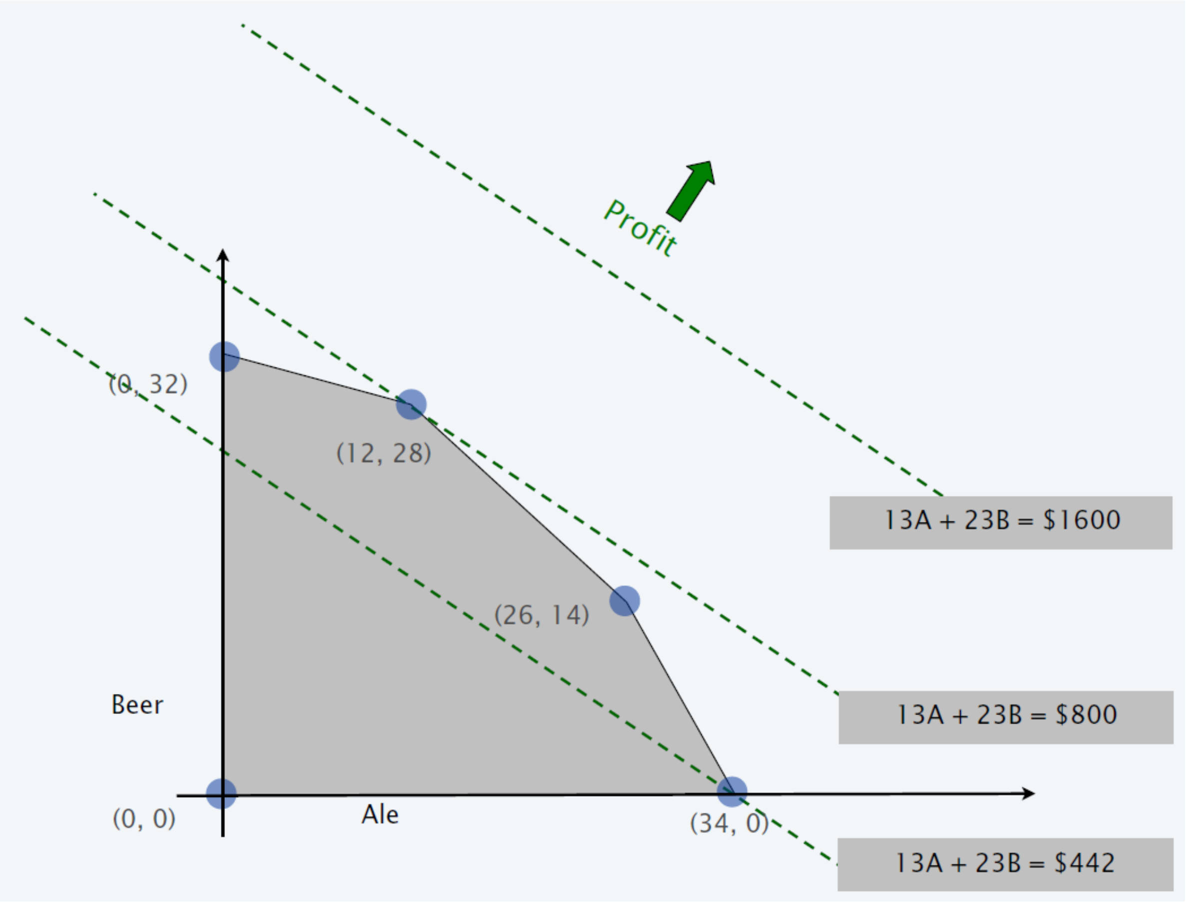


Feasible region – polytope, aka intersection of half-spaces!

Back to Brewery Example

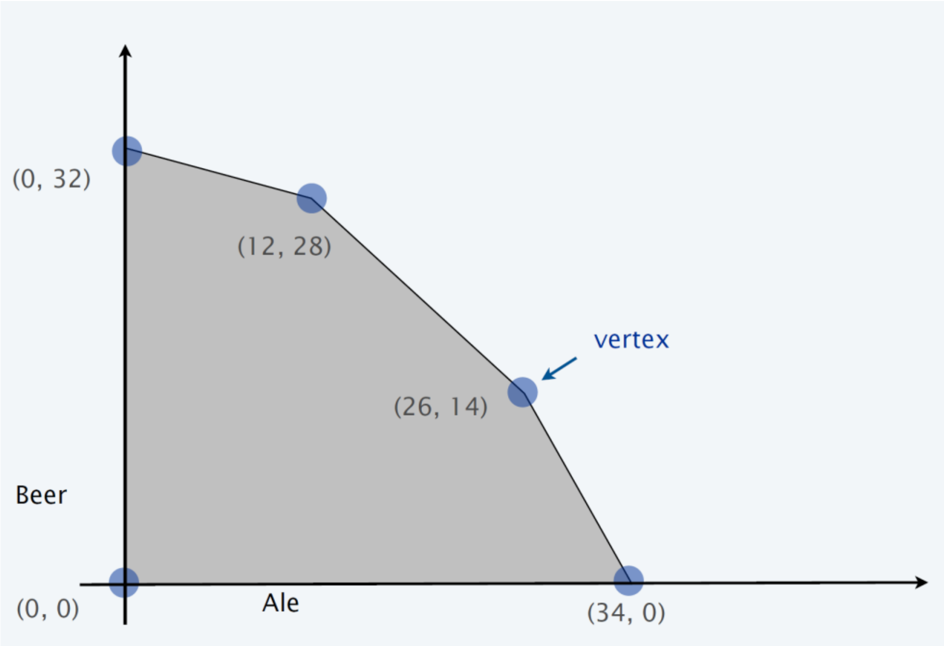


Back to Brewery Example



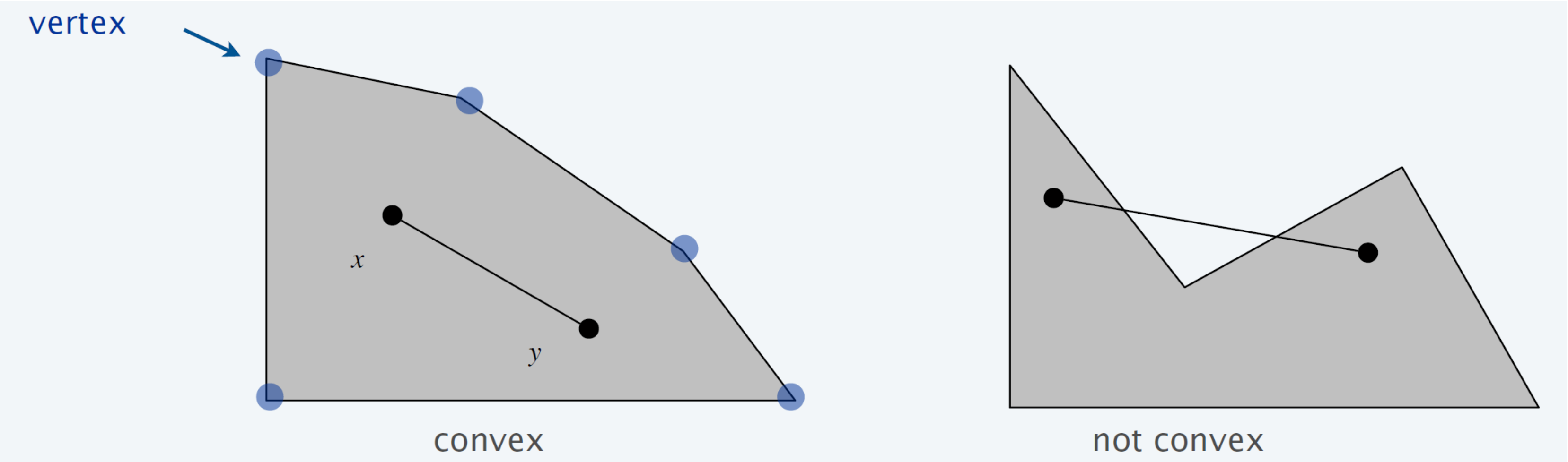
Optimal Solution At A Vertex

- Claim: Regardless of the objective function, there must be a vertex that is an optimal solution



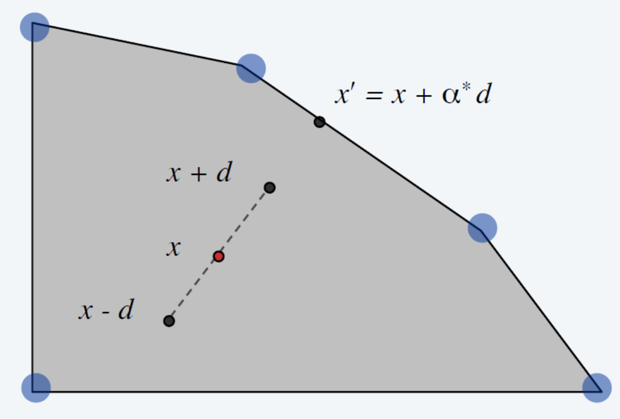
Convexity

- Convex set: S is convex if
$$x, y \in S, \lambda \in [0,1] \Rightarrow \lambda x + (1-\lambda) y \in S$$
- Vertex: A point which cannot be written as a strict convex combination of any two points in the set
- Observation: Feasible region of an LP is a convex set



Optimal Solution At A Vertex

- Intuitive proof of the claim:
  - Start at some point x in the feasible region
  - If x is not a vertex:
    - Find a direction d such that points within a positive distance of ϵ from x in both d and $-d$ directions are within the feasible region
    - Objective must *not decrease* in at least one of the two directions
    - Follow that direction until you reach a new point x for which at least one more constraint is "tight"
  - Repeat until we are at a vertex



LP, Standard Formulation

- Input: $c, a_1, a_2, \ldots, a_m \in \mathbb{R}^n, b \in \mathbb{R}^m$
  - There are n variables and m constraints
- Goal:

$$\begin{aligned} \text{Maximize } & c^T x \\ \text{Subject to } & a_1^T x \leq b_1 \\ & a_2^T x \leq b_2 \\ & \vdots \\ & a_m^T x \leq b_m \\ & x \geq 0 \end{aligned}$$

n variables

m constraints

n more constraints

LP, Standard Matrix Form

- Input: $c, a_1, a_2, \ldots, a_m \in \mathbb{R}^n, b \in \mathbb{R}^m$
  - There are n variables and m constraints
- Goal:

n variables

Maximize $c^T x$

Subject to $Ax \leq b$

$x \geq 0$

m constraints

n more constraints

Convert to Standard Form

- What if the LP is not in standard form?
  - Constraints that use $\geq$
    - $a^T x \geq b \iff -a^T x \leq -b$
  - Constraints that use equality
    - $a^T x = b \iff a^T x \leq b, \; a^T x \geq b$
  - Objective function is a minimization
    - Minimize $c^T x \iff$ Maximize $-c^T x$
  - Variable is unconstrained
    - x with no constraint $\iff$ Replace x by two variables x' and x'', replace every occurrence of x with $x' - x''$, and add constraints $x' \geq 0, \; x'' \geq 0$

LP Transformation Example

$$\begin{array}{lrcrcl} \text{minimize} & -2x_1 & + & 3x_2 & & \\ \text{subject to} & & & & & \\ & x_1 & + & x_2 & = & 7 \\ & x_1 & - & 2x_2 & \leq & 4 \\ & x_1 & & & \geq & 0\,, \end{array}$$

$$\begin{array}{lrcrcl} \text{maximize} & 2x_1 & - & 3x_2 & & \\ \text{subject to} & & & & & \\ & x_1 & + & x_2 & = & 7 \\ & x_1 & - & 2x_2 & \leq & 4 \\ & x_1 & & & \geq & 0\,. \end{array}$$

$$\begin{array}{lrcrcrcl} \text{maximize} & 2x_1 & - & 3x_2' & + & 3x_2'' & & \\ \text{subject to} & & & & & & & \\ & x_1 & + & x_2' & - & x_2'' & = & 7 \\ & x_1 & - & 2x_2' & + & 2x_2'' & \leq & 4 \\ & \multicolumn{5}{c}{x_1, x_2', x_2''} & \geq & 0\,. \end{array}$$

Optimal Solution

- Does an LP always have an optimal solution?

- No! The LP can "fail" for two reasons:
  1. It is *infeasible*, i.e. $\{x \mid Ax \leq b\} = \emptyset$
     - E.g. the set of constraints is $\{x_1 \leq 1, -x_1 \leq -2\}$
  2. It is *unbounded*, i.e. the objective function can be made arbitrarily large (for maximization) or small (for minimization)
     - E.g. "maximize x_1 subject to $x_1 \geq 0$"

- But if the LP has an optimal solution, we know that there must be a vertex which is optimal

Simplex Algorithm

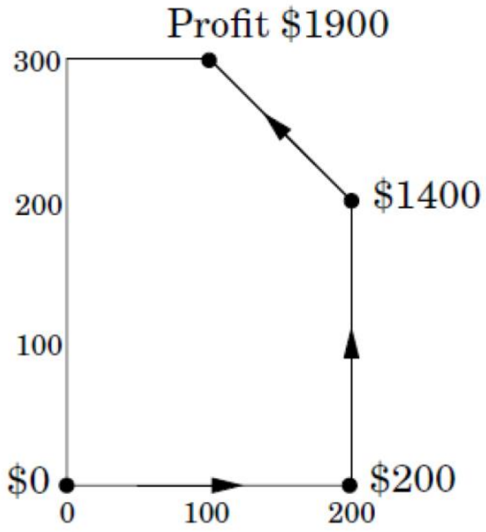
```
let v be any vertex of the feasible region
while there is a neighbor v' of v with better objective value:
    set v = v'
```

- Simple algorithm, easy to specify geometrically
- Worst-case running time is exponential
- Excellent performance in practice

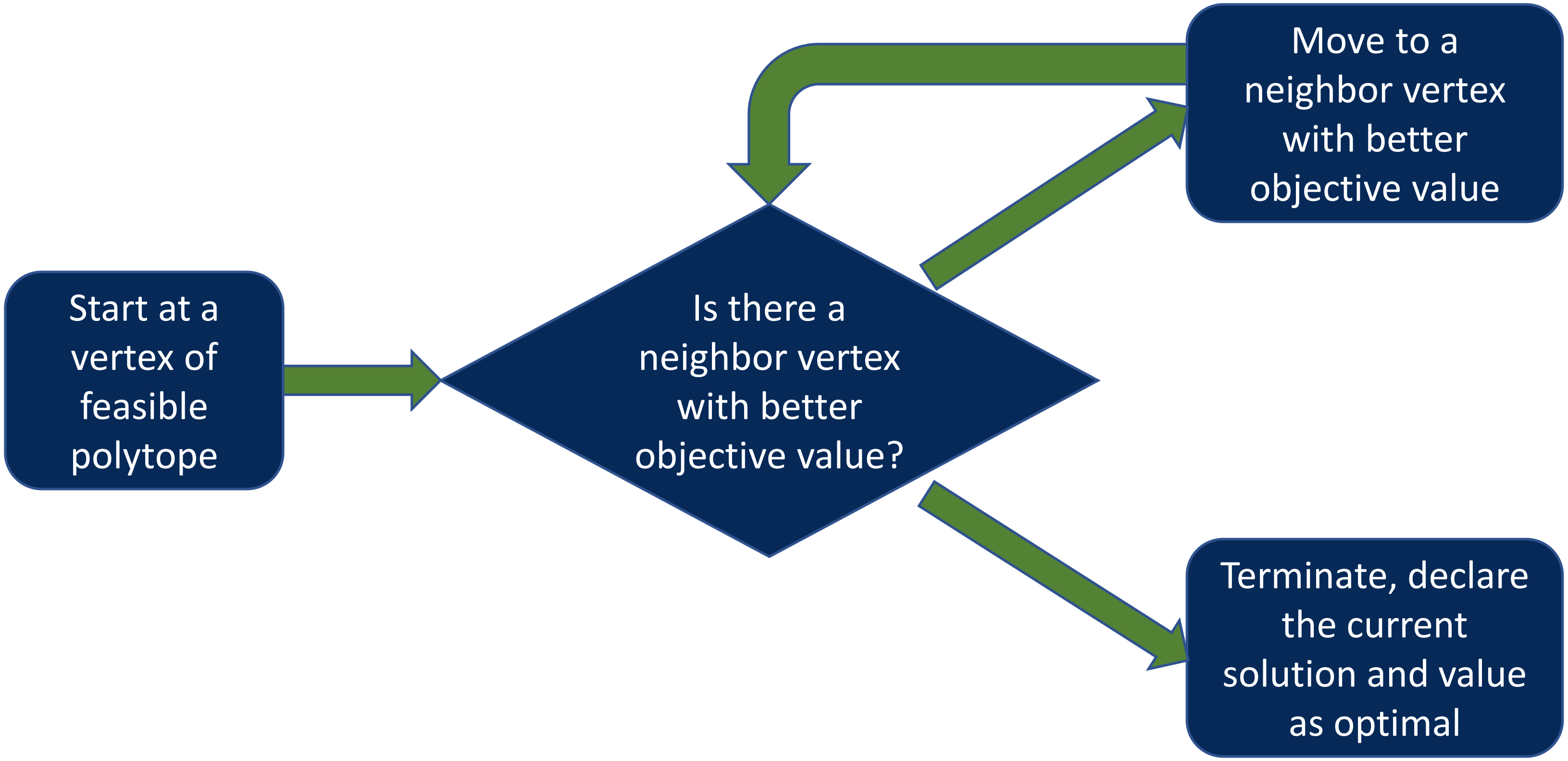
Simplex: Geometric View

```
let v be any vertex of the feasible region
while there is a neighbor v' of v with better objective value:
    set v = v'
```

$$\begin{aligned} \max\ x_1 + 6x_2 \\ x_1 \leq 200 \\ x_2 \leq 300 \\ x_1 + x_2 \leq 400 \\ x_1, x_2 \geq 0 \end{aligned}$$



Algorithmic Implementation



How Do We Implement This?

- We'll work with the slack form of LP
  - Convenient for implementing simplex operations
  - We want to maximize z in the slack form, but for now, forget about the maximization objective

Standard form:

$$\begin{aligned} \text{Maximize } & c^T x \\ \text{Subject to } & Ax \leq b \\ & x \geq 0 \end{aligned}$$

Slack form:

$$\begin{aligned} \mathrm{z} &= c^T x \\ s &= b - Ax \\ s, x &\geq 0 \end{aligned}$$

Slack Form

$$
\begin{array}{lrcrcrcr}
\text{maximize} & 2x_1 & - & 3x_2 & + & 3x_3 & & \\
\text{subject to} & & & & & & & \\
& x_1 & + & x_2 & - & x_3 & \leq & 7 \\
& -x_1 & - & x_2 & + & x_3 & \leq & -7 \\
& x_1 & - & 2x_2 & + & 2x_3 & \leq & 4 \\
& & & x_1, x_2, x_3 & & & \geq & 0 \ .
\end{array}
$$

Nonbasic Variables

$$
\begin{array}{lrcrcrcrcr}
\text{maximize} & & & & & \overbrace{2x_1 \quad - \quad 3x_2 \quad + \quad 3x_3}^{\text{Nonbasic Variables}} & & & & \\
\text{subject to} & & & & & & & & & \\
& x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
& x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
& x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
& & & & & x_1, x_2, x_3, x_4, x_5, x_6 & \geq & 0 & . &
\end{array}
$$

Basic Variables: $\left\{ x_4, x_5, x_6 \right.$

Slack Form

$$
\begin{array}{rcrcrcrcr}
z & = & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
\multicolumn{5}{l}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0 & &
\end{array}
$$

Nonbasic Variables

$$
\begin{array}{lrcrcrcrcr}
\text{maximize} & & & & & 2x_1 & - & 3x_2 & + & 3x_3 \\
\text{subject to} & & & & & & & & & \\
& x_4 & = & 7 & - & x_1 & - & x_2 & + & x_3 \\
& x_5 & = & -7 & + & x_1 & + & x_2 & - & x_3 \\
& x_6 & = & 4 & - & x_1 & + & 2x_2 & - & 2x_3 \\
& \multicolumn{5}{l}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0 & . &
\end{array}
$$

Basic Variables

Simplex: Step 1

- Start at a feasible vertex
  - How do we find a feasible vertex?
  - For now, assume $b \geq 0$ (each $b_i \geq 0$)
    - In this case, $x = 0$ is a feasible vertex.
    - In the slack form, this means setting the nonbasic variables to 0
  - We'll later see what to do in the general case

Standard form:

Maximize $c^T x$
Subject to $Ax \leq b$
$x \geq 0$

Slack form:

$$z = c^T x$$
$$s = b - Ax$$
$$s, x \geq 0$$

Simple: Step 2

- What next? Let's look at an example

$$
\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3 \\
\multicolumn{5}{l}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0 & &
\end{array}
$$

- To increase the value of z:
  - Find a nonbasic variable with a positive coefficient
    - This is called an *entering variable*
  - See how much you can increase its value without violating any constraints

Simple: Step 2

Try to increase!

Obstacles!

$$
\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3 \\
x_1, x_2, x_3, x_4, x_5, x_6 & \geq & 0 & & & & & &
\end{array}
$$

$x_1 \leq 30$

$x_1 \leq 24/2 = 12$

$x_1 \leq 36/4 = 9$

Tightest obstacle!

This is because the current values of x_2 and x_3 are 0, and we need $x_4, x_5, x_6 \geq 0$

Simple: Step 2

$$
\begin{array}{rcrcrcrcr}
z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\
x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\
x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\
x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3
\end{array}
$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Tightest obstacle

- Solve the tightest obstacle for the nonbasic variable

$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4}$$

  - Substitute the entering variable (called pivot) in other equations
  - Now x_1 becomes basic and x_6 becomes non-basic
  - x_6 is called the *leaving variable*

Simplex: Step 2

$$\begin{array}{rcrcrcrcr} z & = & & & 3x_1 & + & x_2 & + & 2x_3 \\ x_4 & = & 30 & - & x_1 & - & x_2 & - & 3x_3 \\ x_5 & = & 24 & - & 2x_1 & - & 2x_2 & - & 5x_3 \\ x_6 & = & 36 & - & 4x_1 & - & x_2 & - & 2x_3 \\ x_1, x_2, x_3, x_4, x_5, x_6 & \geq & 0 \end{array}$$

$$\Rightarrow$$

$$\begin{array}{rcrcrcrcr} z & = & 27 & + & \frac{x_2}{4} & + & \frac{x_3}{2} & - & \frac{3x_6}{4} \\ x_1 & = & 9 & - & \frac{x_2}{4} & - & \frac{x_3}{2} & - & \frac{x_6}{4} \\ x_4 & = & 21 & - & \frac{3x_2}{4} & - & \frac{5x_3}{2} & + & \frac{x_6}{4} \\ x_5 & = & 6 & - & \frac{3x_2}{2} & - & 4x_3 & + & \frac{x_6}{2} \\ x_1, x_2, x_3, x_4, x_5, x_6 & \geq & 0 \end{array}$$

- After one iteration of this step:
  - The basic feasible solution (i.e. substituting 0 for all nonbasic variables) improves from $z = 0$ to $z = 27$
- Repeat!

Simplex: Step 2

Entering variable
Try to increase!

$$
\begin{aligned}
z &= 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \\
x_1 &= 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \\
x_4 &= 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \\
x_5 &= 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2} . \\
x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{aligned}
$$

Leaving variable
Tightest obstacle!

Pivot!

$$
\begin{aligned}
z &= \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \\
x_1 &= \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \\
x_3 &= \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \\
x_4 &= \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16} \\
x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{aligned}
$$

Simplex: Step 2

Entering variable
Try to increase!

$$
\begin{aligned}
z &= \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \\
x_1 &= \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \\
x_3 &= \frac{3}{2} - \frac{3x_2}{8} - \frac{x_5}{4} + \frac{x_6}{8} \\
x_4 &= \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16} . \\
x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{aligned}
$$

Leaving variable
Tightest obstacle!

Pivot!

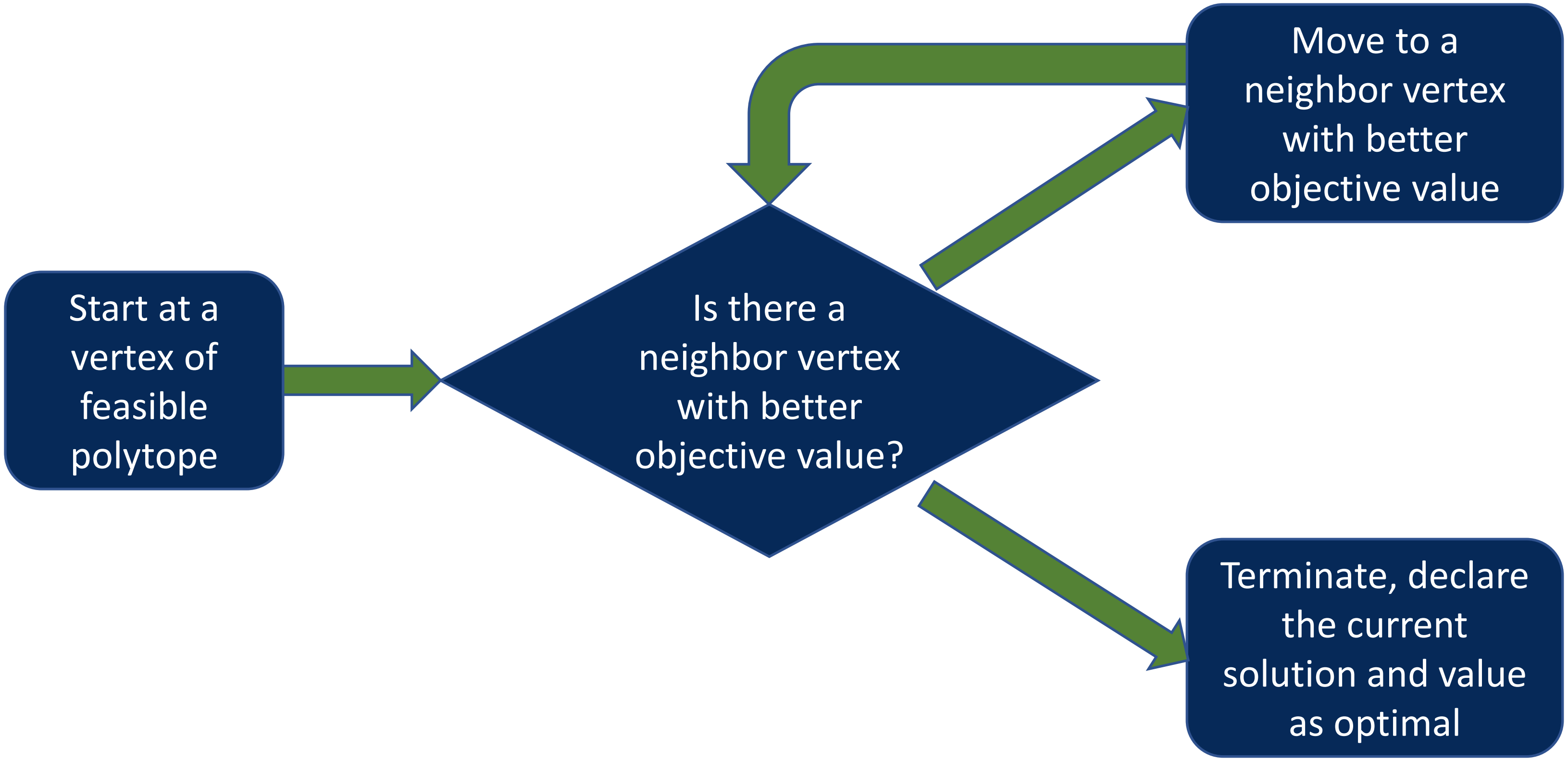
$$
\begin{aligned}
z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2} . \\
x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{aligned}
$$

Simplex: Step 2

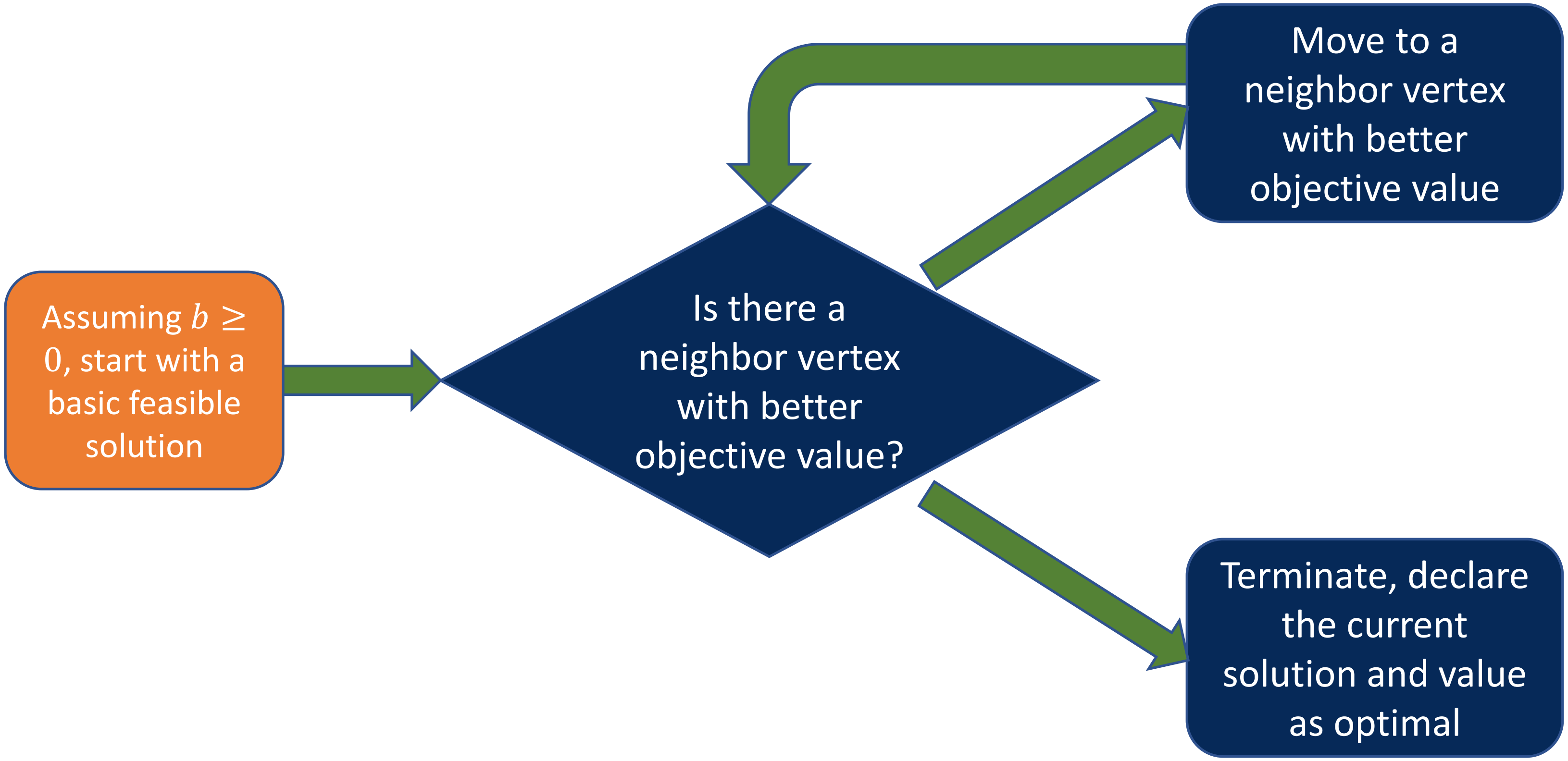
$$
\begin{aligned}
z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2} . \\
x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{aligned}
$$

- There is no leaving variable (nonbasic variable with positive coefficient).
- What now? Nothing! We are done.
- Take the basic feasible solution ($x_3 = x_5 = x_6 = 0$).
- Gives the optimal value $z = 28$
- In the optimal solution, $x_1 = 8$, $x_2 = 4$, $x_3 = 0$

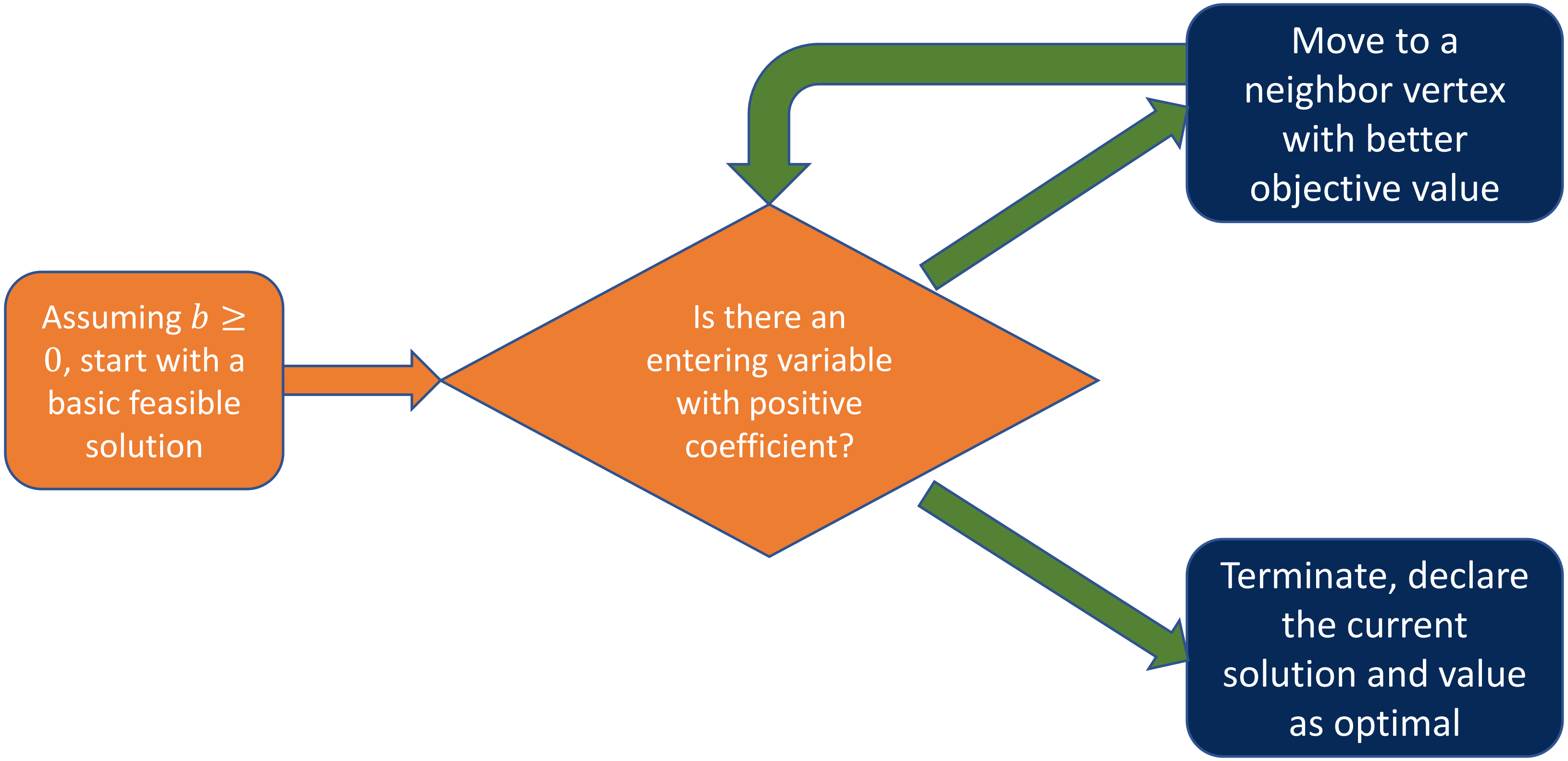
Simplex Overview



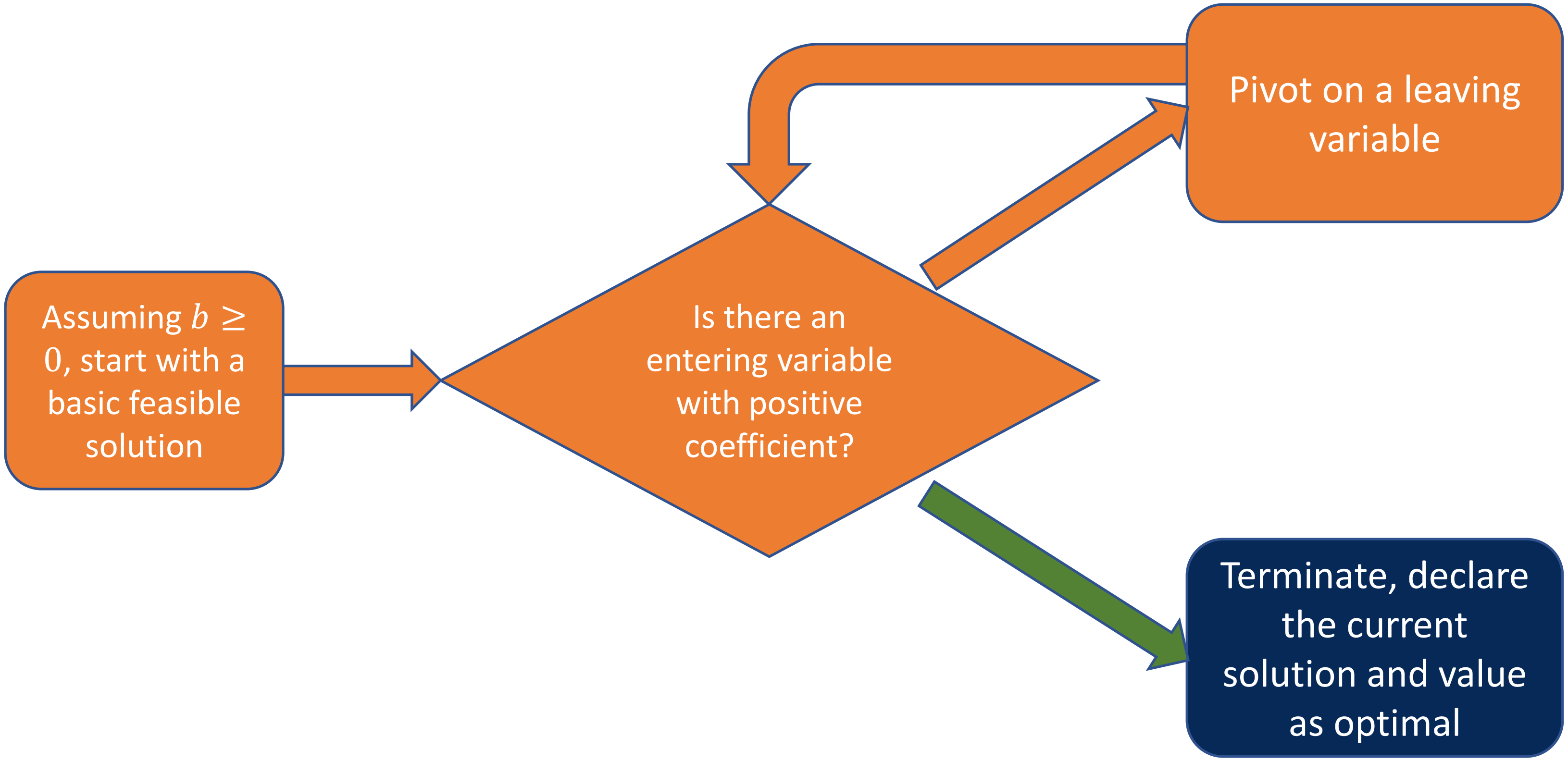
Simplex Overview



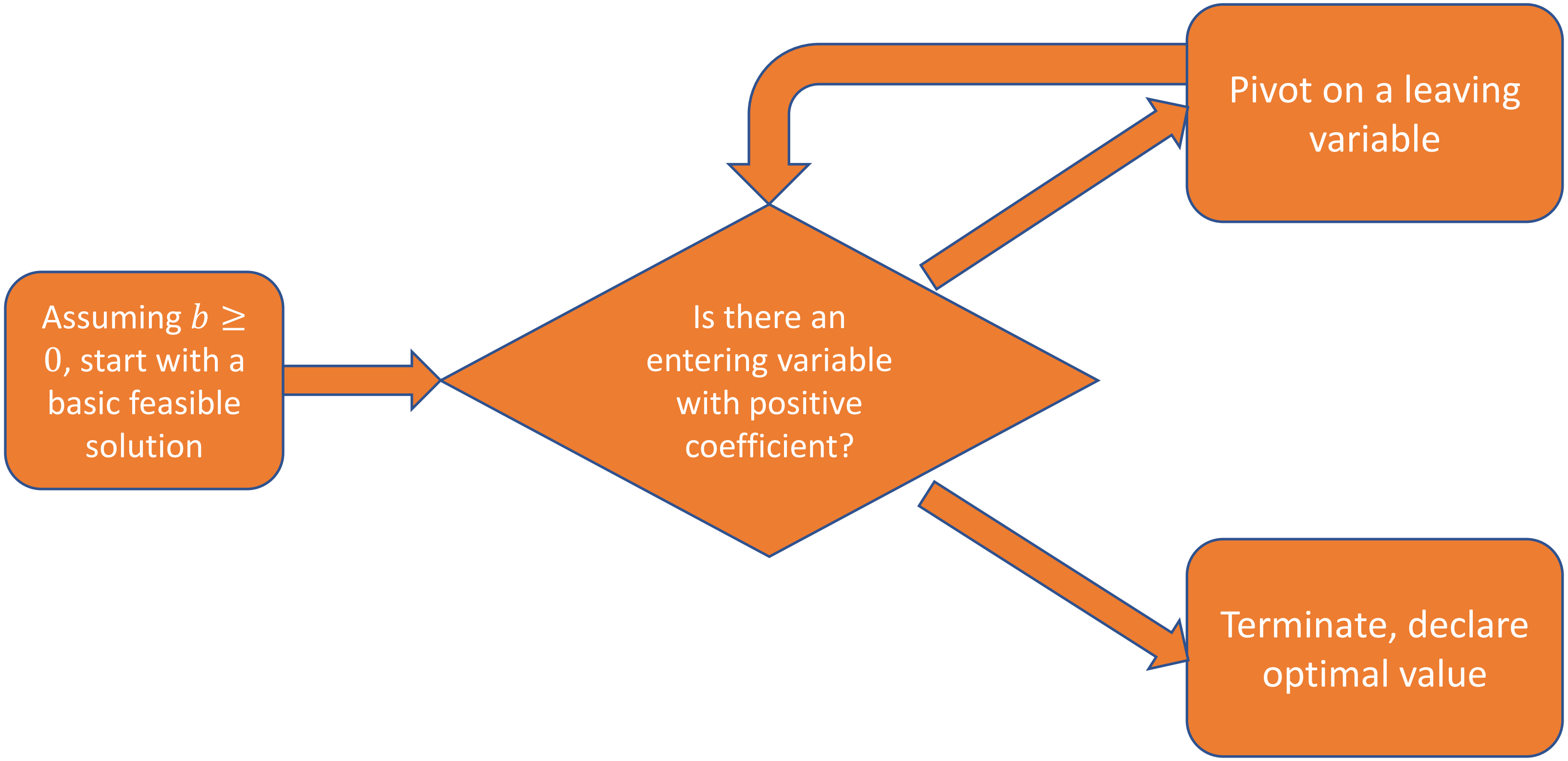
Simplex Overview



Simplex Overview



Simplex Overview



Some Outstanding Issues

- What if the entering variable has no upper bound?
  - If it doesn't appear in any constraints, or only appears in constraints where it can go to ∞
  - Then z can also go to ∞, so declare that LP is unbounded
- What if pivoting doesn't change the constant in z?
  - Known as *degeneracy*, and can lead to infinite loops
  - Can be prevented by "perturbing" b by a small random amount in each coordinate
  - Or by carefully breaking ties among entering and leaving variables, e.g., by smallest index (known as *Bland's rule*)

Some Outstanding Issues

- We assumed $b \geq 0$, and then started with the vertex $x = 0$
- What if this assumption does not hold?

$$
\begin{array}{l}
LP_1 \\
\text{Max } c^T x \\
\text{s.t. } a_1^T x \leq b_1 \\
\quad a_2^T x \leq b_2 \\
\quad \vdots \\
\quad a_m^T x \leq b_m \\
\quad x \geq 0
\end{array}
\Rightarrow
\begin{array}{l}
LP_2 \\
\text{Max } c^T x \\
\text{s.t. } a_1^T x + s_1 = b_1 \\
\quad a_2^T x + s_2 = b_2 \\
\quad \vdots \\
\quad a_m^T x + s_m = b_m \\
\quad x, s \geq 0
\end{array}
\Rightarrow
\begin{array}{l}
LP_3 \\
\text{Max } c^T x \\
\text{s.t. } a_1^T x + s_1 = b_1 \\
\quad -a_2^T x - s_2 = -b_2 \\
\quad \vdots \\
\quad -a_m^T x - s_m = -b_m \\
\quad x, s \geq 0
\end{array}
$$

Multiply every constraint with negative b_i by -1 so RHS is now positive

Some Outstanding Issues

- We assumed $b \geq 0$, and then started with the vertex $x = 0$
- What if this assumption does not hold?

LP_3

Max $c^T x$

s.t. $a_1^T x + s_1 = b_1$

$-a_2^T x - s_2 = -b_2$

$\vdots$

$-a_m^T x - s_m = -b_m$

$x, s \geq 0$

LP_4

Min $\sum_i z_i$

s.t. $a_1^T x + s_1 + z_1 = b_1$

$-a_2^T x - s_2 + z_2 = -b_2$

$\vdots$

$-a_m^T x - s_m + z_m = -b_m$

$x, s, z \geq 0$

Remember: we only want to find a basic feasible solution to LP_1

Remember: RHS is now positive

Some Outstanding Issues

- We assumed $b \geq 0$, and then started with the vertex $x = 0$
- What if this assumption does not hold?

Remember: the RHS is now positive

$$LP_4$$

Min $\sum_i z_i$

s.t. $a_1^T x + s_1 + z_1 = b_1$

$$-a_2^T x - s_2 + z_2 = -b_2$$

$$\vdots$$

$$-a_m^T x - s_m + z_m = -b_m$$

$$x, s, z \geq 0$$

What now?

- Solve LP_4 using simplex with the initial basic solution being $x = s = 0, z = |b|$
- If its optimum value is 0, extract a basic feasible solution x^* from it, use it to solve LP_1 using simplex
- If optimum value for LP_4 is greater than 0, then LP_1 is infeasible

Some Outstanding Issues

- We assumed $b \geq 0$, and then started with the vertex $x = 0$
- What if this assumption does not hold?

LP_1

Max $c^T x$

s.t. $a_1^T x \leq b_1$

$a_2^T x \leq b_2$

$\vdots$

$a_m^T x \leq b_m$

$x \geq 0$

LP_2

Min $\sum_i z_i$

s.t. $a_1^T x + s_1 + z_1 = b_1$

$a_2^T x + s_2 + z_2 = b_2$

$\vdots$

$a_m^T x + s_m + z_m = b_m$

$x, s \geq 0$

- Solve LP_2 using simplex with the initial basic feasible solution $x = s = 0, z = b$
- If its optimum value is 0, extract a basic feasible solution x^* from it, use it to solve LP_1 using simplex
- If optimum value for LP_2 is greater than 0, then LP_1 is infeasible

Some Outstanding Issues

- Curious about pseudocode? Proof of correctness? Running time analysis?

- See textbook for details, but this is <u>NOT</u> in syllabus!

Running Time

- Notes
  - Number of vertices of a polytope can be exponential in the number of constraints
    - There are examples where simplex takes exponential time if you choose your pivots arbitrarily
    - No pivot rule known which guarantees polynomial running time
  - There are other algorithms which run in polynomial time
    - Ellipsoid method, interior point method, …
    - Ellipsoid uses $O(mn^3L)$ arithmetic operations, where L = length of input
    - But no known *strongly polynomial time* algorithm
      - Number of arithmetic operations = poly(m,n)

Certificate of Optimality

- Suppose you design a state-of-the-art LP solver that can solve very large problem instances

- You want to convince someone that you have this new technology without showing them the code
  - Idea: They can give you very large LPs and you can quickly return the optimal solutions
  - Question: But how would they know that your solutions are optimal, if they don't have the technology to solve those LPs?

Certificate of Optimality

$$\max\ x_1 + 6x_2$$
$$x_1 \leq 200$$
$$x_2 \leq 300$$
$$x_1 + x_2 \leq 400$$
$$x_1, x_2 \geq 0$$

- Suppose I tell you that $(x_1, x_2) = (100{,}300)$ is optimal with objective value 1900
- How can you check this?
  - Note: Can easily substitute (x_1, x_2), and verify that it is feasible, and its objective value is indeed 1900

Certificate of Optimality

$$
\begin{aligned}
\max \; x_1 + 6x_2 & \\
x_1 &\le 200 \\
x_2 &\le 300 \\
x_1 + x_2 &\le 400 \\
x_1, x_2 &\ge 0
\end{aligned}
$$

- Claim: $(x_1, x_2) = (100{,}300)$ is optimal with objective value 1900

- Any solution that satisfies these inequalities also satisfies their positive combinations
  - E.g. 2*first_constraint + 5*second_constraint + 3*third_constraint
  - Try to take combinations which give you $x_1 + 6x_2$ on LHS

Certificate of Optimality

$$\begin{aligned} \max \ x_1 + 6x_2 & \\ x_1 &\leq 200 \\ x_2 &\leq 300 \\ x_1 + x_2 &\leq 400 \\ x_1, x_2 &\geq 0 \end{aligned}$$

- Claim: $(x_1, x_2) = (100{,}300)$ is optimal with objective value 1900

- first_constraint + 6*second_constraint
  - $x_1 + 6x_2 \leq 200 + 6 * 300 = 2000$
  - This shows that no feasible solution can beat 2000

Certificate of Optimality

$$\begin{aligned} \max \ x_1 + 6x_2 & \\ x_1 &\le 200 \\ x_2 &\le 300 \\ x_1 + x_2 &\le 400 \\ x_1, x_2 &\ge 0 \end{aligned}$$

- Claim: $(x_1, x_2) = (100{,}300)$ is optimal with objective value 1900

- 5*second_constraint + third_constraint
  - $5x_2 + (x_1 + x_2) \le 5 * 300 + 400 = 1900$
  - This shows that no feasible solution can beat 1900
    - No need to proceed further
    - We already know one solution that achieves 1900, so it must be optimal!

Is There a General Algorithm?

- Introduce variables y_1, y_2, y_3 by which we will be multiplying the three constraints
  - Note: These need not be integers. They can be reals.

Multiplier	Inequality
y_1	$x_1 \leq 200$
y_2	$x_2 \leq 300$
y_3	$x_1 + x_2 \leq 400$

- After multiplying and adding constraints, we get:
  $(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400y_3$

Is There a General Algorithm?

Multiplier	Inequality
y_1	$x_1 \leq 200$
y_2	$x_2 \leq 300$
y_3	$x_1 + x_2 \leq 400$

- We have:
  $$(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400y_3$$

- What do we want?
  - $y_1, y_2, y_3 \geq 0$ because otherwise direction of inequality flips
  - LHS to look like objective $x_1 + 6x_2$
    - In fact, it is sufficient for LHS to be an upper bound on objective
    - So we want $y_1 + y_3 \geq 1$ and $y_2 + y_3 \geq 6$

Is There a General Algorithm?

Multiplier	Inequality
y_1	$x_1 \leq 200$
y_2	$x_2 \leq 300$
y_3	$x_1 + x_2 \leq 400$

- We have:
  $(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400y_3$
- What do we want?
  - $y_1, y_2, y_3 \geq 0$
  - $y_1 + y_3 \geq 1,\ y_2 + y_3 \geq 6$
  - Subject to these, we want to minimize the upper bound $200y_1 + 300y_2 + 400y_3$

Is There a General Algorithm?

Multiplier	Inequality
y_1	$x_1 \leq 200$
y_2	$x_2 \leq 300$
y_3	$x_1 + x_2 \leq 400$

- We have:
  $$(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \leq 200y_1 + 300y_2 + 400y_3$$

- What do we want?
  - This is just another LP!
  - Called the dual
  - Original LP is called the primal

$$\begin{aligned} \min\ & 200y_1 + 300y_2 + 400y_3 \\ & y_1 + y_3 \geq 1 \\ & y_2 + y_3 \geq 6 \\ & y_1, y_2, y_3 \geq 0 \end{aligned}$$

Is There a General Algorithm?

PRIMAL

$$\begin{aligned} \max \;\; x_1 + 6x_2 & \\ x_1 &\le 200 \\ x_2 &\le 300 \\ x_1 + x_2 &\le 400 \\ x_1, x_2 &\ge 0 \end{aligned}$$

DUAL

$$\begin{aligned} \min \;\; 200y_1 + 300y_2 + 400y_3 & \\ y_1 + y_3 &\ge 1 \\ y_2 + y_3 &\ge 6 \\ y_1, y_2, y_3 &\ge 0 \end{aligned}$$

- The problem of verifying optimality is another LP
  - For any (y_1, y_2, y_3) that you can find, the objective value of the dual is an upper bound on the objective value of the primal
  - If you found a specific (y_1, y_2, y_3) for which this dual objective becomes equal to the primal objective for the (x_1, x_2) given to you, then you would know that the given (x_1, x_2) is optimal for primal (and your (y_1, y_2, y_3) is optimal for dual)

Is There a General Algorithm?

PRIMAL

$$\begin{aligned} \max\ x_1 + 6x_2 \\ x_1 \le 200 \\ x_2 \le 300 \\ x_1 + x_2 \le 400 \\ x_1, x_2 \ge 0 \end{aligned}$$

DUAL

$$\begin{aligned} \min\ 200y_1 + 300y_2 + 400y_3 \\ y_1 + y_3 \ge 1 \\ y_2 + y_3 \ge 6 \\ y_1, y_2, y_3 \ge 0 \end{aligned}$$

- The problem of verifying optimality is another LP
  - Issue 1: But…but…if I can't solve large LPs, how will I solve the dual to verify if optimality of (x_1, x_2) given to me?
    - You don't. Ask the other party to give you both (x_1, x_2) and the corresponding (y_1, y_2, y_3) for proof of optimality
  - Issue 2: What if there are no (y_1, y_2, y_3) for which dual objective matches primal objective under optimal solution (x_1, x_2)?
    - As we will see, this can't happen!

Is There a General Algorithm?

Primal LP	**Dual LP**
$\max\ \mathbf{c}^T\mathbf{x}$	$\min\ \mathbf{y}^T\mathbf{b}$
$\mathbf{A}\mathbf{x} \leq \mathbf{b}$	$\mathbf{y}^T\mathbf{A} \geq \mathbf{c}^T$
$\mathbf{x} \geq 0$	$\mathbf{y} \geq 0$

- General version, in our standard form for LPs

Is There a General Algorithm?

Primal LP

$$\max \ \mathbf{c}^T \mathbf{x}$$
$$\mathbf{A}\mathbf{x} \leq \mathbf{b}$$
$$\mathbf{x} \geq 0$$

Dual LP

$$\min \ \mathbf{y}^T \mathbf{b}$$
$$\mathbf{y}^T \mathbf{A} \geq \mathbf{c}^T$$
$$\mathbf{y} \geq 0$$

- $c^T x$ for any feasible $x \ \leq \ y^T b$ for any feasible y
- $\max\limits_{\text{primal feasible } x} c^T x \ \leq \ \min\limits_{\text{dual feasible } y} y^T b$
- If there is (x^*, y^*) with $c^T x^* = (y^*)^T b$, then both must be optimal
- In fact, for optimal (x^*, y^*), we claim that this must happen!
  - Does this remind you of something? Max-flow, min-cut…

Weak Duality

Primal LP	**Dual LP**
$\max \ \mathbf{c}^T \mathbf{x}$	$\min \ \mathbf{y}^T \mathbf{b}$
$\mathbf{A}\mathbf{x} \leq \mathbf{b}$	$\mathbf{y}^T \mathbf{A} \geq \mathbf{c}^T$
$\mathbf{x} \geq 0$	$\mathbf{y} \geq 0$

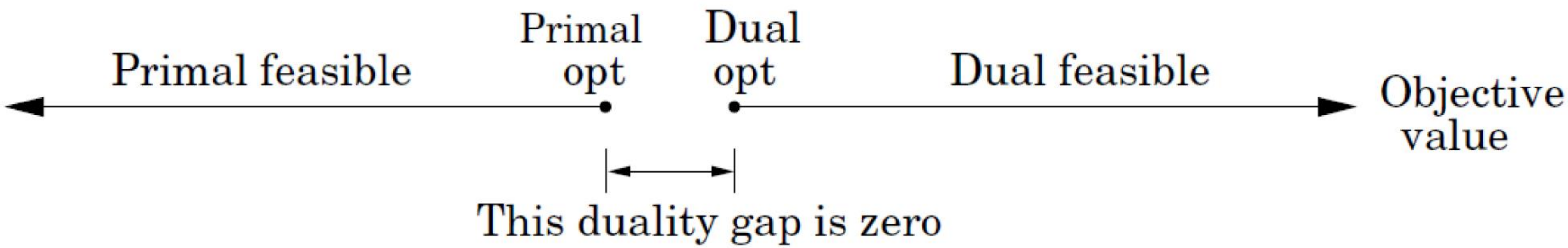
- From here on, assume primal LP is feasible and bounded
- Weak duality theorem:
  - For any primal feasible x and dual feasible y, $c^T x \leq y^T b$
- Proof:

$$c^T x \leq (y^T A)x = y^T(Ax) \leq y^T b$$

Strong Duality

Primal LP	**Dual LP**
$\max\ \mathbf{c}^T\mathbf{x}$	$\min\ \mathbf{y}^T\mathbf{b}$
$\mathbf{A}\mathbf{x} \le \mathbf{b}$	$\mathbf{y}^T\mathbf{A} \ge \mathbf{c}^T$
$\mathbf{x} \ge 0$	$\mathbf{y} \ge 0$

- Strong duality theorem:
  - For any primal optimal x^* and dual optimal y^*, $c^T x^* = (y^*)^T b$



Strong Duality Proof

This slide is not in the scope of the course

- Farkas' lemma (one of many, many versions):
  - Exactly one of the following holds:
  1) There exists x such that $Ax \leq b$
  2) There exists y such that $y^T A = 0,\ y \geq 0,\ y^T b < 0$

- Geometric intuition:
  - Define image of A = set of all possible values of Ax
  - It is known that this is a "linear subspace" (e.g. a line in a plane, a line or plane in 3D, etc)

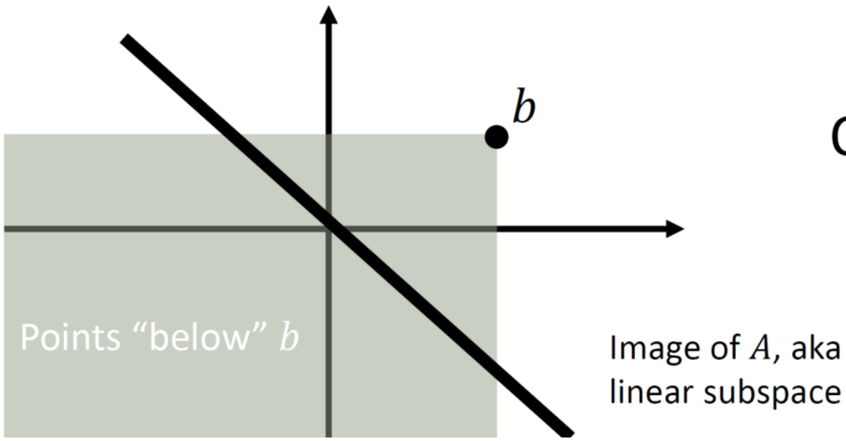
Strong Duality Proof

This slide is not in the scope of the course

- Farkas' lemma: Exactly one of the following holds:
  1) There exists x such that $Ax \leq b$
  2) There exists y such that $y^T A = 0,\ y \geq 0,\ y^T b < 0$

1) Image of A contains a point "below" b

2) The region "below" b doesn't intersect image of A this is witnessed by normal vector to the image of A



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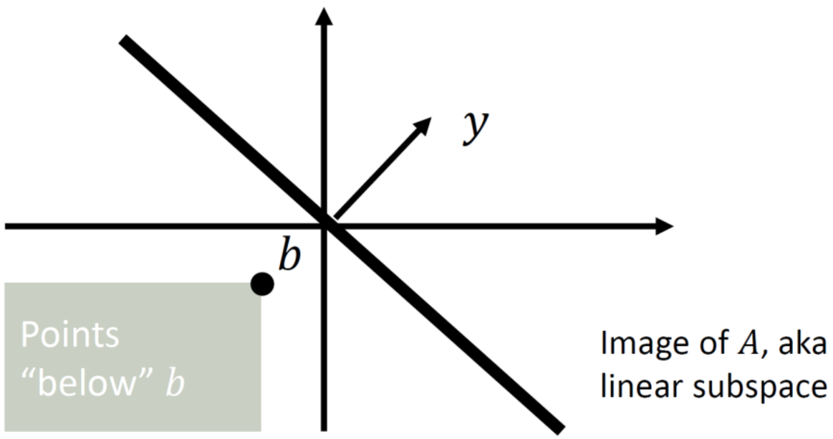

Strong Duality

This slide is not in the scope of the course

Primal LP	**Dual LP**
$\max\ \mathbf{c}^T\mathbf{x}$	$\min\ \mathbf{y}^T\mathbf{b}$
$\mathbf{A}\mathbf{x} \leq \mathbf{b}$	$\mathbf{y}^T\mathbf{A} \geq \mathbf{c}^T$
$\mathbf{x} \geq 0$	$\mathbf{y} \geq 0$

- Strong duality theorem:
  - For any primal optimal x^* and dual optimal y^*, $c^T x^* = (y^*)^T b$
  - Proof (by contradiction):
    - Let $z^* = c^T x^*$ be the optimal primal value.
    - Suppose optimal dual objective value $> z^*$
    - So there is no y such that $y^T A \geq c^T$ and $y^T b \leq z^*$, i.e.,

$$\begin{pmatrix} -A^T \\ b^T \end{pmatrix} y \leq \begin{pmatrix} c \\ z^* \end{pmatrix}$$

Strong Duality

This slide is not in the scope of the course

- There is no y such that $\begin{pmatrix} -A^T \\ b^T \end{pmatrix} y \leq \begin{pmatrix} c \\ z^* \end{pmatrix}$
- By Farkas' lemma, there is x and λ such that

$$(x^T \quad \lambda)\begin{pmatrix} -A^T \\ b^T \end{pmatrix} = 0, x \geq 0, \lambda \geq 0, -x^T c + \lambda z^* < 0$$

- Case 1: $\lambda > 0$
  - Note: $c^T x > \lambda z^*$ and $Ax = 0 = \lambda b$.
  - Divide both by λ to get $A\left(\frac{x}{\lambda}\right) = b$ and $c^T\left(\frac{x}{\lambda}\right) > z^*$
    - Contradicts optimality of z^*
- Case 2: $\lambda = 0$
  - We have $Ax = 0$ and $c^T x > 0$
  - Adding x to optimal x^* of primal improves objective value beyond $z^* \Rightarrow$ contradiction

Exercise: Formulating LPs

- A canning company operates two canning plants (A and B).
- Three suppliers of fresh fruits:
  - S1: 200 tonnes at \$11/tonne
  - S2: 310 tonnes at \$10/tonne
  - S3: 420 tonnes at \$9/tonne
- Shipping costs in \$/tonne:

	To:	Plant A	Plant B
From:	S1	3	3.5
	S2	2	2.5
	S3	6	4

- Plant capacities and labour costs:

	Plant A	Plant B
Capacity	460 tonnes	560 tonnes
Labour cost	\$26/tonne	\$21/tonne

- Selling price: \$50/tonne, no limit
- Objective: Find which plant should get how much supply from each grower to maximize profit

Exercise: Formulating LPs

- Similarly to the brewery example from the beginning:
  - A brewery can invest its inventory of corn, hops and malt into producing three types of beer
  - Per unit resource requirement and profit are as given below
  - <u>The brewery cannot produce positive amounts of *both* A and B</u>
  - Goal: maximize profit

Beverage	Corn (kg)	Hops (kg)	Malt (kg)	Profit ($)
A	5	4	35	13
B	15	4	20	23
C	10	7	25	15
Limit	500	300	1000	

Exercise: Formulating LPs

- Similarly to the brewery example from the beginning:
  - A brewery can invest its inventory of corn, hops and malt into producing three types of beer
  - Per unit resource requirement and profit are as given below
  - The brewery can only produce C in integral quantities up to 100
  - Goal: maximize profit

Beverage	Corn (kg)	Hops (kg)	Malt (kg)	Profit ($)
A	5	4	35	13
B	15	4	20	23
C	10	7	25	15
Limit	500	300	1000	

Exercise: Formulating LPs

- Similarly to the brewery example from the beginning:
  - A brewery can invest its inventory of corn, hops and malt into producing three types of beer
  - Per unit resource requirement and profit are as given below
  - Goal: maximize profit, <u>but if there are multiple profit-maximizing solutions, then…</u>
    - <u>Break ties to choose those with the largest quantity of A</u>
    - <u>Break any further ties to choose those with the largest quantity of B</u>

Beverage	Corn (kg)	Hops (kg)	Malt (kg)	Profit ($)
A	5	4	35	13
B	15	4	20	23
C	10	7	25	15
Limit	500	300	1000	

More Tricks

- Constraint: $|x| \leq 3$
  - Replace with constraints $x \leq 3$ and $-x \leq 3$
  - What if the constraint is $|x| \geq 3$?
- Objective: minimize $3|x| + y$
  - Add a variable t
  - Add the constraints $t \geq x$ and $t \geq -x$ (so $t \geq |x|$)
  - Change the objective to minimize $3t + y$
  - What if the objective is to *maximize* $3|x| + y$?
- Objective: minimize $\max(3x + y, x + 2y)$
  - Hint: minimizing $3|x| + y$ in the earlier bullet was equivalent to minimizing $\max(3x + y, -3x + y)$
- …


NOW I KNOW
LINEAR PROGRAMMING