CSC373

Week 5:
Network Flow (contd)

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Recap

- Some more DP
  - Traveling salesman problem (TSP)

- Start of network flow
  - Problem statement
  - Ford-Fulkerson algorithm
  - Running time
  - Correctness using max-flow, min-cut

This Lecture

- Network flow in polynomial time
  - Edmonds-Karp algorithm (shortest augmenting path)

- Applications of network flow
  - Bipartite matching & Hall's theorem
  - Edge-disjoint paths & Menger's theorem
  - Multiple sources/sinks
  - Circulation networks
  - Lower bounds on flows
  - Survey design
  - Image segmentation

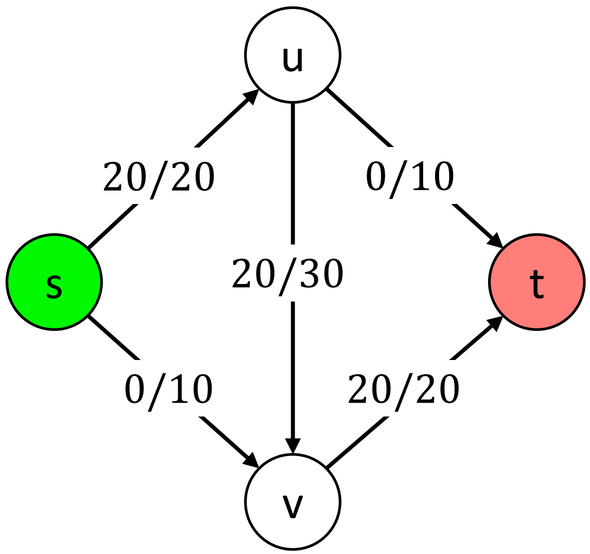
Ford-Fulkerson Recap

- Define the residual graph G_f of flow f
  - G_f has the same vertices as G
  - For each edge e $= (u, v)$ in G, G_f has at most two edges
    - Forward edge $e = (u, v)$ with capacity $c(e) - f(e)$
      - We can send this much additional flow on e
    - Reverse edge $e^{rev} = (v, u)$ with capacity $f(e)$
      - The maximum "reverse" flow we can send is the maximum amount by which we can reduce flow on e, which is $f(e)$
    - We only add each edge if its capacity > 0

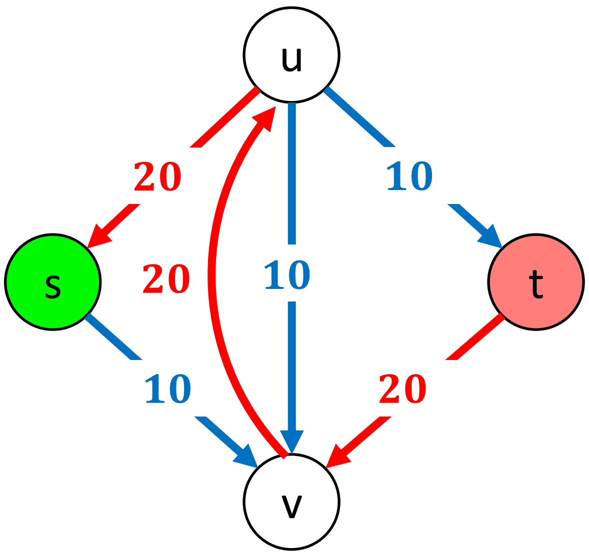
Ford-Fulkerson Recap

- Example!

Flow f



Residual graph G_f



Ford-Fulkerson Recap

```
MaxFlow(G):
  // initialize:
  Set f(e) = 0 for all e in G

  // while there is an s-t path in G_f:
  While P = FindPath(s,t,Residual(G,f))!=None:
    f = Augment(f,P)
    UpdateResidual(G,f)
  EndWhile
  Return f
```

Ford-Fulkerson Recap

- Running time:
  - #Augmentations:
    - At every step, flow and capacities remain integers
    - For path P in G_f, $\text{bottleneck}(P, f) > 0$ implies $\text{bottleneck}(P, f) \geq 1$
    - Each augmentation increases flow by at least 1
    - At most $C = \sum_{e \text{ leaving } s} c(e)$ augmentations
  - Time for an augmentation:
    - G_f has n vertices and at most $2m$ edges
    - Finding an s-t path in G_f takes $O(m+n)$ time
  - Total time: $O((m+n) \cdot C)$

Edmonds-Karp Algorithm

- At every step, find the shortest path from s to t in G_f, and augment.

Minimum number of edges

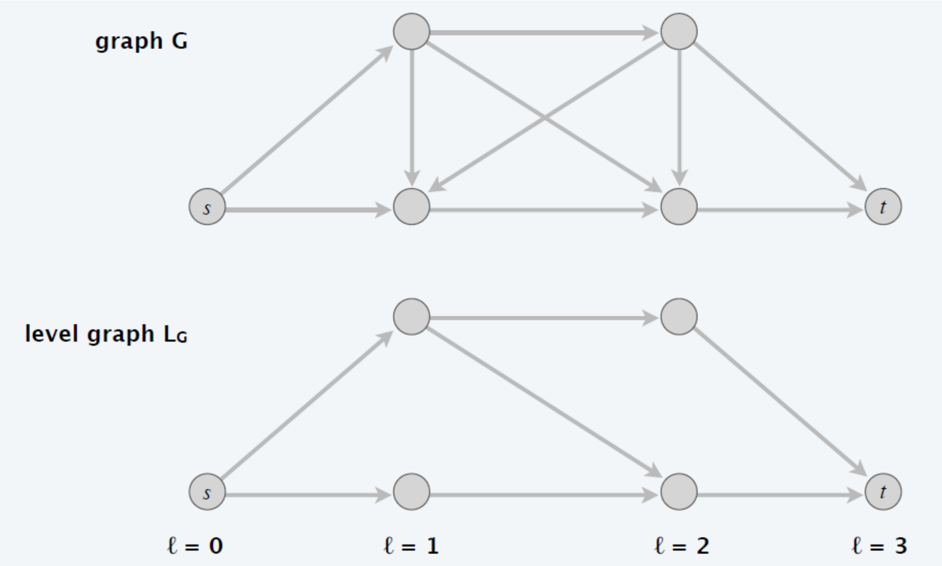
```
MaxFlow(G):
  // initialize:
  Set f(e) = 0 for all e in G

  // Find shortest s-t path in G_f & augment:
  While P = BFS(s,t,Residual(G,f))!=None:
    f = Augment(f,P)
    UpdateResidual(G,f)
  EndWhile
  Return f
```

Proof Overview

- Overview
  - Lemma 1: The length of the shortest $s \to t$ path in G_f never decreases.
    - (Proof ahead)
  - Lemma 2: After at most m augmentations, the length of the shortest $s \to t$ path in G_f must strictly increase.
    - (Proof ahead)
  - Theorem: The algorithm takes $O(m^2 n)$ time.
    - Proof:
      - Length of shortest $s \to t$ path in G_f can go from 0 to $n-1$
      - Using Lemma 2, there can be at most $m \cdot n$ augmentations
      - Each takes $O(m)$ time using BFS. ■

Level Graph

- Level graph L_G of a directed graph $G = (V, E)$:
  - Level: $\ell(v)$ = length of shortest $s \to v$ path
  - Level graph $L_G = (V, E_L)$ is a subgraph of G where we only retain edges $(u, v) \in E$ where $\ell(v) = \ell(u) + 1$
    - Intuition: Keep only the edges useful for shortest paths



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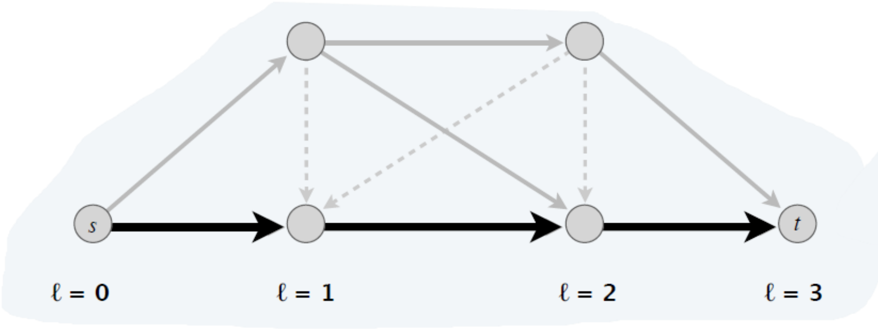
    - Intuition: Keep only the edges useful for shortest paths

- Property: P is a shortest $s \to v$ path in G if and only if P is an $s \to v$ path in L_G.

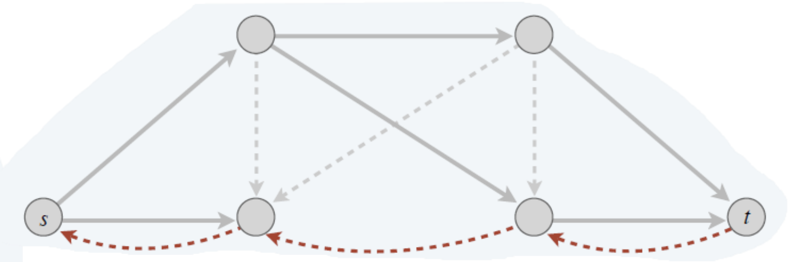
Edmonds-Karp Proof

- Lemma 1:
  - Length of the shortest $s \to t$ path in G_f never decreases.
- Proof:
  - Let f and f' be flows before and after an augmentation step, and G_f and $G_{f'}$ be their residual graphs.

Residual graph G_f



Residual graph $G_{f'}$



Edmonds-Karp Proof

NOT IN SYLLABUS

- Lemma 1:
  - Length of the shortest $s \to t$ path in G_f never decreases.
- Proof:
  - Let f and f' be flows before and after an augmentation step, and G_f and $G_{f'}$ be their residual graphs.
  - Augmentation happens along a path in L_{G_f}
  - For each edge on the path, we either remove it, add an opposite direction edge, or both.
  - Opposite direction edges can't help reduce the length of the shortest $s \to t$ path (exercise!).
  - QED!

Edmonds-Karp Proof

NOT IN SYLLABUS

- **Lemma 2:**
  - After at most m augmentations, the length of the shortest $s \to t$ path in G_f must strictly increase.
- **Proof:**
  - In each augmentation step, we remove at least one edge from L_{G_f}
    - Because we make the flow on at least one edge on the shortest path equal to its capacity
  - No new edges are added in L_{G_f} unless the length of the shortest $s \to t$ path strictly increases
  - This cannot happen more than m times! ■

Edmonds-Karp Proof Overview

- Overview
  - Lemma 1: The length of the shortest $s \to t$ path in G_f never decreases.
  - Lemma 2: After at most m augmentations, the length of the shortest $s \to t$ path in G_f must strictly increase.
  - Theorem: The algorithm takes $O(m^2 n)$ time.

Edmonds-Karp Proof Overview

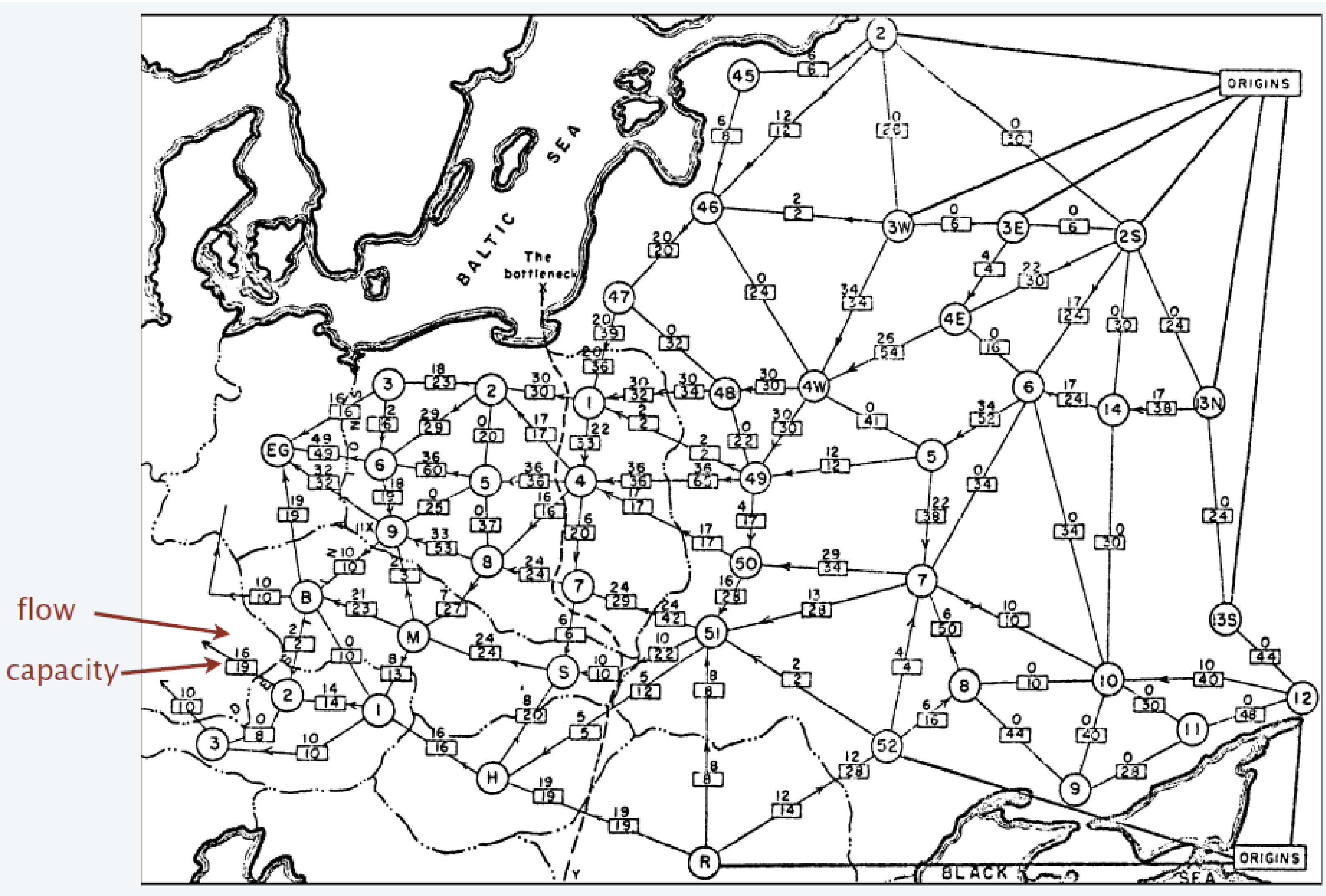
- Note:
  - Some graphs require $\Omega(mn)$ augmentation steps
  - But we may be able to reduce the time to run each augmentation step

- Two algorithms use this idea to reduce run time
  - Dinitz's algorithm [1970] $\Rightarrow O(mn^2)$
  - Sleator–Tarjan algorithm [1983] $\Rightarrow O(m\, n \log n)$
    - Using the dynamic trees data structure

Network Flow Applications

Rail network connecting Soviet Union with Eastern European countries

(Tolstoǐ 1930s)



Rail network connecting Soviet Union with Eastern European countries

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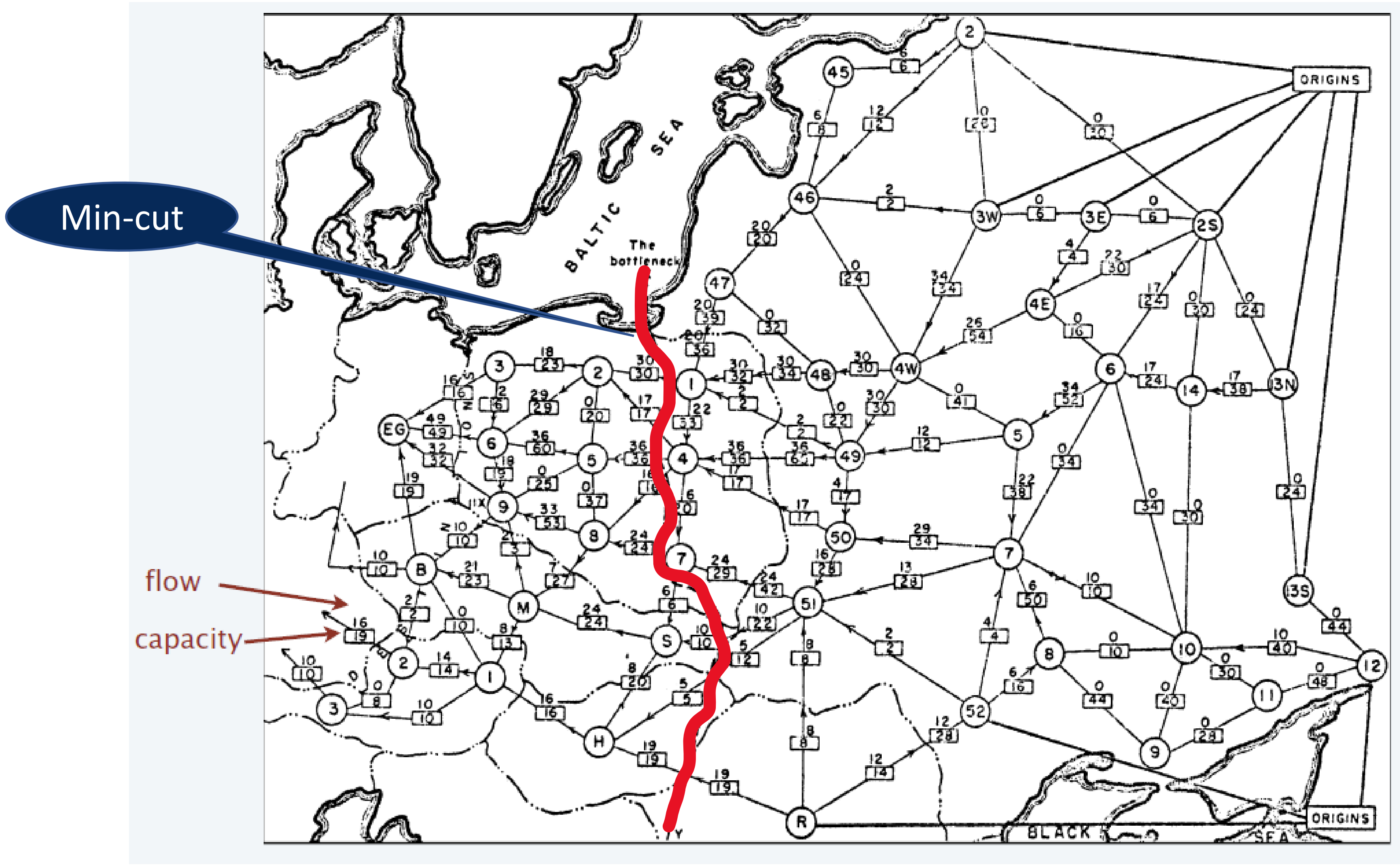


Integrality Theorem

- Before we look at applications, we need the following special property of the max-flow computed by Ford-Fulkerson and its variants

- Observation:
  - If edge capacities are integers, then the max-flow computed by Ford-Fulkerson and its variants are also integral (i.e. the flow on each edge is an integer).
  - Easy to check that each augmentation step preserves integral flow

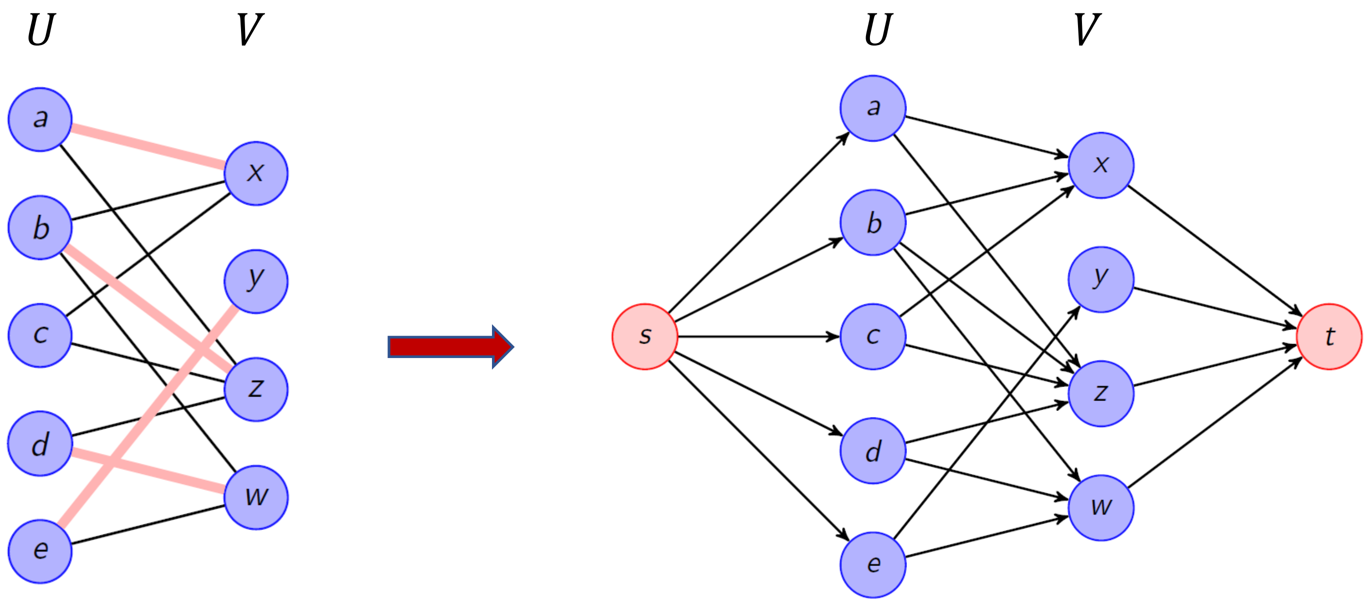
Bipartite Matching

- Problem
  - Given a bipartite graph $G = (U \cup V, E)$, find a maximum cardinality matching

- We do not know any efficient greedy or dynamic programming algorithm for this problem.

- But it can be reduced to max-flow.

Bipartite Matching



- Create a directed flow graph where we…
    - ➢ Add a source node s and target node t
    - ➢ Add edges, all of capacity 1:
        - ○ $s \to u$ for each $u \in U$, $v \to t$ for each $v \in V$
        - ○ $u \to v$ for each $(u, v) \in E$

Bipartite Matching

- Observation
  - There is a 1-1 correspondence between matchings of size k in the original graph and flows with value k in the corresponding flow network.
- Proof: (matching $\Rightarrow$ integral flow)
  - Take a matching $M = \{(u_1, v_1), \dots, (u_k, v_k)\}$ of size k
  - Construct the corresponding unique flow f_M where…
    - Edges $s \to u_i$, $u_i \to v_i$, and $v_i \to t$ have flow 1, for all $i = 1, \dots, k$
    - The rest of the edges have flow 0
  - This flow has value k

Bipartite Matching

- Observation
  - There is a 1-1 correspondence between matchings of size k in the original graph and flows with value k in the corresponding flow network.
- Proof: (integral flow $\Rightarrow$ matching)
  - Take any flow f with value k
  - The corresponding unique matching M_f = set of edges from U to V with a flow of 1
    - Since flow of k comes out of s, unit flow must go to k distinct vertices in U
    - From each such vertex in U, unit flow goes to a distinct vertex in V
    - Uses integrality theorem

Bipartite Matching

- Perfect matching = flow with value n
  - where $n = |U| = |V|$
- Recall naïve Ford-Fulkerson running time:
  - $O((m+n) \cdot C)$, where $C =$ sum of capacities of edges leaving s
  - Q: What's the runtime when used for bipartite matching?
- Some variants are faster…
  - Dinitz's algorithm runs in time $O(m\sqrt{n})$ when all edge capacities are 1

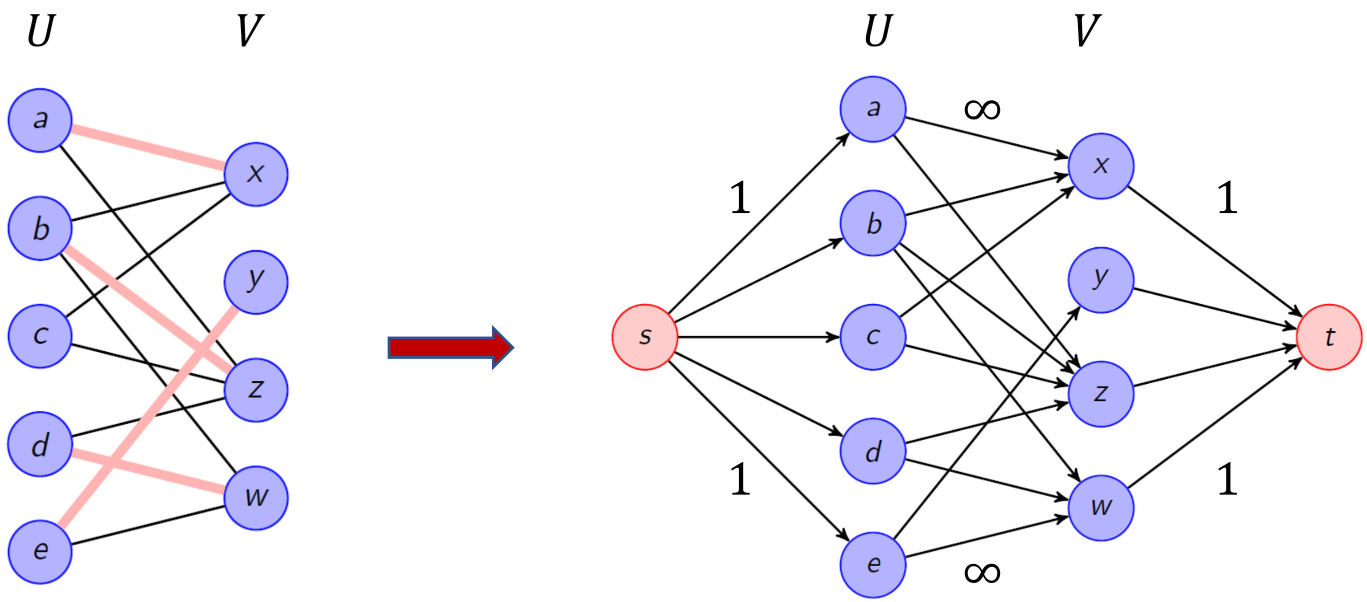
Hall's Marriage Theorem

- When does a bipartite graph have a perfect matching?
  - Well, when the corresponding flow network has value n
  - But can we interpret this condition in terms of edges of the original bipartite graph?
  - For $S \subseteq U$, let $N(S) \subseteq V$ be the set of all nodes in V adjacent to some node in S

- Observation:
  - If G has a perfect matching, $|N(S)| \geq |S|$ for each $S \subseteq U$
  - Because each node in S must be matched to a distinct node in $N(S)$

Hall’s Marriage Theorem

- We’ll consider a slightly different flow network, which is still equivalent to bipartite matching
  - All $U \to V$ edges now have ∞ capacity
  - $s \to U$ and $V \to t$ edges are still unit capacity



Hall's Marriage Theorem

- Hall's Theorem:
  - G has a perfect matching iff $|N(S)| \geq |S|$ for each $S \subseteq V$

- Proof (reverse direction, via network flow):
  - Suppose G doesn't have a perfect matching
  - Hence, max-flow = min-cut $< n$
  - Let (A, B) be the min-cut
    - Can't have any $U \rightarrow V$ (∞ capacity edges)
    - Has unit capacity edges $s \rightarrow U \cap B$ and $V \cap A \rightarrow t$

Hall's Marriage Theorem

- Hall's Theorem:
  - G has a perfect matching iff $|N(S)| \geq |S|$ for each $S \subseteq V$

- Proof (reverse direction, via network flow):
  - $cap(A, B) = |U \cap B| + |V \cap A| < n = |U|$
  - So $|V \cap A| < |U \cap A|$
  - But $N(U \cap A) \subseteq V \cap A$ because the cut doesn't include any ∞ edges
  - So $|N(U \cap A)| \leq |V \cap A| < |U \cap A|$. ■

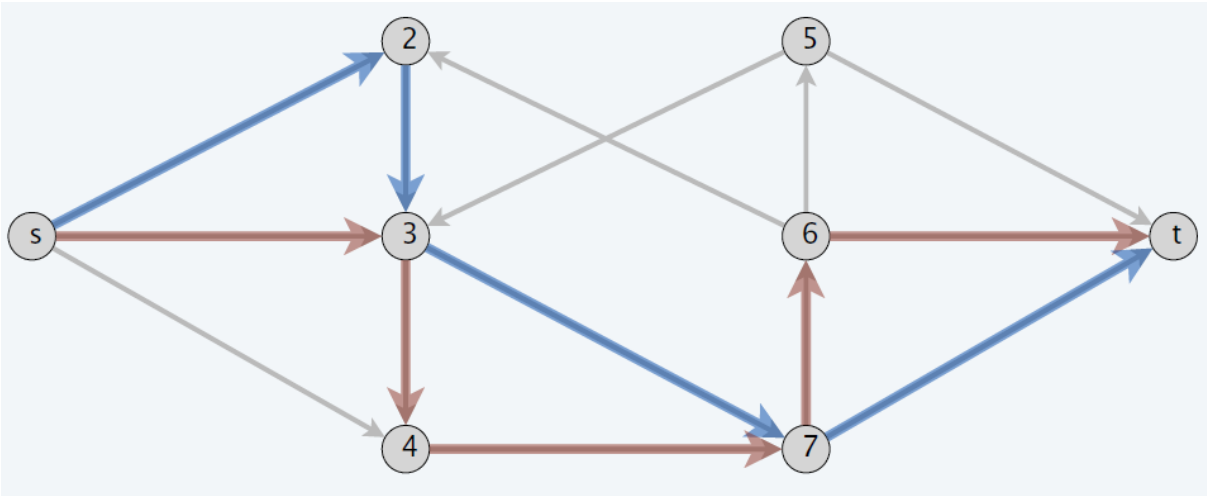
Some Notes

- Runtime for bipartite perfect matching
  - 1955: $O(mn) \rightarrow$ Ford-Fulkerson
  - 1973: $O(m\sqrt{n}) \rightarrow$ blocking flow (Hopcroft-Karp, Karzanov)
  - 2004: $O(n^{2.378}) \rightarrow$ fast matrix multiplication (Mucha–Sankowsi)
  - 2013: $\tilde{O}\left(m^{10/7}\right) \rightarrow$ electrical flow (Mądry)
  - Best running time is still an open question
- Nonbipartite graphs
  - Hall's theorem $\rightarrow$ Tutte's theorem
  - 1965: $O(n^4) \rightarrow$ Blossom algorithm (Edmonds)
  - 1980/1994: $O(m\sqrt{n}) \rightarrow$ Micali-Vazirani

Edge-Disjoint Paths

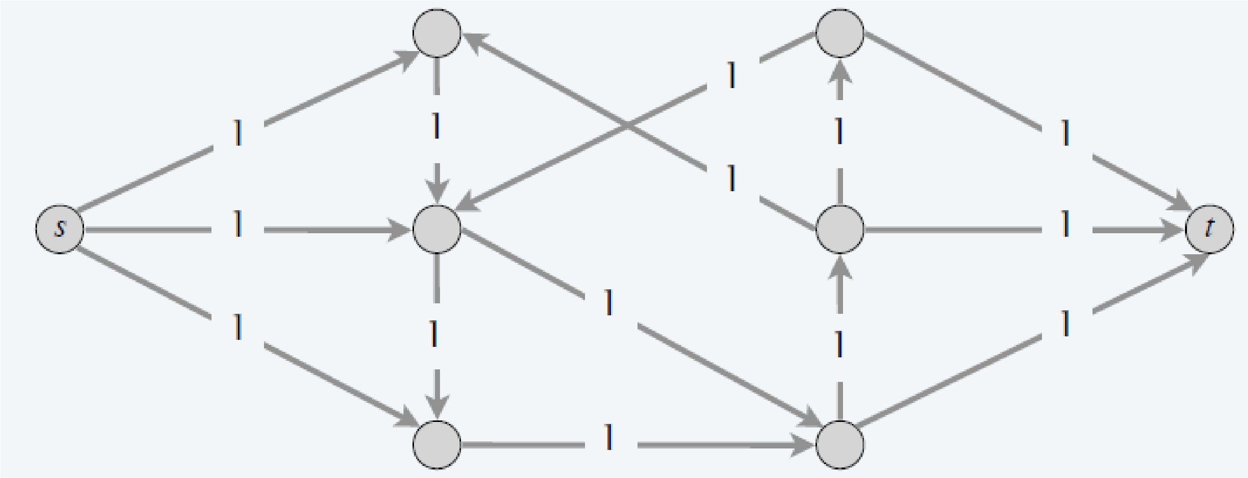
- **Problem**
  - Given a directed graph $G = (V, E)$, two nodes s and t, find the maximum number of edge-disjoint $s \to t$ paths

- Two $s \to t$ paths P and P' are edge-disjoint if they don't share an edge



Edge-Disjoint Paths

- Application:
  - Communication networks
- Max-flow formulation
  - Assign unit capacity on all edges



Edge-Disjoint Paths

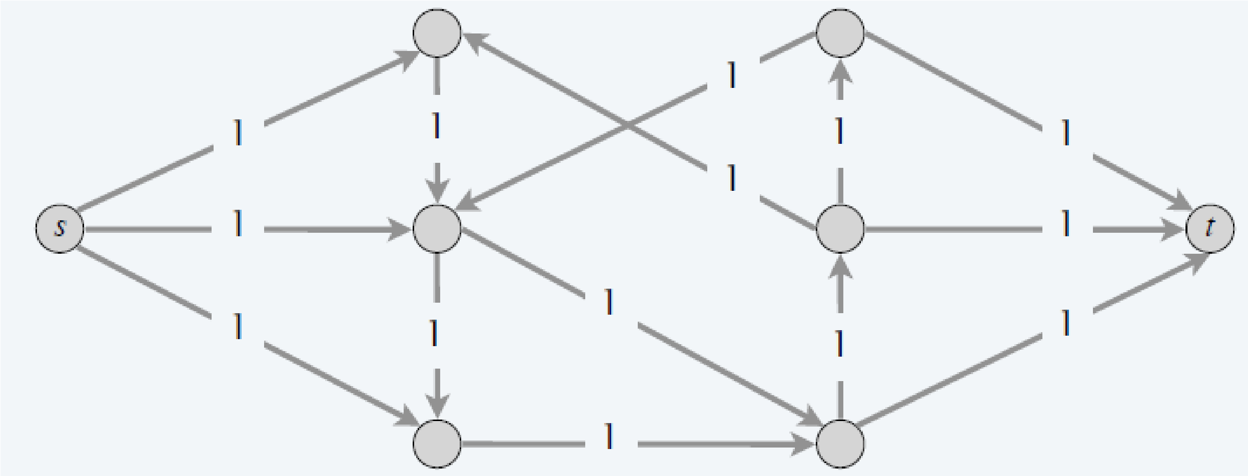
- Theorem:
  - There is 1-1 correspondence between sets of k edge-disjoint $s \to t$ paths and integral flows of value k
- Proof (paths → flow)
  - Let $\{P_1, \dots, P_k\}$ be a set of k edge-disjoint $s \to t$ paths
  - Define flow f where $f(e) = 1$ whenever $e \in P_i$ for some i, and 0 otherwise
  - Since paths are edge-disjoint, flow conservation and capacity constraints are satisfied
  - Unique integral flow of value k

Edge-Disjoint Paths

- Theorem:
  - There is 1-1 correspondence between k edge-disjoint $s \to t$ paths and integral flows of value k
- Proof (flow → paths)
  - Let f be an integral flow of value k
  - k outgoing edges from s have unit flow
  - Pick one such edge (s, u_1)
    - By flow conservation, u_1 must have unit outgoing flow (which we haven't used up yet).
    - Pick such an edge and continue building a path until you hit t
  - Repeat this for the other $k - 1$ edges coming out of s with unit flow. ■

Edge-Disjoint Paths

- Maximum number of edge-disjoint $s \to t$ paths
  - Equals max flow in this network
  - By max-flow min-cut theorem, also equals minimum cut
  - Exercise: minimum cut = minimum number of edges we need to delete to disconnect s from t
    - Hint: Show each direction separately ($\leq$ and $\geq$)



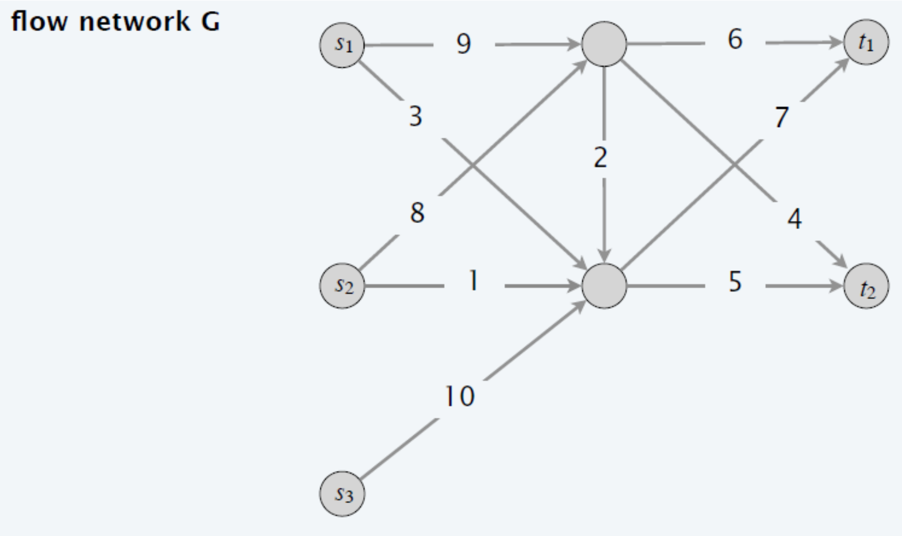
Edge-Disjoint Paths

- Exercise!
  - Show that to compute the maximum number of edge-disjoint s-t paths in an undirected graph, you can create a directed flow network by adding each undirected edge in both directions and setting all capacities to 1

- Menger's Theorem
  - In any directed/undirected graph, the maximum number of edge-disjoint (resp. vertex-disjoint) $s \rightarrow t$ paths equals the minimum number of edges (resp. vertices) whose removal disconnects s and t

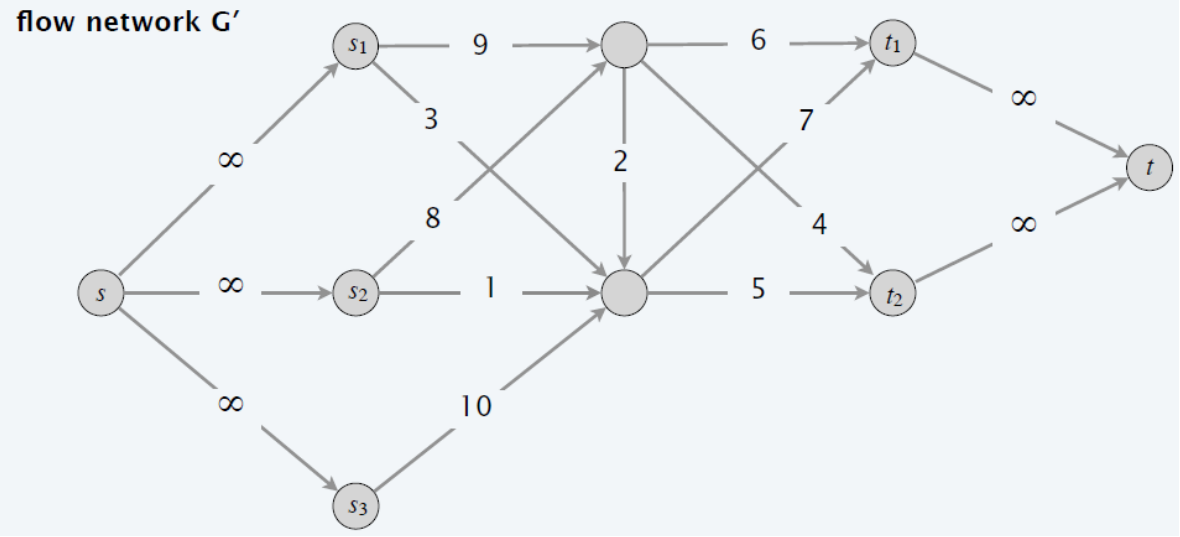
Multiple Sources/Sinks

- Problem
  - Given a directed graph $G = (V, E)$ with edge capacities $c: E \to \mathbb{N}$, sources $s_1, \dots, s_k$ and sinks $t_1, \dots, t_\ell$, find the maximum total flow from sources to sinks.



Multiple Sources/Sinks

- Network flow formulation
  - Add a new source s, edges from s to each s_i with ∞ capacity
  - Add a new sink t, edges from each t_j to t with ∞ capacity
  - Find max-flow from s to t
  - Claim: $1 - 1$ correspondence between flows in two networks



Circulation

- Input
  - Directed graph $G = (V, E)$
  - Edge capacities $c : E \to \mathbb{N}$
  - Node demands $d : V \to \mathbb{Z}$
- Output
  - Some circulation $f : E \to \mathbb{N}$ satisfying
    - For each $e \in E : 0 \le f(e) \le c(e)$
    - For each $v \in V : \sum_{e \text{ entering } v} f(v) - \sum_{e \text{ leaving } v} f(v) = d(v)$

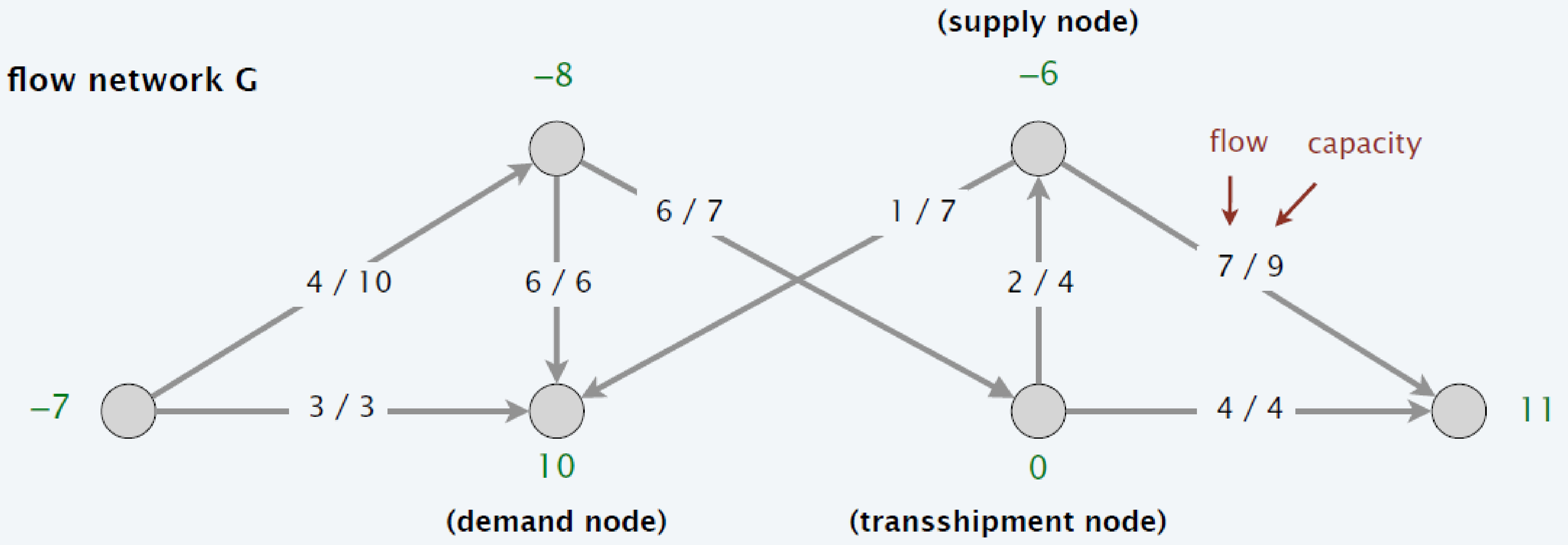
- Note that you need $\sum_{v:d(v)>0} d(v) = \sum_{v:d(v)<0} -d(v)$
- What are demands?

Circulation

- Demand at v = amount of flow you need to take out at node v
  - $d(v) > 0$: You need to take some flow out at v
    - So there should be $d(v)$ *more* incoming flow than outgoing flow
    - "Demand node"
  - $d(v) < 0$: You need to put some flow in at v
    - So there should be $|d(v)|$ *more* outgoing flow than incoming flow
    - "Supply node"
  - $d(v) = 0$: Node has flow conservation
    - Equal incoming and outgoing flows
    - "Transshipment node"

Circulation

- Example



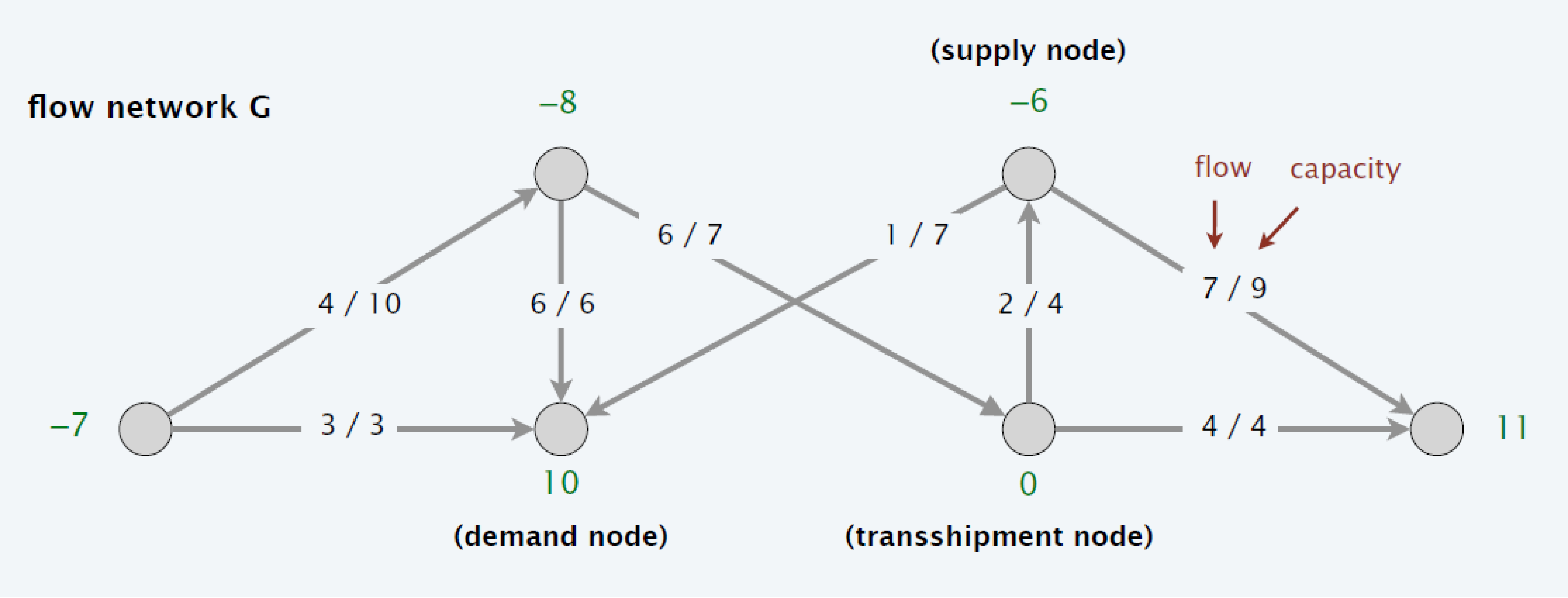
Circulation

- Network-flow formulation G'
  - Add a new source s and a new sink t
  - For each "supply" node v with $d(v) < 0$, add edge (s, v) with capacity $-d(v)$
  - For each "demand" node v with $d(v) > 0$, add edge (v, t) with capacity $d(v)$

- Claim: G has a circulation iff G' has max flow of value $\sum_{v:d(v)>0} d(v) = \sum_{v:d(v)<0} -d(v)$

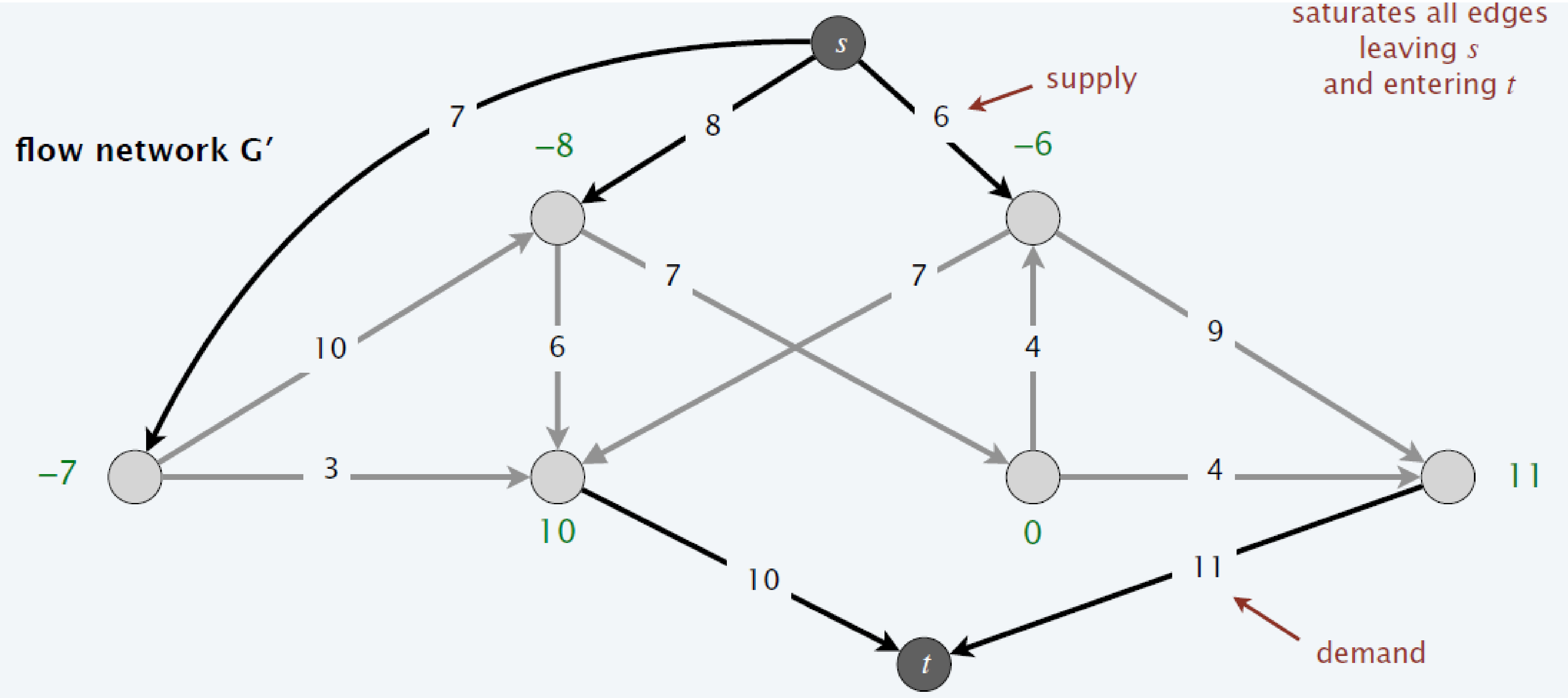
Circulation

- Example



Circulation

- Example



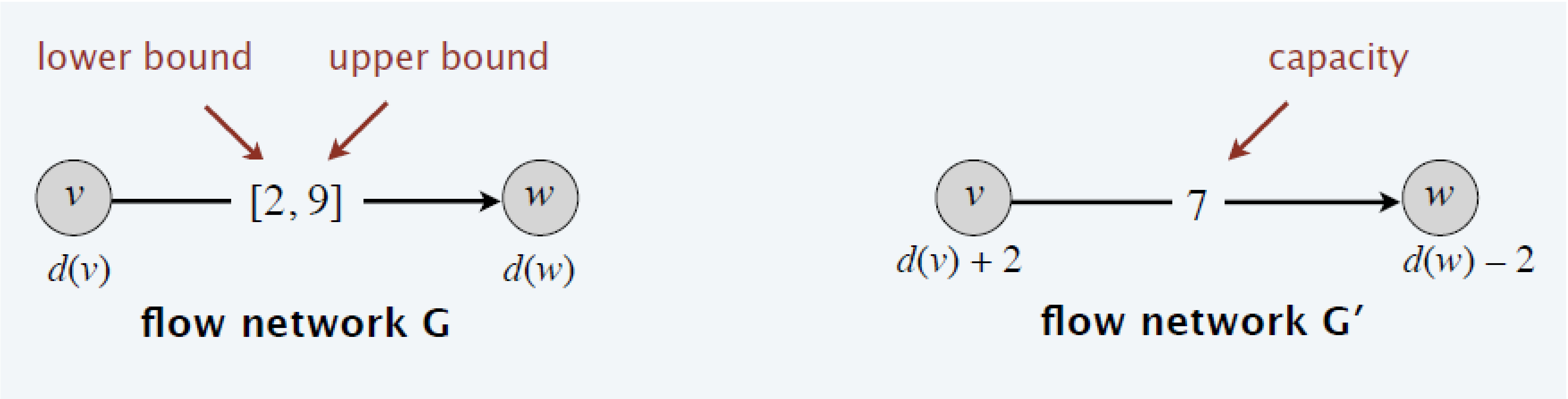
Circulation with Lower Bounds

- Input
  - Directed graph $G = (V, E)$
  - Edge capacities $c : E \to \mathbb{N}$ and lower bounds $\ell : E \to \mathbb{N}$
  - Node demands $d : V \to \mathbb{Z}$
- Output
  - Some circulation $f : E \to \mathbb{N}$ satisfying
    - For each $e \in E$: $\ell(e) \leq f(e) \leq c(e)$
    - For each $v \in V$: $\sum_{e \text{ entering } v} f(v) - \sum_{e \text{ leaving } v} f(v) = d(v)$

- Note that you still need $\sum_{v:d(v)>0} d(v) = \sum_{v:d(v)<0} -d(v)$

Circulation with Lower Bounds

- Transform to circulation without lower bounds
  - Do the following operation to each edge



- Claim: Circulation in G iff circulation in G'
  - Proof sketch: $f(e)$ gives a valid circulation in G iff $f(e) - \ell(e)$ gives a valid circulation in G'

Survey Design

- Problem
  - We want to design a survey about m products
    - We have one question in mind for each product
    - Need to ask product j's question to between p_j and p_j' consumers
  - There are a total of n consumers
    - Consumer i owns a subset of products O_i
    - We can ask consumer i questions about only these products
    - We want to ask consumer i between c_i and c_i' questions
  - Is there a survey meeting all these requirements?

Survey Design

- Bipartite matching is a special case
  - $c_i = c_i' = p_j = p_j' = 1$ for all i and j

- Formulate as circulation with lower bounds
  - Create a network with special nodes s and t
  - Edge from s to each consumer i with flow $\in [c_i, c_i']$
  - Edge from each consumer i to each product $j \in O_i$ with flow $\in [0,1]$
  - Edge from each product j to t with flow $\in [p_j, p_j']$
  - Edge from t to s with flow in $[0, \infty]$
  - All demands and supplies are 0

Survey Design

- Max-flow formulation:
  - Feasible survey iff feasible circulation in this network

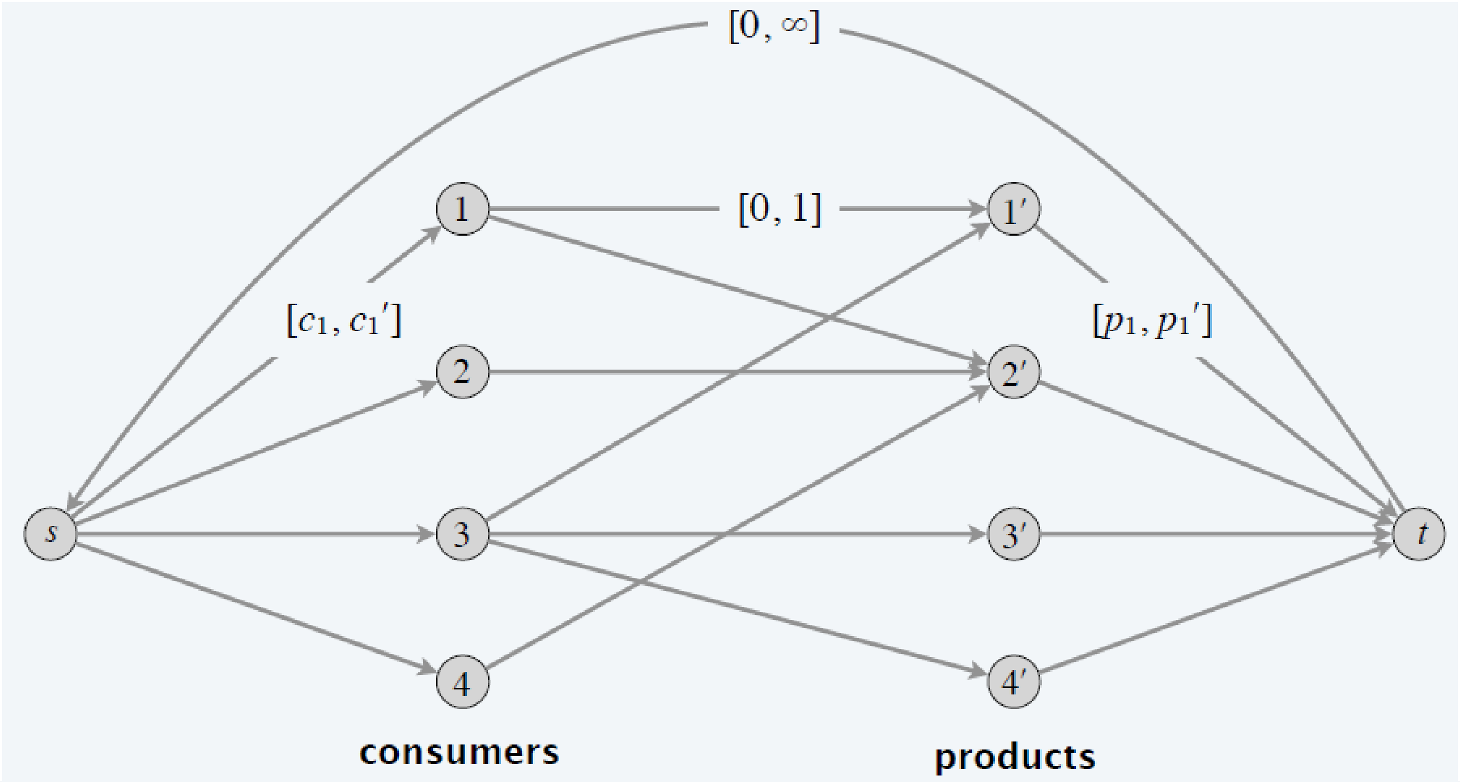


Image Segmentation

- Foreground/background segmentation
  - Given an image, separate "foreground" from "background"
- Here's the power of PowerPoint (or the lack thereof)

Remove background

Image Segmentation

- Foreground/background segmentation
  - Given an image, separate “foreground” from “background”
- Here’s what remove.bg gets using AI

Remove background

Image Segmentation

- Informal problem
  - Given an image (2D array of pixels), and likelihood estimates of different pixels being foreground/background, label each pixel as foreground or background
  - Want to prevent having too many neighboring pixels where one is labeled foreground but the other is labeled background

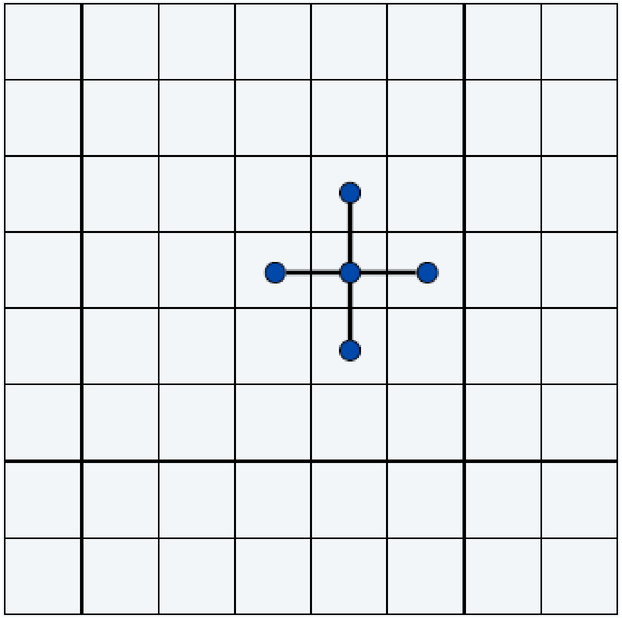


Image Segmentation

- Input
  - An image (2D array of pixels)
  - a_i = likelihood of pixel i being in foreground
  - b_i = likelihood of pixel i being in background
  - $p_{i,j}$ = penalty for "separating" pixels i and j (i.e. labeling one of them as foreground and the other as background)
- Output
  - Label each pixel as "foreground" or "background"
  - Minimize "total penalty"
    - Want it to be high if a_i is high but i is labeled background, b_i is high but i is labeled foreground, or $p_{i,j}$ is high but i and j are separated

Image Segmentation

- Recall
  - a_i = likelihood of pixels i being in foreground
  - b_i = likelihood of pixels i being in background
  - $p_{i,j}$ = penalty for separating pixels i and j
  - Let $E =$ pairs of neighboring pixels
- Output
  - Minimize total penalty
    - $A =$ set of pixels labeled foreground
    - $B =$ set of pixels labeled background
    - Penalty =

$$\sum_{i \in A} b_i + \sum_{j \in B} a_j + \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{i,j}$$

Image Segmentation

- Formulate as a min-cut problem
  - Want to divide the set of pixels V into (A, B) to minimize

$$\sum_{i \in A} b_i + \sum_{j \in B} a_j + \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{i,j}$$

  - Nodes:
    - source s, target t, and v_i for each pixel i
  - Edges:
    - (s, v_i) with capacity a_i for all i
    - (v_i, t) with capacity b_i for all i
    - (v_i, v_j) and (v_j, v_i) with capacity $p_{i,j}$ each for all neighboring (i, j)

Image Segmentation

- Formulate as min-cut problem
  - Here's what the network looks like

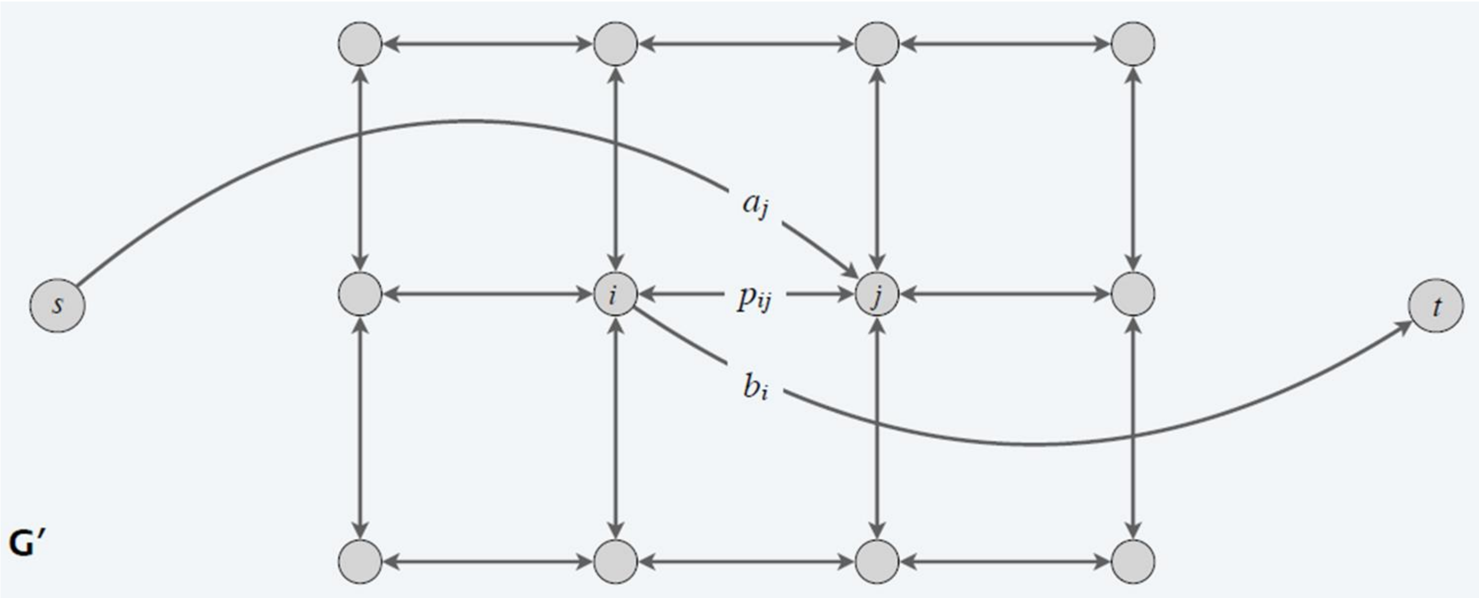


Image Segmentation

If i and j are labeled differently, it will add $p_{i,j}$ exactly once

- Consider the min-cut (A, B)

$$cap(A,B) = \sum_{i \in A} b_i + \sum_{j \in B} a_j + \sum_{\substack{(i,j) \in E \\ i \in A, j \in B}} p_{i,j}$$

- Exactly what we want to minimize!

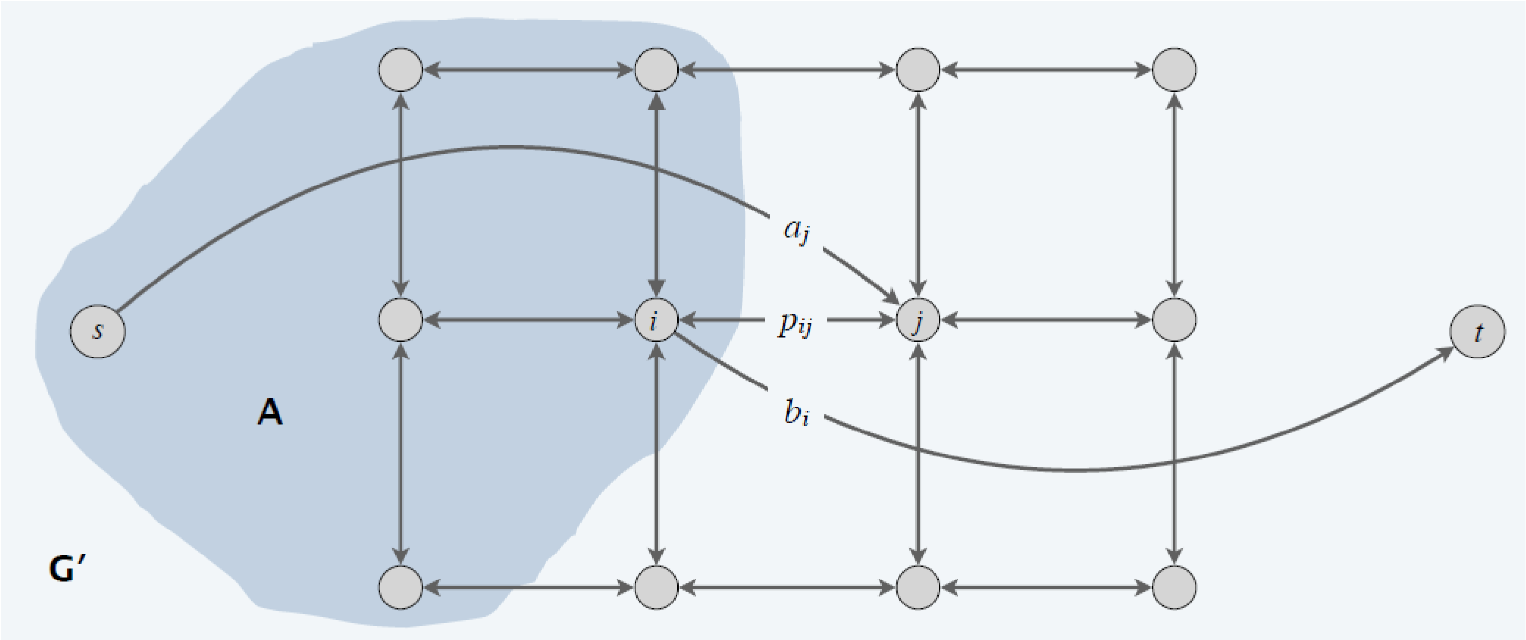


Image Segmentation

- GrabCut [Rother-Kolmogorov-Blake 2004]

"GrabCut" — Interactive Foreground Extraction using Iterated Graph Cuts

Carsten Rother* Vladimir Kolmogorov† Andrew Blake‡
Microsoft Research Cambridge, UK

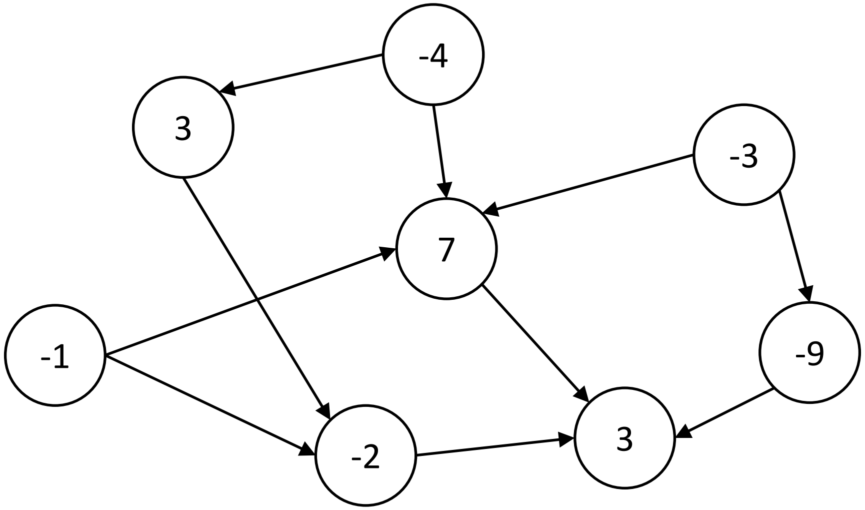
Figure 1: **Three examples of GrabCut .** The user drags a rectangle loosely around an object. The object is then extracted automatically.

Profit Maximization (Yeaa…!)

- Problem
  - There are n tasks
  - Performing task i generates a profit of p_i
    - We allow $p_i < 0$ (i.e. performing task i may be costly)
  - There is a set E of precedence relations
    - $(i, j) \in E$ indicates that if we perform i, we must also perform j
- Goal
  - Find a subset of tasks S which, subject to the precedence constraints, maximizes $profit(S) = \sum_{i \in S} p_i$

Profit Maximization

- We can represent the input as a graph
  - Nodes = tasks, node weights = profits,
  - Edges = precedence constraints
  - Goal: find a subset of nodes S with highest total weight s.t. if $i \in S$ and $(i, j) \in E$, then $j \in S$ as well

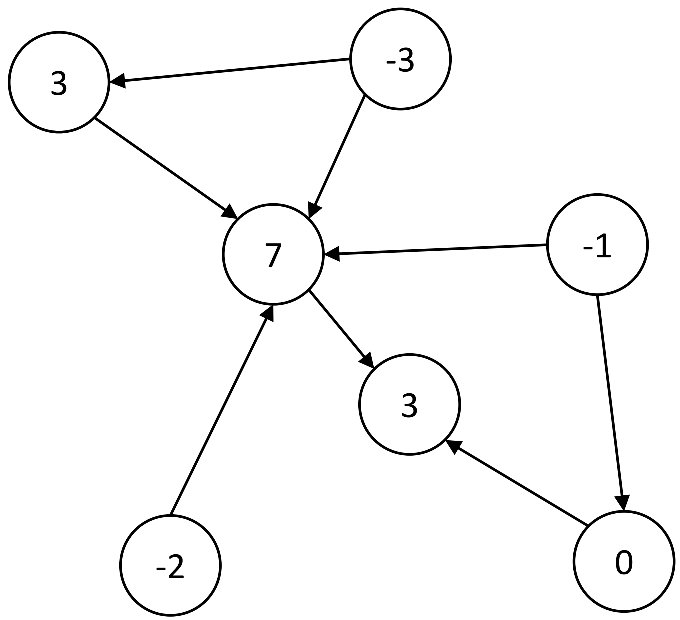


Profit Maximization

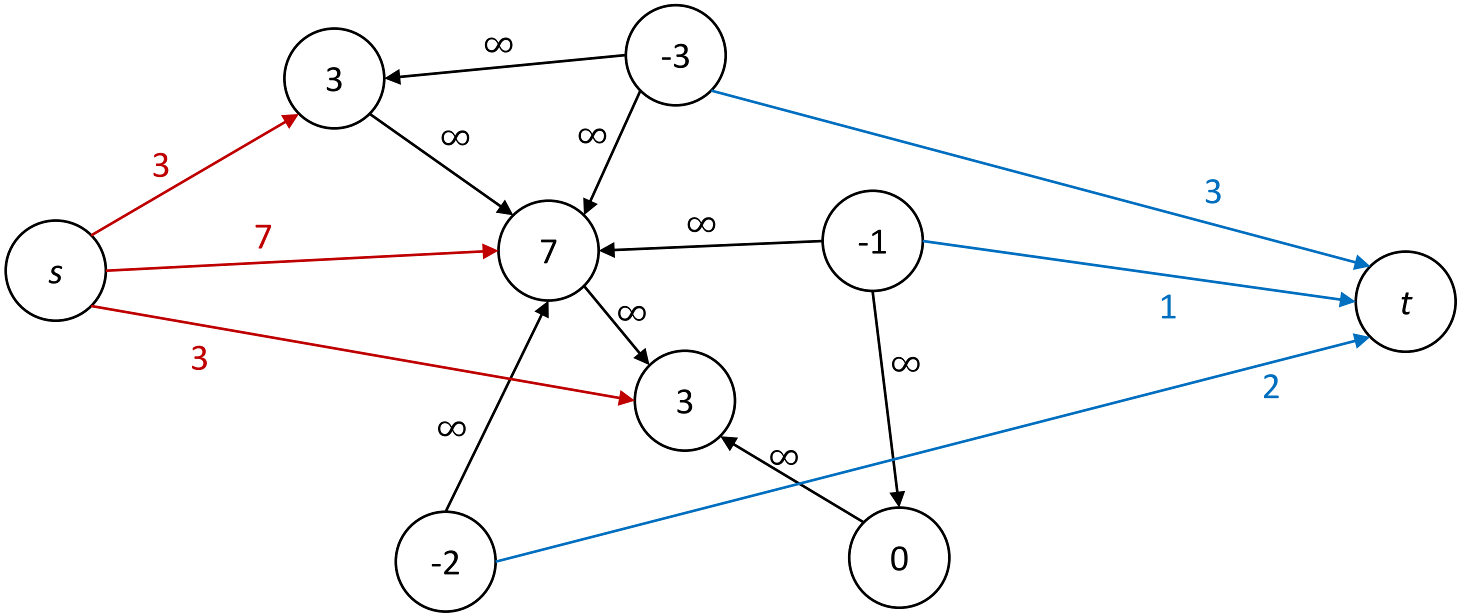
- Want to formulate as a min-cut
  - Add source s and target t
  - min-cut $(A, B) \Rightarrow$ want desired solution to be $S = A \setminus \{s\}$
  - Goals:
    - $cap(A, B)$ should nicely relate to $profit(S)$
    - Precedence constraints *must be* respected
      - "Hard" constraints are usually enforced using infinite capacity edges

- Construction:
  - Add each $(i, j) \in E$ with *infinite* capacity
  - For each i:
    - If $p_i > 0$, add (s, i) with capacity p_i
    - If $p_i < 0$, add (i, t) with capacity $-p_i$

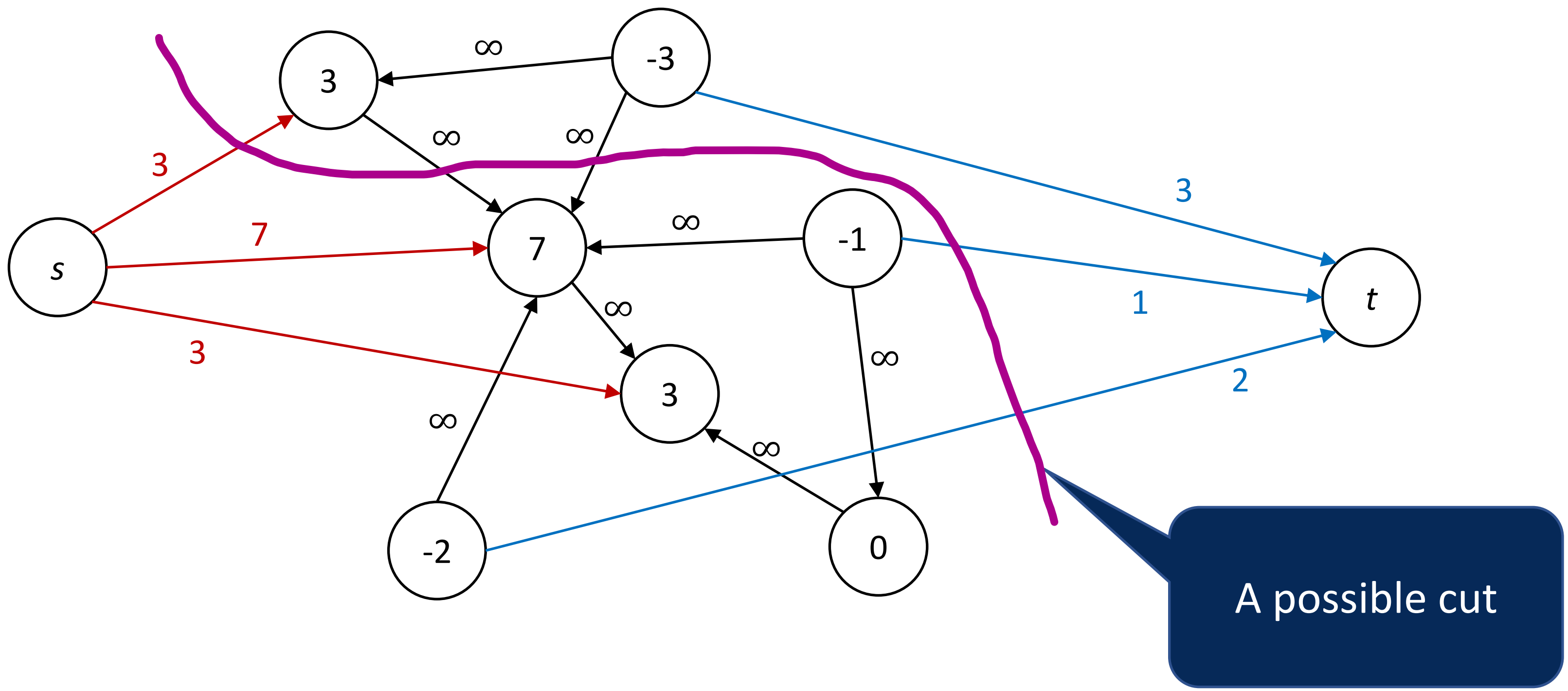
Profit Maximization



Profit Maximization



Profit Maximization



QUESTION: What is the capacity of this cut?

Profit Maximization

Exercise: Show that…

1. A finite capacity cut exists.
2. If $cap(A, B)$ is finite, then $A \backslash \{s\}$ is a valid solution;
3. Minimizing $cap(A, B)$ maximizes $profit(A \backslash \{s\})$
   - Show that $cap(A, B) =$ constant $- \, profit(A \backslash \{s\})$, where the constant is independent of the choice of (A, B)