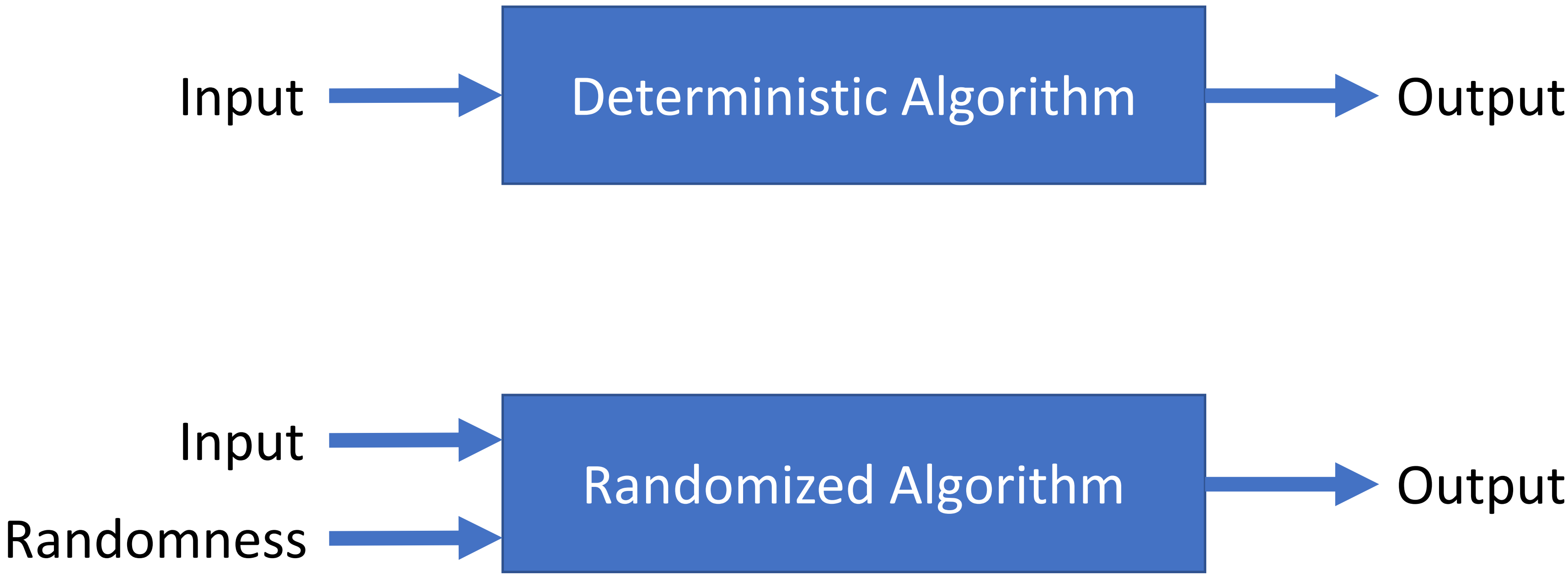
CSC373

Week 11: Randomized Algorithms

Randomized Algorithms



Randomized Algorithms

- Running time
  - Harder goal: the running time should *always* be small
    - Regardless of both the input and the random coin flips
  - Easier goal: the running time should be small *in expectation*
    - Expectation over random coin flips
    - But it should still be small for every input (i.e. worst-case)
- Approximation Ratio
  - The objective value of the solution returned should, *in expectation*, be close to the optimum objective value
    - Once again, the expectation is over random coin flips
    - The approximation ratio should be small for every input

Derandomization

- After coming up with a randomized approximation algorithm, one might ask if it can be "derandomized"
  - Informally, the randomized algorithm is making random choices that, in expectation, turn out to be good
  - Can we make these "good" choices deterministically?

- For some problems…
  - It may be easier to first design a simple randomized approximation algorithm and then de-randomize it…
  - Than to try to directly design a deterministic approximation algorithm

Recap: Probability Theory

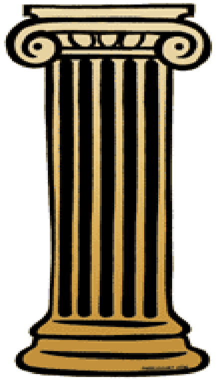
- Random variable X
  - Discrete
    - Takes value v_1 with probability p_1, v_2 w.p. p_2, …
    - Expected value $E[X] = p_1 \cdot v_1 + p_2 \cdot v_2 + \cdots$
    - Examples: coin toss, the roll of a six-sided die, …
  - Continuous
    - Has a probability density function (pdf) f
    - Its integral is the cumulative density function (cdf) F
      - $F(x) = \Pr[X \le x] = \int_{-\infty}^{x} f(t)\, dt$
    - Expected value $E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$
    - Examples: normal distribution, exponential distribution, uniform distribution over $[0,1]$, …

Recap: Probability Theory

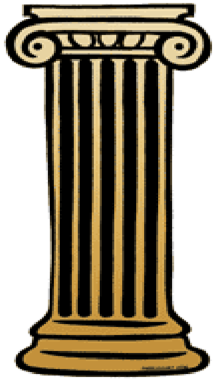
- Things you should be aware of…
  - Conditional probabilities
  - Conditional expectations
  - Independence among random variables
  - Moments of random variables
  - Standard discrete distributions: uniform over a finite set, Bernoulli, binomial, geometric, Poisson, …
  - Standard continuous distributions: uniform over intervals, Gaussian/normal, exponential, …

Three Pillars

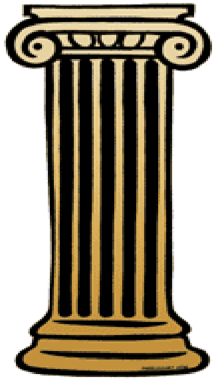
Linearity of Expectation



Union Bound



Chernoff Bound



- Deceptively simple, but incredibly powerful!
- Many many many many probabilistic results are just interesting applications of these three results

Three Pillars

- Linearity of expectation
  - $E[X+Y] = E[X] + E[Y]$
  - This does *not* require any independence assumptions about X and Y
  - E.g. if you want to find out how many people will attend your party on average, just ask each person the probability with which they will attend and sum up the probabilities
    - It does not matter whether some of them are friends and either all will attend together or none will attend

Three Pillars

- Union bound
  - For any two events A and B, $\Pr[A \cup B] \leq \Pr[A] + \Pr[B]$
  - "Probability that at least one of the n events $A_1, \ldots, A_n$ will occur is at most $\sum_i \Pr[A_i]$"
  - Typically, $A_1, \ldots, A_n$ are "bad events"
    - You do not want any of them to occur
    - If you can individually bound $\Pr[A_i] \leq {}^{1}/_{2n}$ for each i, then probability that at least one them occurs $\leq 1/2$
    - Thus, with probability $\geq {}^{1}/_{2}$, *none* of the bad events will occur

- Chernoff bound & Hoeffding's inequality
  - Read up!

Exact Max-k-SAT

Exact Max-k-SAT

- Problem (recall)
  - Input: An exact k-SAT formula $\varphi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$, where each clause C_i has exactly k literals, and a weight $w_i \geq 0$ of each clause C_i
  - Output: A truth assignment τ maximizing the number (or total weight) of clauses satisfied under τ

- Let us denote by $W(\tau)$ the total weight of clauses satisfied under τ

Exact Max-k-SAT

- Recall our local search
  - $N_d(\tau)$ = set of all truth assignments which can be obtained by changing the value of at most d variables in τ

- Result 1: Neighborhood $N_1(\tau) \Rightarrow {}^2/_3$-apx for Exact Max-2-SAT.
- Result 2: Neighborhood $N_1(\tau) \cup \tau^c \Rightarrow {}^3/_4$-apx for Exact Max-2-SAT.
- Result 3: Neighborhood $N_1(\tau)$ + oblivious local search $\Rightarrow {}^3/_4$-apx for Exact Max-2-SAT.

Exact Max-k-SAT

- Recall our local search
  - $N_d(\tau)$ = set of all truth assignments which can be obtained by changing the value of at most d variables in τ

- We claimed that ¾-apx for Exact Max-2-SAT can be generalized to $\frac{2^k-1}{2^k}$-apx for Exact Max-k-SAT
  - Algorithm becomes slightly more complicated

- What can we do with randomized algorithms?

Exact Max-k-SAT

- Recall:
  - We have a formula $\varphi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$
  - Variables = $x_1, \dots, x_n$, literals = variables or their negations
  - Each clause contains exactly k literals

- The most naïve randomized algorithm
  - Set each variable to TRUE with probability ½ and to FALSE with probability ½

- How good is this?

Exact Max-k-SAT

- Recall:
  - We have a formula $\varphi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$
  - Variables = $x_1, \dots, x_n$, literals = variables or their negations
  - Each clause contains exactly k literals

- Let τ be a random assignment
  - For each clause C_i: $\Pr[C_i \text{ is not satisfied}] = {}^{1}/_{2^k}$ (WHY?)
    - Hence, $\Pr[C_i \text{ is satisfied}] = {}^{(2^k-1)}/_{2^k}$
  - $E[W(\tau)] = \sum_{i=1}^{m} w_i \cdot \Pr[C_i \text{ is satisfied}]$ (WHY?)
  - $E[W(\tau)] = \frac{2^k-1}{2^k} \cdot \sum_{i=1}^{m} w_i \geq \frac{2^k-1}{2^k} \cdot OPT$

Derandomization

- Can we derandomize this algorithm?
  - What are the choices made by the algorithm?
    - Setting the values of $x_1, x_2, \dots, x_n$
  - How do we know which set of choices is good?

- Idea:
  - Do not think about all the choices at once.
  - Think about them one by one.
  - Goal: Gradually convert the random assignment τ to a deterministic assignment $\hat{\tau}$ such that $W(\hat{\tau}) \geq E[W(\tau)]$
    - Combining with $E[W(\tau)] \geq \frac{2^k-1}{2^k} \cdot OPT$ will give the desired deterministic approximation ratio

Derandomization

- Start with the random assignment τ and write…

$$E[W(\tau)] = \Pr[x_1 = T] \cdot E[W(\tau) | x_1 = T] + \Pr[x_1 = F] \cdot E[W(\tau) | x_1 = F]$$
$$= {}^1/_2 \cdot E[W(\tau) | x_1 = T] + {}^1/_2 \cdot E[W(\tau) | x_1 = F]$$

- Hence, $\max(E[W(\tau) | x_1 = T], E[W(\tau) | x_1 = F]) \geq E[W(\tau)]$
  - What is $E[W(\tau) | x_1 = T]$?
    - It is the expected weight when setting $x_1 = T$ deterministically but still keeping $x_2, \ldots, x_n$ random

- If we can compute both $E[W(\tau) | x_1 = T]$ and $E[W(\tau) | x_1 = F]$, and pick the better one…
  - Then we can set x_1 deterministically without degrading the expected objective value

Derandomization

- After deterministically making the right choice for x_1 (say T), we can apply the same logic to x_2

$$E[W(\tau)|x_1 = T] = {}^1/_2 \cdot E[W(\tau)|x_1 = T, x_2 = T] \\ + {}^1/_2 \cdot E[W(\tau)|x_1 = T, x_2 = F]$$

  - Pick the better of the two conditional expectations

- **Derandomized Algorithm:**
  - For $i = 1, \dots, n$
    - Let $z_i = T$ if $E[W(\tau)|x_1 = z_1, \dots, x_{i-1} = z_{i-1}, x_i = T] \geq E[W(\tau)|x_1 = z_1, \dots, x_{i-1} = z_{i-1}, x_i = F]$, and $z_i = F$ otherwise
    - Set $x_i = z_i$

Derandomization

- This is called *the method of conditional expectations*
  - If we're happy when making a choice at random, we should be at least as happy conditioned on at least one of the possible values of that choice

- Remaining question:
  - How do we compute & compare the two conditional expectations: $E[W(\tau) | x_1 = z_1, \dots, x_{i-1} = z_{i-1}, x_i = T]$ and $E[W(\tau) | x_1 = z_1, \dots, x_{i-1} = z_{i-1}, x_i = F]$?

Derandomization

- $E[W(\tau)|x_1 = z_1, \dots, x_{i-1} = z_{i-1}, x_i = T]$
  - $\sum_r w_r \cdot \Pr[C_r$ is satisfied $|x_1 = z_1, \dots, x_{i-1} = z_{i-1}, x_i = T]$
  - Set the values of $x_1, \dots, x_{i-1}, x_i$
  - If C_r resolves to TRUE already, the corresponding probability is 1
  - If C_r resolves to FALSE already, the corresponding probability is 0
  - Otherwise, if there are ℓ literals left in C_r after setting $x_1, \dots, x_{i-1}, x_i$, the corresponding probability is $\frac{2^\ell - 1}{2^\ell}$

- Compute $E[W(\tau)|x_1 = z_1, \dots, x_{i-1} = z_{i-1}, x_i = F]$ similarly

Max-SAT

- Simple randomized algorithm
  - $\frac{2^k-1}{2^k}$ −approximation for Max-k-SAT
  - Max-3-SAT ⇒ $^7/_8$
    - [Håstad]: This is the best possible assuming P ≠ NP
  - Max-2-SAT ⇒ $^3/_4 = 0.75$
    - The best known approximation is 0.9401 using semi-definite programming and randomized rounding
  - Max-SAT ⇒ $^1/_2$
    - Max-SAT = no restriction on the number of literals in each clause
    - The best known approximation is 0.7968, also using semi-definite programming and randomized rounding

Max-SAT

- Better approximations for Max-SAT
  - Semi-definite programming is out of the scope
  - But we will see the simpler "LP relaxation + randomized rounding" approach that gives $1 - {}^{1}/_{e} \approx 0.6321$ approximation

- Max-SAT:
  - Input: $\varphi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$, where each clause C_i has weight $w_i \geq 0$ (and can have any number of literals)
  - Output: Truth assignment that approximately maximizes the weight of clauses satisfied

LP Formulation of Max-SAT

- First, IP formulation:
  - Variables:
    - $y_1, \ldots, y_n \in \{0,1\}$
      - $y_i = 1$ iff variable $x_i =$ TRUE in Max-SAT
    - $z_1, \ldots, z_m \in \{0,1\}$
      - $z_j = 1$ iff clause C_j is satisfied in Max-SAT
    - Program:

      Maximize $\Sigma_j \, w_j \cdot z_j$
      s.t.
      $\Sigma_{x_i \in C_j} \, y_i + \Sigma_{\bar{x}_i \in C_j} \, (1 - y_i) \geq z_j \quad \forall j \in \{1, \ldots, m\}$
      $y_i, z_j \in \{0,1\} \quad \forall i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}$

LP Formulation of Max-SAT

- LP relaxation:
  - Variables:
    - $y_1, \dots, y_n \in [0,1]$
      - $y_i = 1$ iff variable $x_i = \text{TRUE}$ in Max-SAT
    - $z_1, \dots, z_m \in [0,1]$
      - $z_j = 1$ iff clause C_j is satisfied in Max-SAT
    - Program:

      Maximize $\Sigma_j \, w_j \cdot z_j$
      s.t.
      $\Sigma_{x_i \in C_j} \, y_i + \Sigma_{\bar{x}_i \in C_j} \, (1 - y_i) \geq z_j \quad \forall j \in \{1, \dots, m\}$
      $y_i, z_j \in [0,1] \quad \forall i \in \{1, \dots, n\}, j \in \{1, \dots, m\}$

Randomized Rounding

- Randomized rounding
  - ➢ Find the optimal solution (y^*, z^*) of the LP
  - ➢ Compute a random IP solution $\hat{y}$ such that
    - Each $\hat{y}_i = 1$ with probability y_i^* and 0 with probability $1 - y_i^*$
    - Independently of other $\hat{y}_i$'s
    - The output of the algorithm is the corresponding truth assignment
  - ➢ What is $\Pr[C_j \text{ is satisfied}]$ if C_j has k literals?

$$1 - \Pi_{x_i \in C_j} (1 - y_i^*) \cdot \Pi_{\bar{x}_i \in C_j} (y_i^*)$$

$$\underbrace{\geq 1 - \left(\frac{\Sigma_{x_i \in C_j} (1 - y_i^*) + \Sigma_{\bar{x}_i \in C_j} (y_i^*)}{k}\right)^k}_{\text{AM-GM inequality}} \underbrace{\geq 1 - \left(\frac{k - z_j^*}{k}\right)^k}_{\text{LP constraint}}$$

Randomized Rounding

- Claim
  - $1 - \left(1 - \frac{z}{k}\right)^k \geq \left(1 - \left(1 - \frac{1}{k}\right)^k\right) \cdot z$ for all $z \in [0,1]$ and $k \in \mathbb{N}$

- Assuming the claim:

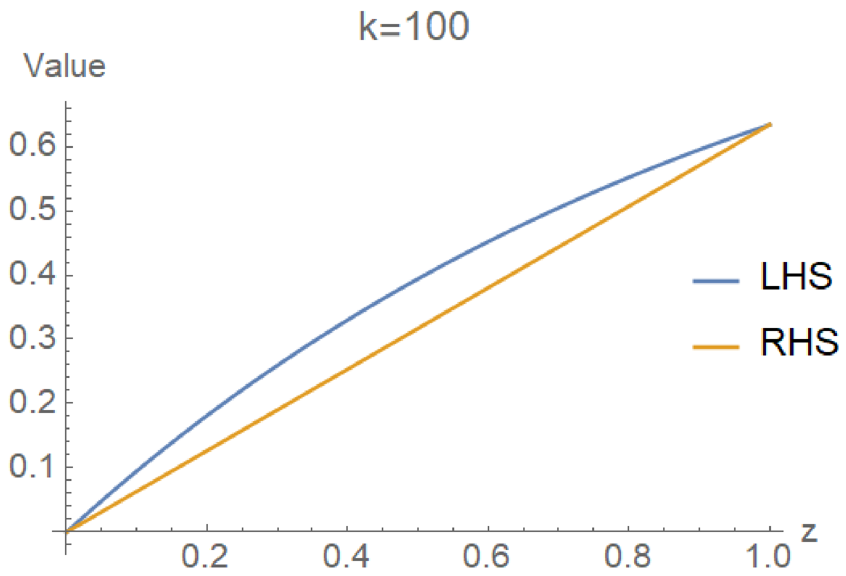
$$\Pr\left[C_j \text{ is satisfied}\right] \geq 1 - \left(\frac{k - z_j^*}{k}\right)^k \geq \underbrace{\left(1 - \left(1 - \frac{1}{k}\right)^k\right) \cdot z_j^* \geq \left(1 - \frac{1}{e}\right) \cdot z_j^*}_{\text{Standard inequality}}$$

- Hence,

$$\mathbb{E}[\text{\#weight of clauses satisfied}] \geq \underbrace{\left(1 - \frac{1}{e}\right) \sum_j w_j \cdot z_j^* \geq \left(1 - \frac{1}{e}\right) \cdot OPT}_{\text{Optimal LP objective} \geq \text{optimal ILP objective}}$$

Randomized Rounding

- Claim
  - $1 - \left(1 - \frac{z}{k}\right)^k \geq \left(1 - \left(1 - \frac{1}{k}\right)^k\right) \cdot z$ for all $z \in [0,1]$ and $k \in \mathbb{N}$
- Proof of claim:
  - True at $z = 0$ and $z = 1$ (same quantity on both sides)
  - For $0 \leq z \leq 1$:
    - LHS is a convex function
    - RHS is a linear function
    - Hence, LHS $\geq$ RHS ∎



Improving Max-SAT Apx

- Best of both worlds:
  - Run both “LP relaxation + randomized rounding” and “naïve randomized algorithm”
  - Return the best of the two solutions
  - Claim without proof: This achieves a $^3/_4 = 0.75$ approximation!
    - This algorithm can be derandomized.
  - Recall:
    - “naïve randomized” = independently set each variable to TRUE/FALSE with probability 0.5 each, which only gives $^1/_2 = 0.5$ approximation by itself

Back to 2-SAT

- Max-2-SAT is NP-hard (we didn't prove this!)
- But 2-SAT can be efficiently solved
  - "Given a 2-CNF formula, check whether *all* clauses can be satisfied simultaneously."

- Algorithm:
  - Repeatedly eliminate a clause with one literal & set the literal to true
  - Create a graph with each remaining literal as a vertex
  - For every clause $(x \vee y)$, add two edges: $\bar{x} \to y$ and $\bar{y} \to x$
    - $u \to v$ means if u is true, v must be true
  - Formula is satisfiable iff no path from x to $\bar{x}$ or $\bar{x}$ to x for any x
    - Can be checked in polynomial time

Random Walk + 2-SAT

- Here's a cute randomized algorithm by Papadimitriou [1991]

- Algorithm:
  - Start with an arbitrary assignment.
  - While there is an unsatisfied clause $C = (x \vee y)$
    - Pick one of the two literals with equal probability.
    - Flip the variable value so that C is satisfied.

- But can't this hurt the other clauses?
  - In a given step, yes.
  - But in expectation, we will still make progress.

Random Walk + 2-SAT

- Theorem:
  - If there is a satisfying assignment τ^*, then this algorithm reaches a satisfying assignment in $O(n^2)$ expected time.
- Proof:
  - Fix a satisfying assignment τ^*
  - Let τ_0 be the starting assignment
  - Let τ_i be the assignment after i iterations
  - Consider the "hamming distance" d_i between τ_i and τ^*
    - Number of coordinates in which the two differ
    - $d_i \in \{0,1,\ldots,n\}$
  - Claim: the algorithm hits $d_i = 0$ in $O(n^2)$ iterations in expectation, unless it stops before that

Random Walk + 2-SAT

- Observation: $d_{i+1} = d_i - 1$ or $d_{i+1} = d_i + 1$
  - Because we change one variable in each iteration.

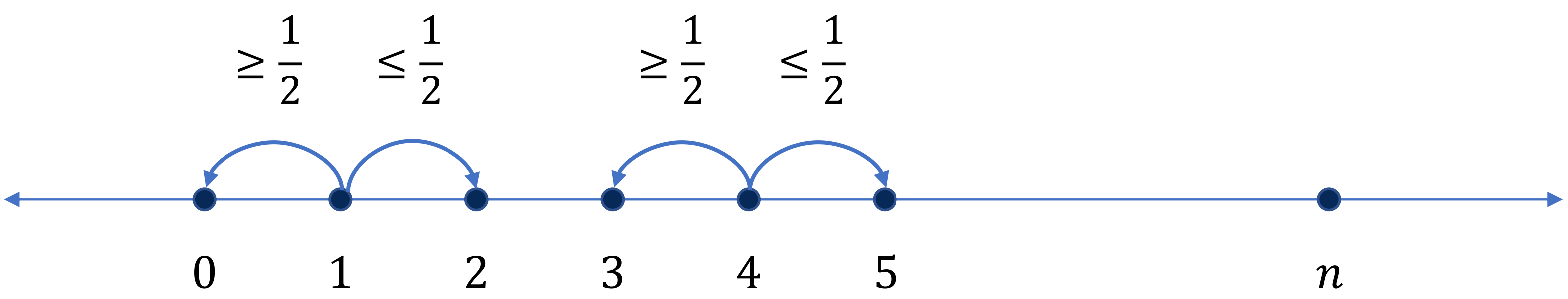
- Claim: $\Pr[d_{i+1} = d_i - 1] \geq 1/2$
- Proof:
  - Iteration i considers an unsatisfied clause $C = (x \vee y)$
  - τ^* satisfies at least one of x or y, while τ_i satisfies neither
  - Because we pick a literal randomly, w.p. at least ½ we pick one where τ_i and τ^* differ and decrease the distance
  - Q: Why did we need an unsatisfied clause? What if we pick one of n variables randomly and flip it?

Random Walk 2-SAT

- Answer:
  - We want the distance to decrease with probability at least $\frac{1}{2}$ no matter how close or far we are from τ^*
  - If we are already close, choosing a variable at random will likely choose one where τ and τ^* already match
  - Flipping this variable will increase the distance with high probability
  - An unsatisfied clause narrows it down to two variables s.t. τ and τ^* differ on at least one of them

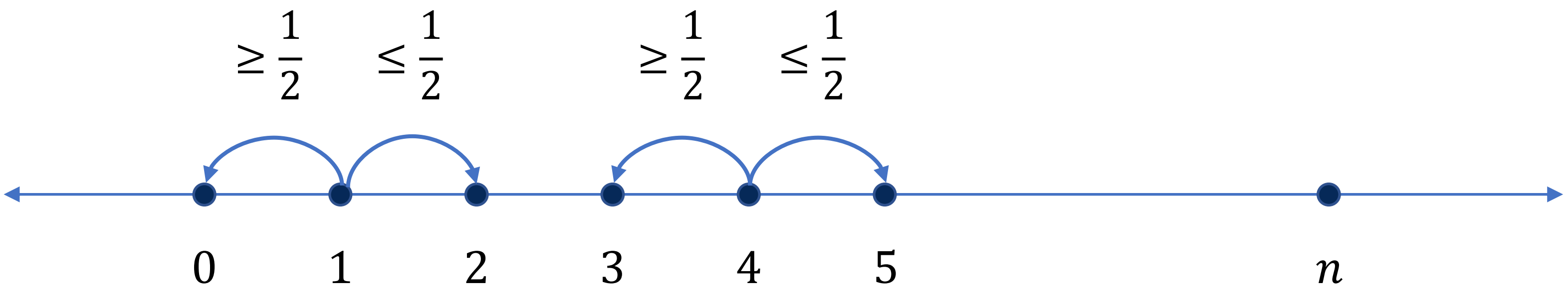
Random Walk + 2-SAT

- Observation: $d_{i+1} = d_i - 1$ or $d_{i+1} = d_i + 1$
- Claim: $\Pr[d_{i+1} = d_i - 1] \geq 1/2$



- How does this help?

Random Walk + 2-SAT



- How does this help?
  - Can view this as a "Markov chain" and use known results on "hitting time"
  - But let's prove it using elementary methods

Random Walk + 2-SAT

- For $k > \ell$, define:
  - $T_{k,\ell}$ = expected number of iterations it takes to hit distance ℓ for the first time when you start at distance k

- $T_{i+1,i} \leq \frac{1}{2} * 1 + \frac{1}{2} * \left(1 + T_{i+2,i}\right)$
  $= \frac{1}{2} * (1) + \frac{1}{2} * \left(1 + T_{i+2,i+1} + T_{i+1,i}\right)$

Which pillar did we use?

- Simplifying:
  - $T_{i+1,i} \leq 2 + T_{i+2,i+1} \leq 4 + T_{i+3,i+2} \leq \cdots \leq O(n) + T_{n,n-1} \leq O(n)$
    - Uses $T_{n,n-1} = 1$ (Why?)

- $T_{n,0} \leq T_{n,n-1} + \cdots + T_{1,0} = O(n^2)$

Random Walk + 2-SAT

- Can view this algorithm as a “drunken local search”
  - We are searching the local neighborhood
  - But we don’t ensure that we necessarily improve
  - We just ensure that in expectation, we aren’t hurt
  - Hope to reach a feasible solution in polynomial time
- Schöning extended this technique to k-SAT
  - Schöning’s algorithm no longer runs in polynomial time, but this is okay because k-SAT is NP-hard
  - It still improves upon the naïve 2^n
  - Later derandomized by Moser and Scheder [2011]

Schöning's Algorithm for k-SAT

- Algorithm:
  - Choose a random assignment τ
  - Repeat $3n$ times ($n =$ #variables)
    - If τ satisfies the CNF, stop
    - Else, pick an arbitrary unsatisfied clause and flip a random literal in the clause

Schöning's Algorithm

- Randomized algorithm with one-sided error
  - If the CNF is satisfiable, it finds an assignment with probability at least $\left(\frac{1}{2} \cdot \frac{k}{k-1}\right)^n$
  - If the CNF is unsatisfiable, it never finds an assignment

- Expected #times we need to repeat in order to find a satisfying assignment when one exists: $\left(2\left(1-\frac{1}{k}\right)\right)^n$
  - For $k = 3$, this gives $O(1.3333^n)$
  - For $k = 4$, this gives $O(1.5^n)$

Best Known Results

- 3-SAT

- Deterministic
  - Derandomized Schöning's algorithm: $O(1.3333^n)$
  - Best known: $O(1.3303^n)$ [HSSW]
    - If we are assured that there is a unique satisfying assignment: $O(1.3071^n)$ [PPSZ]

- Randomized
  - Nothing better known without one-sided error
  - With one-sided error, best known is $O(1.30704^n)$ [Modified PPSZ]

Random Walk + 2-SAT

- Random walks are not only of theoretical interest
  - WalkSAT is a practical SAT algorithm
  - At each iteration, pick an unsatisfied clause *at random*
  - Pick a variable in the unsatisfied clause to flip:
    - With some probability, pick at random.
    - With the remaining probability, pick one that will make the fewest previously satisfied clauses unsatisfied
  - Restart a few times (avoids being stuck in local minima)

- Faster than "intelligent local search" (GSAT)
  - Flip the variable that satisfies most clauses

Random Walks on Graphs

- Aleliunas et al. [1979]
  - Let G be a connected undirected graph. Then a random walk starting from any vertex will cover the entire graph (visit each vertex at least once) in $O(mn)$ steps.

- Limiting probability distribution
  - In the limit, the random walk will visit a vertex with degree d_i in $\frac{d_i}{2m}$ fraction of the steps

- Markov chains
  - Generalize to directed (possibly infinite) graphs with unequal edge traversal probabilities

Randomization for Sublinear Running Time

Sublinear Running Time

- Given an input of length n, we want an algorithm that runs in time $o(n)$
  - $o(n)$ examples: $\log n$, $\sqrt{n}$, $n^{0.999}$, $\frac{n}{\log n}$, …
  - The algorithm doesn't even get to read the full input!

- There are four possibilities:
  - Exact vs inexact: whether the algorithm always returns the correct/optimal solution or only does so with high probability (or gives some approximation)
  - Worst-case versus expected running time: whether the algorithm always takes $o(n)$ time or only does so in expectation (but still on every instance)

Exact algorithms, expected sublinear time

Searching in Sorted List

- Input: A sorted doubly linked list with n elements.
  - Imagine you have an array A with $O(1)$ access to $A[i]$
  - $A[i]$ is a tuple (x_i, p_i, n_i)
    - Value, index of previous element, index of next element.
  - Sorted: $x_{p_i} \leq x_i \leq x_{n_i}$

- Task: Given x, check if there exists i s.t. $x = x_i$

- Goal: We will give a randomized + exact algorithm with expected running time $O(\sqrt{n})$!

Searching in Sorted List

- Motivation:
  - Often we deal with large datasets that are stored in a large file on disk, or possibly broken into multiple files
  - Creating a new, sorted version of the dataset is expensive
  - It is often preferred to "implicitly sort" the data by simply adding previous-next pointers along with each element

  - Would like algorithms that can operate on such implicitly sorted versions and yet achieve sublinear running time
    - Just like binary search achieves for an explicitly sorted array

Searching in Sorted List

Algorithm:

- Select $\sqrt{n}$ random indices R
- Access x_j for each $j \in R$
- Find "accessed x_j nearest to x in either direction"
  - either the largest among all $x_j \leq x$…
  - or the smallest among all $x_j \geq x$
- If you take the largest $x_j \leq x$, start from there and keep going "next" until you find x or go past its value
- If you take the smallest $x_j \geq x$, start from there and keep going "previous" until you find x or go past its value

Searching in Sorted List

- Analysis sketch:
  - Suppose you find the largest $x_j \leq x$ and keep going "next"
  - Let x_i be smallest value $\geq x$
  - Algorithm stops when it hits x_i
  - Algorithm throws $\sqrt{n}$ random "darts" on the sorted list
  - Chernoff bound:
    - Expected distance of x_i to the closest dart to its left is $O(\sqrt{n})$
    - We'll assume this without proof!
  - Hence, the algorithm only does "next" $O(\sqrt{n})$ times in expectation

Searching in Sorted List

- Note:
  - We don't *really* require the list to be doubly linked. Just "next" pointer suffices if we have a pointer to the first element of the list (a.k.a. "anchored list").

- This algorithm is optimal!
- Theorem: No algorithm that always returns the correct answer can run in $o(\sqrt{n})$ expected time.
  - Can be proved using "Yao's minimax principle"
  - Beyond the scope of the course, but this is a fundamental result with wide-ranging applications

Sublinear Geometric Algorithms

- Chazelle, Liu, and Magen [2003] proved the $\Theta(\sqrt{n})$ bound for searching in a sorted linked list
  - Their main focus was to generalize these ideas to come up with sublinear algorithms for geometric problems
  - Polygon intersection: Given two convex polyhedra, check if they intersect.
  - Point location: Given a Delaunay triangulation (or Voronoi diagram) and a point, find the cell in which the point lies.
  - They provided optimal $O(\sqrt{n})$ algorithms for both these problems.

Inexact algorithms, expected sublinear time

Estimating Avg Degree in Graph

- Input:
  - Undirected graph G with n vertices
  - $O(1)$ access to the degree of any queried vertex
- Output:
  - Estimate the average degree of all vertices
  - More precisely, we want to find a $(2+\epsilon)$-approximation in expected time $O\left(\epsilon^{-O(1)}\sqrt{n}\right)$
- Wait!
  - Isn't this equivalent to "given an array of n numbers between 1 and $n-1$, estimate their average"?
  - No! That requires $\Omega(n)$ time for any constant approximation!
    - Consider an instance with constantly many $n-1$'s, and all other 1's: you may not discover any $n-1$ until you query $\Omega(n)$ numbers

Estimating Avg Degree in Graph

- Why are degree sequences more special?

- Erdős–Gallai theorem:
    - $d_1 \geq \cdots \geq d_n$ is a degree sequence iff their sum is even and $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} d_i$

- Intuitively, we will sample $O(\sqrt{n})$ vertices
    - We may not discover the few high degree vertices but we'll find their neighbors and thus account for their edges anyway!

Estimating Avg Degree in Graph

- Algorithm:
  - Take ${}^{8}/_{\epsilon}$ random subsets $S_i \subseteq V$ with $|S_i| = O\left(\frac{\sqrt{n}}{\epsilon}\right)$
  - Compute the average degree d_{S_i} in each S_i.
  - Return $\widehat{d} = \min_i d_{S_i}$

- Analysis beyond the scope of this course
  - This gets the approximation right with probability at least $\frac{5}{6}$
  - By repeating the experiment $\Omega(\log n)$ times and reporting the median answer, we can get the approximation right with probability at least $1 - 1/O(n)$ and a bad approximation with the other $1/O(n)$ probability cannot hurt much