CSC373

Algorithm Design, Analysis & Complexity

Nisarg Shah

Introduction

Totally useless this semester!

- Instructors
  - Nisarg Shah
    - cs.toronto.edu/~nisarg, nisarg@cs, SF 2301C
    - LEC 0101 and 0102
- TAs: Too many to list
- Disclaimer!
  - First online version of the course, so expect a bumpy ride at the start, but hopefully, we'll get through together
  - Use any of the feedback mediums (email, Piazza, …) to let me know if you have any suggestions for improvement

Course Information

- Course Page www.cs.toronto.edu/~nisarg/teaching/373f20/
  - All the information below is in the course information sheet, available on Piazza
- Discussion Board piazza.com/utoronto.ca/fall2020/csc373
- Grading – MarkUs
  - Link will be distributed after about a week or two
  - LaTeX preferred, scans are OK!
- All times in Eastern time zone, all zoom links on the course page

Lectures

- Time & Place: Tue 4-5pm, Thu 1-3pm, Zoom
- Details
  - Delivered live
  - 10 minute break after every 50 minutes of lecture
  - Students can ask questions using Zoom's chat feature
  - One TA will be present to continuously answer questions
  - I might also answer questions once in a while

Tutorials

- Time & Place: Tue 5-6pm, Zoom
- Details
  - Delivered live by TAs
  - Problem sets will be posted early on the course webpage
    - Easier problems that are warm-up to assignments/exams
  - Please try them before coming to the tutorials
  - TAs will explain the problems, allow you to discuss them in breakout rooms, and then go over key parts of the solutions
  - Solutions will be posted later on the course webpage

Tutorials

- Further details
  - Each section is divided into three parts (A,B,C)
  - Students divided by birth month: A = Jan-Apr, B = May-Aug, C = Sep-Dec
  - Feel free to attend a different tutorial than the one you’re assigned
    - EXCEPT when the tutorial slot is being used for a test
  - If the attendance is low, the number of tutorials per section may be reduced

Office Hours

- Time & Place: Wed 4-5pm, Fri 10-11am, Zoom
  - Do you have conflicts with these slots? Poll!
- Details
  - I will conduct them
  - Use the “raise hand” feature
  - I will call upon the raised hands in order
  - When called upon, unmute and ask the question
  - Always phrase your question in a way that doesn’t give away your solutions or approach to an assignment problem
    - Just like in a physical office

Tests

- 2 term tests, one end-of-term test (final exam)
- Time & Place: Tue 5-6pm (tutorial slot)
  - Need to be able to attend live!
  - I’m considering using part of the Tue 4-5pm lecture slot to give you more time
- Tentative Plan
  - Open book, closed internet
  - You may be asked to join a zoom link and keep your video on
  - If you have a question, you can “raise hand”, and I or a TA can take you to a breakout room to answer your question
  - Upload scanned answer sheet at the end (we’ll do a mock run of this)

Assignments

- 4 assignments, best 3 out of 4
- Group work
  - In groups of up to three students
  - Best way to learn is for each member to try each problem
- Questions will be more difficult
  - May need to mull them over for several days; do *not* expect to start and finish the assignment on the same day!
  - May include bonus questions
- Submission on MarkUs, more details later
  - May need to compress the PDF

Grading Policy

• 3 homeworks	*	10%	=	30%
• 2 term tests	*	20%	=	40%
• Final exam	*	30%	=	30%

- NOTE: To pass, you must earn at least 40% on the final exam

Approximate Due Dates

- Please note the word approximate!
  - Assignment 1: Apx. Oct 9
  - Assignment 2: Apx. Oct 30
  - Assignment 3: Apx. Nov 13
  - Assignment 4: Apx. Nov 27
  - Midterm 1: Apx. Oct 20
  - Midterm 2: Apx. Nov 17
- Conflicts
  - The tests are during the tutorial slot, so there should ideally be no conflict
  - That said, if you think you'll have a conflict, let me know at the earliest

Textbook

- Primary reference: lecture slides

- Primary textbook (required)
  - [CLRS] Cormen, Leiserson, Rivest, Stein: *Introduction to Algorithms*.

- Supplementary textbooks (optional)
  - [DPV] Dasgupta, Papadimitriou, Vazirani: *Algorithms*.
  - [KT] Kleinberg; Tardos: *Algorithm Design*.

Other Policies

- Collaboration
  - Free to discuss with classmates or read online material
  - Must write solutions in your own words
    - Easier if you do not take any pictures/notes from discussions
- Citation
  - For each question, must cite the peer (write the name) or the online sources (provide links), if you obtained a significant insight directly pertinent to the question
  - Failing to do this is plagiarism!

Other Policies

- "No Garbage" Policy
  - Borrowed from: Prof. Allan Borodin (citation!)

1. Partial marks for viable approaches
2. Zero marks if the answer makes no sense
3. 20% marks if you admit to not knowing how to approach the question ("I do not know how to approach this question")

- 20% > 0% !!

Other Policies

- Late Days
  - 4 total late days across all 4 assignments
  - Managed by MarkUs
  - At most 2 late days can be applied to a single assignment
  - Already covers legitimate reasons such as illness, university activities, etc.
    - Petitions will only be granted for circumstances which cannot be covered by this

Zoom Features

- Just to get acquainted, let’s try out the following features:
  - Polls (already tried)
  - Chat
  - Reactions
  - Raise hand
  - Yes/No
  - Breakout rooms

Enough with the boring stuff.

What will we study?

Why will we study it?

Muhammad ibn Musa al-Khwarizmi
c. 780 – c. 850

What is this course about?

- Algorithms
  - Ubiquitous in the real world
    - From your smartphone to self-driving cars
    - From graph problems to graphics problems
    - …
  - Important to be able to design and analyze algorithms
  - For some problems, good algorithms are hard to find
    - For some of these problems, we can formally establish complexity results
    - We'll often find that one problem is easy, but its minor variants are suddenly hard

What is this course about?

- Algorithms
  - Algorithms in specialized environments or using advanced techniques
    - Distributed, parallel, streaming, sublinear time, spectral, genetic…
  - Other concerns with algorithms
    - Fairness, ethics, …
  - …mostly beyond the scope of this course

What is this course about?

- Topics in this course
  - Divide and Conquer
  - Greedy
  - Dynamic programming
  - Network flow
  - Linear programming
  - NP-completeness (not really an algorithm design paradigm)
  - Approximation algorithms (if time permits)
  - Randomized algorithms (if time permits)

What is this course about?

- How do we know which paradigm is right for a given problem?
  - A very interesting question!
  - Subject of much ongoing research…
    - Sometimes, you just know it when you see it…
- How do we analyze an algorithm?
  - Proof of correctness
  - Proof of running time
    - We'll try to prove the algorithm is *efficient* in the *worst case*
    - In practice, average case matters just as much (or even more)

What is this course about?

- What does it mean for an algorithm to be efficient in the worst case?
  - Polynomial time
  - It should use at most poly(n) steps on *any* n-bit input
    - $n, n^2, n^{100}, 100n^6 + 237n^2 + 432, \ldots$
  - If the input to an algorithm is a number x, the number of bits of input is $\log x$
    - This is because it takes $\log x$ bits to represent the input x in binary
    - So the running time should be polynomial in $\log x$, not in x
  - How much is too much?

What is this course about?

Picture-Hanging Puzzles*

Erik D. Demaine[†] Martin L. Demaine[†] Yair N. Minsky[‡] Joseph S. B. Mitchell[§]
Ronald L. Rivest[†] Mihai Pătraşcu[¶]

Theorem 7 *For any $n \geq k \geq 1$, there is a picture hanging on n nails, of length $n^{c'}$ for a constant c', that falls upon the removal of any k of the nails.*

$n^{6,100 \log_2 c}$. Using the $c \leq 1{,}078$ upper bound, we obtain an upper bound of $c' \leq 6{,}575{,}800$. Using

So, while this construction is polynomial, it is a rather large polynomial. For small values of n, we can use known small sorting networks to obtain somewhat reasonable constructions.

What is this course about?

Better Balance by Being Biased:
A 0.8776-Approximation for Max Bisection

Per Austrin*, Siavosh Benabbas*, and Konstantinos Georgiou†

has a lot of flexibility, indicating that further improvements may be possible. We remark that, while polynomial, the running time of the algorithm is somewhat abysmal; loose estimates places it somewhere around $O\left(n^{10^{100}}\right)$; the running time of the algorithm of [RT12] is similar.

What is this course about?

- What if we can't find an efficient algorithm for a problem?
  - Try to prove that the problem is hard
  - Formally establish complexity results
  - NP-completeness, NP-hardness, …

- We'll often find that one problem may be easy, but its simple variants may suddenly become hard
  - Minimum spanning tree (MST) vs bounded degree MST
  - 2-colorability vs 3-colorability

I’m not convinced.

Will I really ever need to know how to design abstract algorithms?

At the very least…

This will help you prepare for your technical job interview!

Real Microsoft interview question:

- Given an array a, find indices (i, j) with the largest $j - i$ such that $a[j] > a[i]$
- Greedy? Divide & conquer?

Disclaimer

- The course is theoretical in nature
  - You'll be working with abstract notations, proving correctness of algorithms, analyzing the running time of algorithms, designing new algorithms, and proving complexity results.
- Something for everyone…
  - If you're somewhat scared going into the course
  - If you're already comfortable with the proofs, and want challenging problems

Related/Follow-up Courses

- Direct follow-up
  - CSC473: Advanced Algorithms
  - CSC438: Computability and Logic
  - CSC463: Computational Complexity and Computability

- Algorithms in other contexts
  - CSC304: Algorithmic Game Theory and Mechanism Design (self promotion!)
  - CSC384: Introduction to Artificial Intelligence
  - CSC436: Numerical Algorithms
  - CSC418: Computer Graphics

Divide & Conquer

History?

- Maybe you saw a subset of these algorithms?
  - Mergesort - $O(n \log n)$
  - Karatsuba algorithm for fast multiplication - $O\left(n^{\log_2 3}\right)$ rather than $O(n^2)$
  - Largest subsequence sum in $O(n)$
  - …
- Have you seen some divide & conquer algorithms before?
  - Maybe in CSC236/CSC240 and/or CSC263/CSC265
  - Write “yes”/”no” in chat

Divide & Conquer

- General framework
  - Break (a large chunk of) a problem into two smaller subproblems of the same type
  - Solve each subproblem recursively and independently
  - At the end, quickly combine solutions from the two subproblems and/or solve any remaining part of the original problem

- Hard to formally define when a given algorithm is divide-and-conquer…
- Let’s see some examples!

Master Theorem

- Here's the master theorem, as it appears in CLRS
  - ➢ Useful for analyzing divide-and-conquer running time
  - ➢ If you haven't already seen it, please spend some time understanding it

Theorem 4.1 (Master theorem)

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence

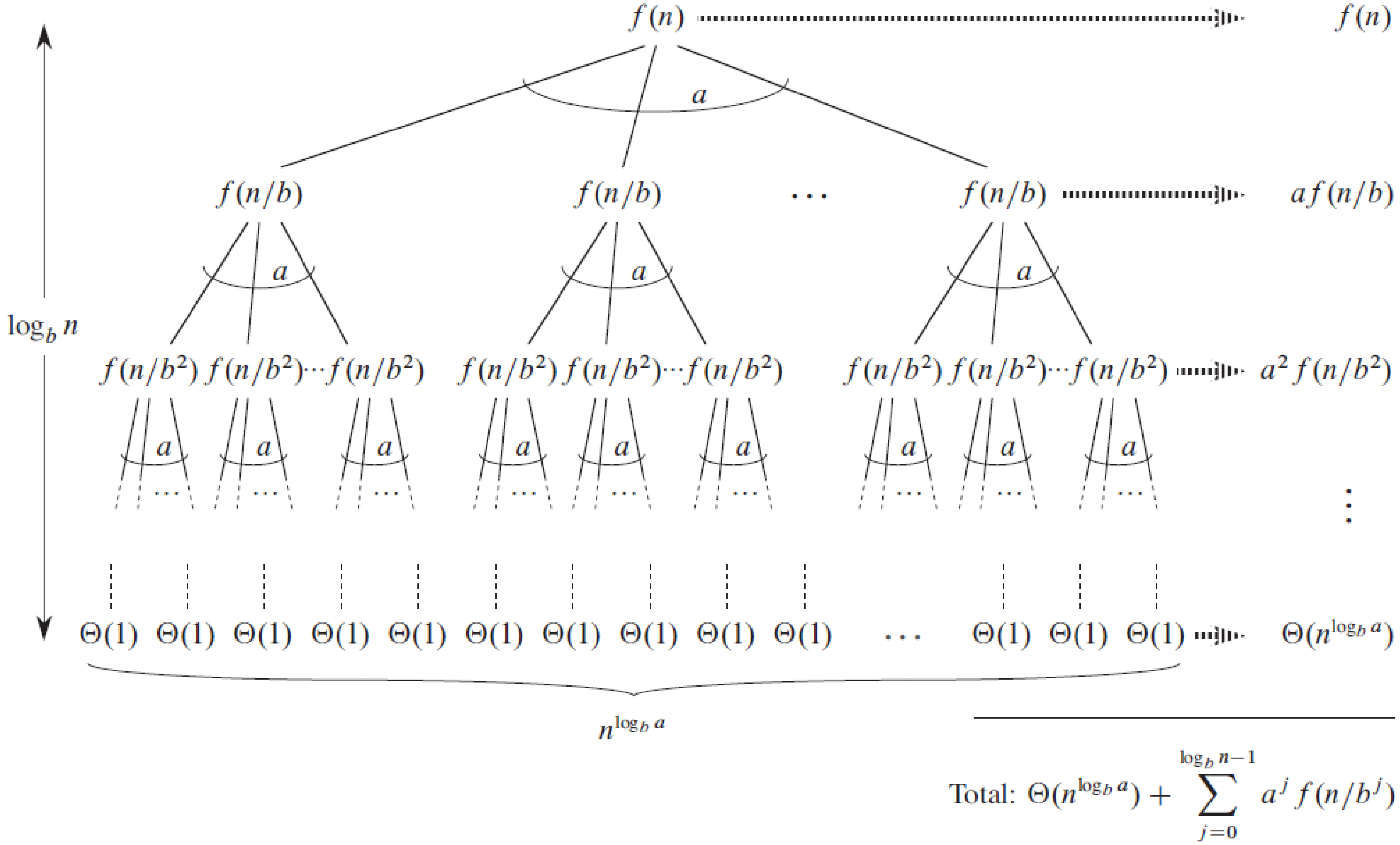
$$T(n) = aT(n/b) + f(n) ,$$

where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ has the following asymptotic bounds:

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$. ∎

Master Theorem

Intuition: Compare *f(n)* with $n^{\log_b a}$. The larger determines the recurrence solution.



Counting Inversions

- Problem
  - Given an array a of length n, count the number of pairs (i, j) such that $i < j$ but $a[i] > a[j]$
- Applications
  - Voting theory
  - Collaborative filtering
  - Measuring the "sortedness" of an array
  - Sensitivity analysis of Google's ranking function
  - Rank aggregation for meta-searching on the Web
  - Nonparametric statistics (e.g., Kendall's tau distance)

Counting Inversions

- Problem
  - Count (i, j) such that $i < j$ but $a[i] > a[j]$
- Brute force
  - Check all $\Theta(n^2)$ pairs
- Divide & conquer
  - Divide: break array into two equal halves x and y
  - Conquer: count inversions in each half recursively
  - Combine:
    - Solve (we'll see how): count inversions with one entry in x and one in y
    - Merge: add all three counts

Counting Inversions

- From Kevin Wayne's slides

SORT-AND-COUNT (L)

IF list L has one element

    RETURN $(0, L)$.

DIVIDE the list into two halves A and B.

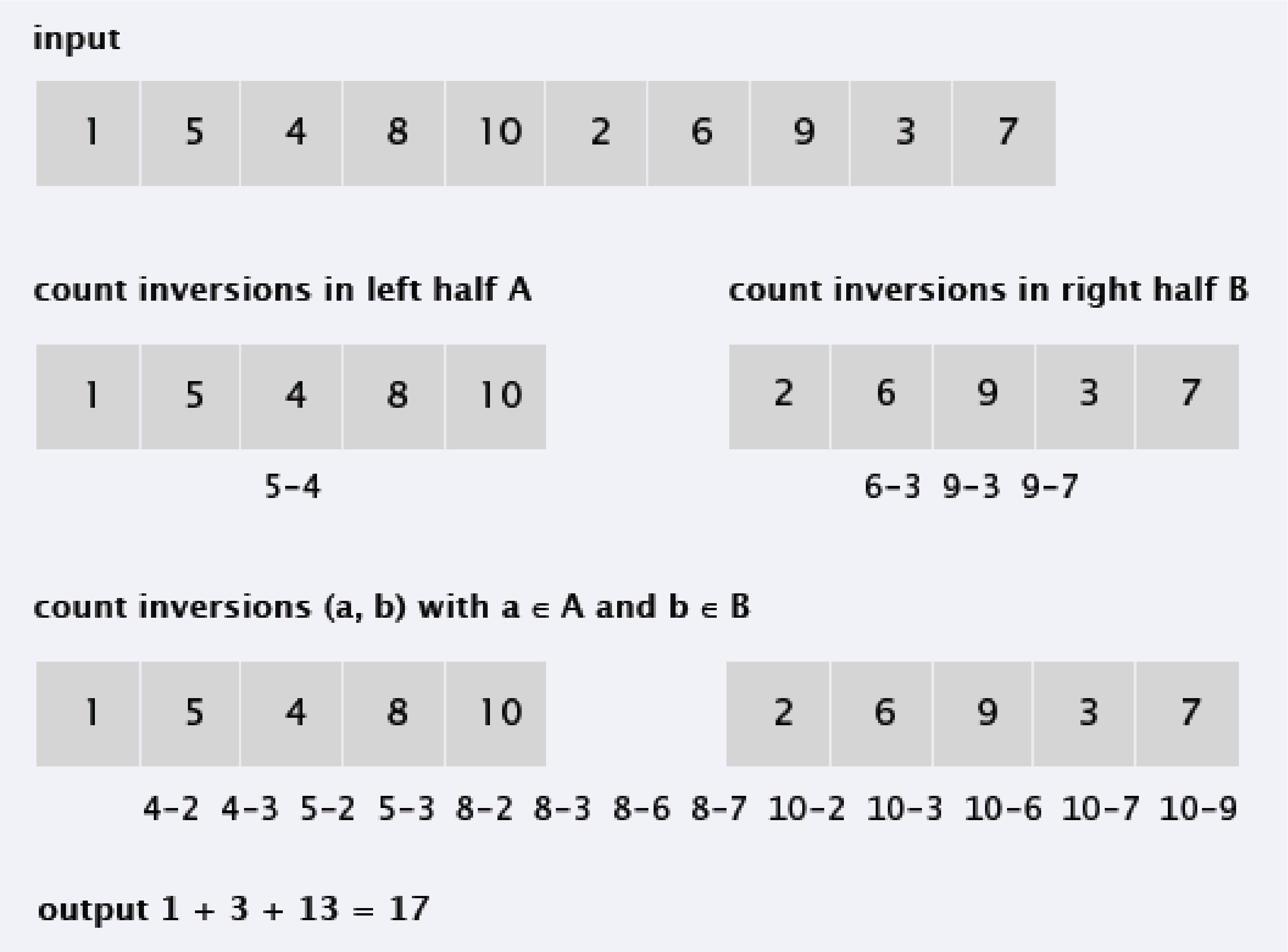
$(r_A, A) \leftarrow$ SORT-AND-COUNT(A).

$(r_B, B) \leftarrow$ SORT-AND-COUNT(B).

$(r_{AB}, L') \leftarrow$ MERGE-AND-COUNT(A, B).

RETURN $(r_A + r_B + r_{AB}, L')$.

Counting Inversions



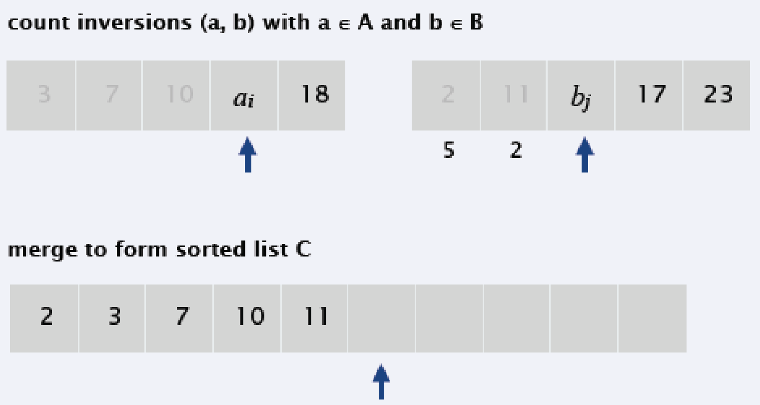
Counting Inversions

Q. How to count inversions (a, b) with $a \in A$ and $b \in B$?

A. Easy if A and B are sorted!

Count inversions (a, b) with $a \in A$ and $b \in B$, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
- If $a_i < b_j$, then a_i is not inverted with any element left in B.
- If $a_i > b_j$, then b_j is inverted with every element left in A.
- Append smaller element to sorted list C.



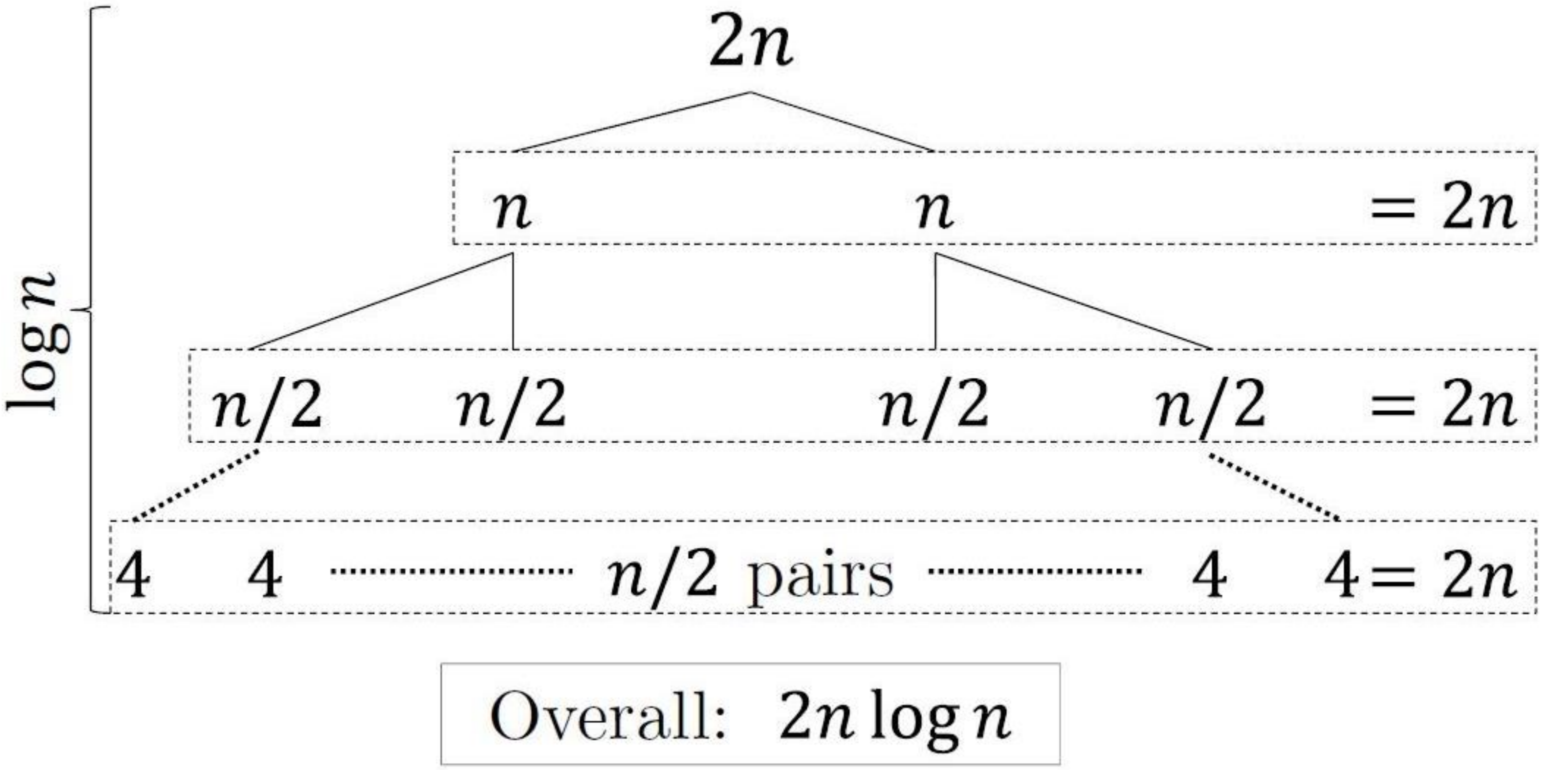
Counting Inversions

- How do we formally prove correctness?
  - Induction on n is usually very helpful
  - Allows you to assume correctness of subproblems

- Running time analysis
  - Suppose $T(n)$ is the running time for inputs of size n
  - Our algorithm satisfies $T(n) = 2\,T(n/2) + O(n)$
  - Master theorem says this is $T(n) = O(n \log n)$

Without Master Theorem

Let's say $T(n) = 2\,T(n/2) + 2n$



Closest Pair in $\mathbb{R}^2$

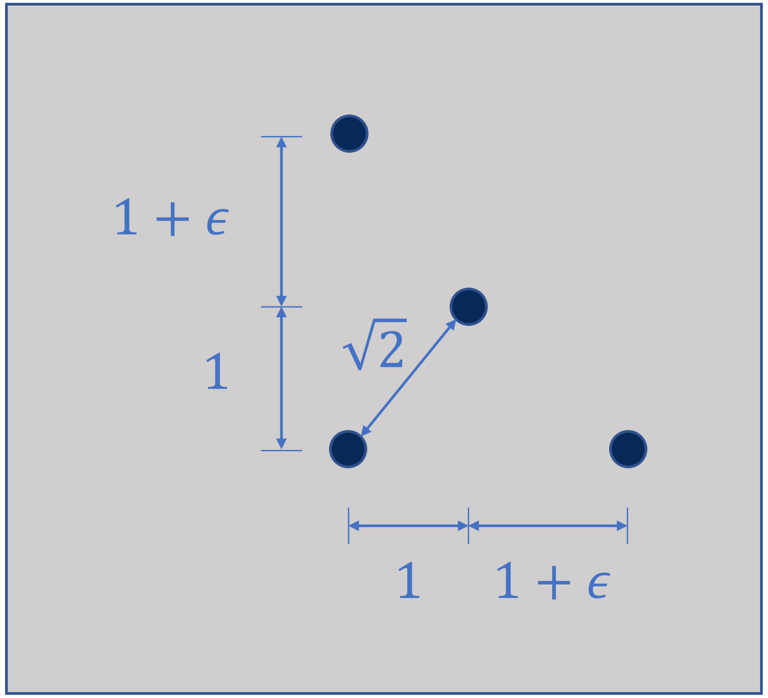
- Problem:
  - Given n points of the form (x_i, y_i) in the plane, find the closest pair of points.
- Applications:
  - Basic primitive in graphics and computer vision
  - Geographic information systems, molecular modeling, air traffic control
  - Special case of nearest neighbor
- Brute force: $\Theta(n^2)$

Intuition from 1D?

- In 1D, the problem would be easily $O(n \log n)$
  - Sort and check!
- Sorting attempt in 2D
  - Find closest points by x coordinate
  - Find closest points by y coordinate
- Non-degeneracy assumption
  - No two points have the same x or y coordinate

Intuition from 1D?

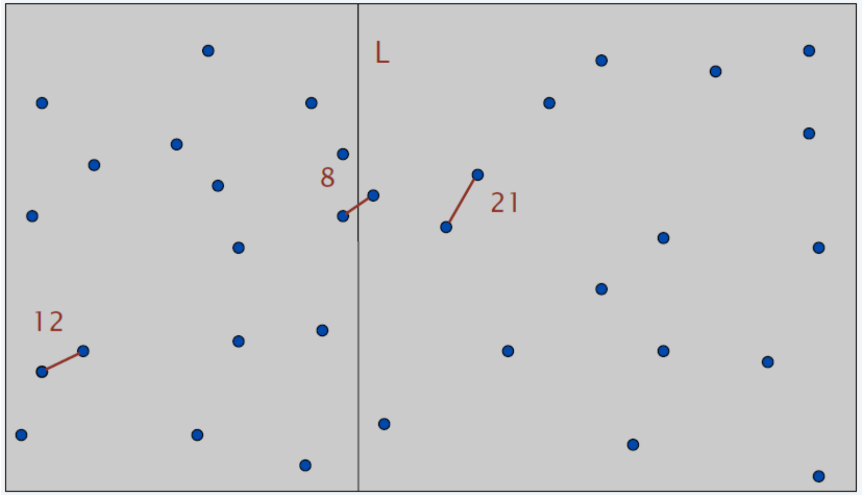
- Sorting attempt in 2D
  - Find closest points by x or y coordinate
  - Doesn't work!



Closest Pair in $\mathbb{R}^2$

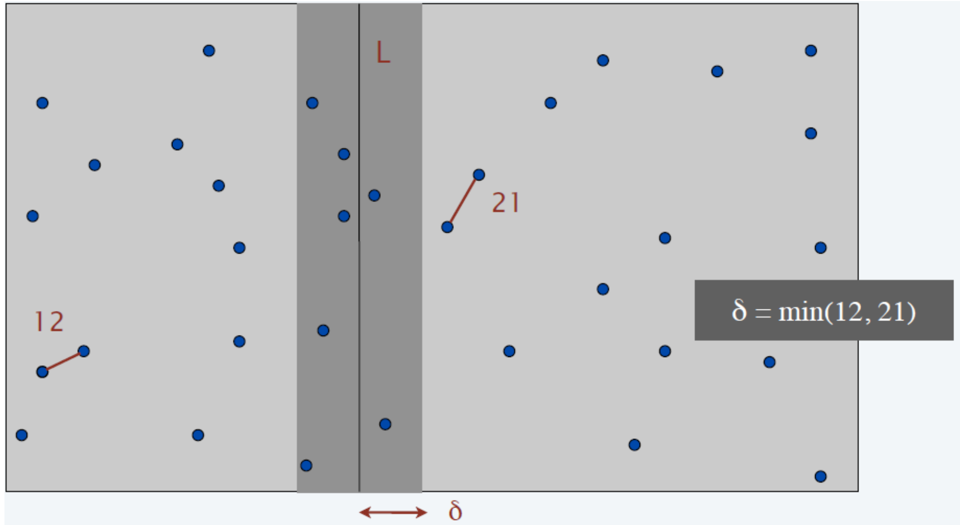
- Let's try divide-and-conquer!
  - Divide: points in equal halves by drawing a vertical line L
  - Conquer: solve each half recursively
  - Combine: find closest pair with one point on each side of L
  - Return the best of 3 solutions

Seems like $\Omega(n^2)$ ☹



Closest Pair in $\mathbb{R}^2$

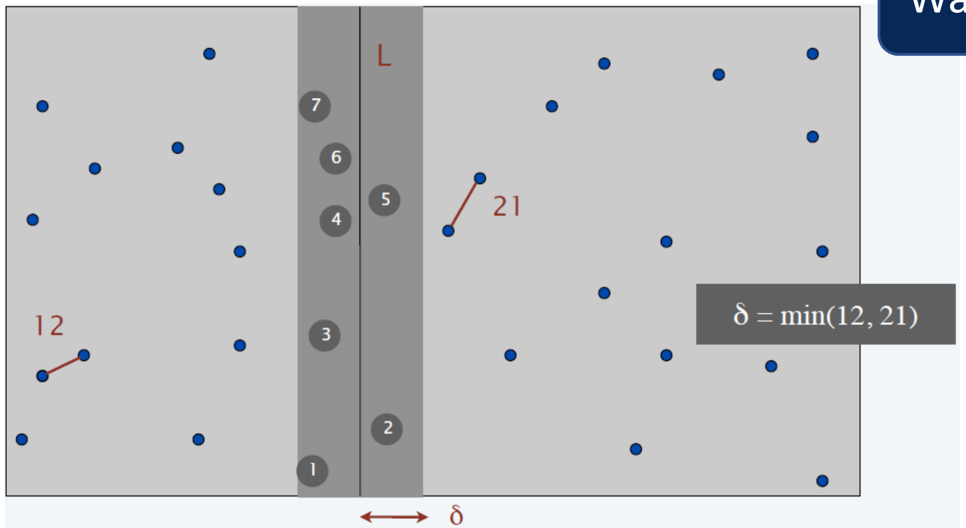
- Combine
  - We can restrict our attention to points within δ of L on each side, where δ = best of the solutions in two halves



Closest Pair in $\mathbb{R}^2$

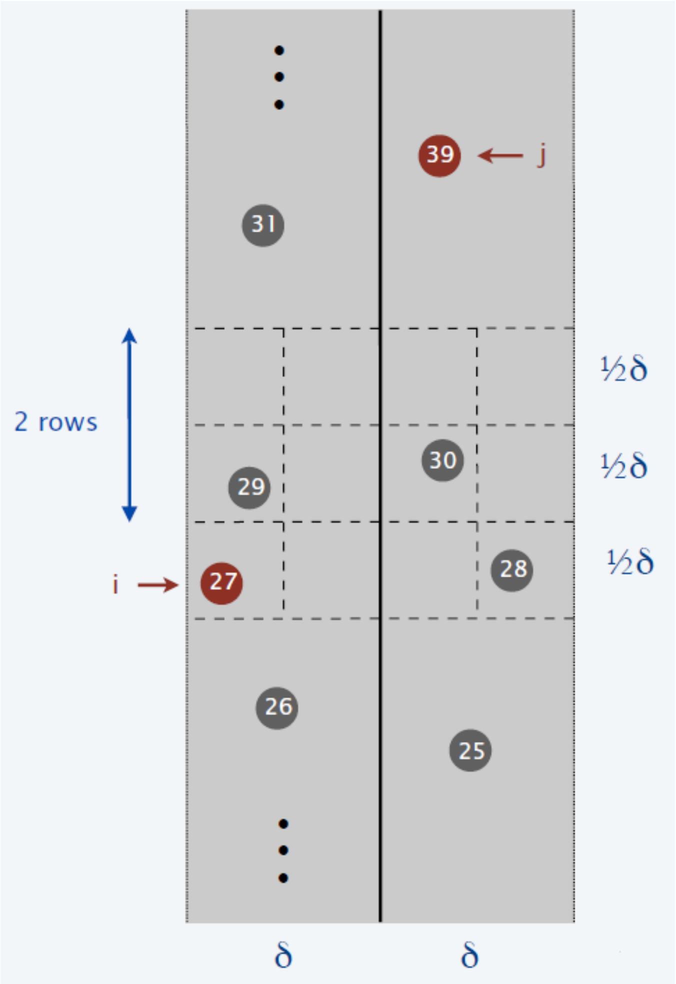
- Combine (let δ = best of solutions in two halves)
  - Only need to look at points within δ of L on each side,
  - Sort points on the strip by y coordinate
  - Only need to check each point with next 11 points in sorted list!

Wait, what? Why 11?



Why 11?

- Claim:
  - If two points are at least 12 positions apart in the sorted list, their distance is at least δ
- Proof:
  - No two points lie in the same $\delta/2 \times \delta/2$ box
  - Two points that are more than two rows apart are at distance at least δ



Recap: Karatsuba's Algorithm

- Fast way to multiply two n digit integers x and y
- Brute force: $O(n^2)$ operations
- Karatsuba's observation:
  - ➢ Divide each integer into two parts
    - $x = x_1 * 10^{n/2} + x_2, y = y_1 * 10^{n/2} + y_2$
    - $xy = (x_1 y_1) * 10^n + (x_1 y_2 + x_2 y_1) * 10^{n/2} + (x_2 y_2)$
  - ➢ Four $n/2$-digit multiplications can be replaced by three
    - $x_1 y_2 + x_2 y_1 = (x_1 + x_2)(y_1 + y_2) - x_1 y_1 - x_2 y_2$
  - ➢ Running time
    - $T(n) = 3\,T(n/2) + O(n) \Rightarrow T(n) = O\left(n^{\log_2 3}\right)$

								1	1	0	1	0	1	0	1
							×	0	1	1	1	1	1	0	1
								1	1	0	1	0	1	0	1
							0	0	0	0	0	0	0	0	
						1	1	0	1	0	1	0	1		
					1	1	0	1	0	1	0	1			
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		1	1	0	1	0	1	0	1						
	0	0	0	0	0	0	0	0							
0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	1

Strassen's Algorithm

- Generalizes Karatsuba's insight to design a fast algorithm for multiplying two $n \times n$ matrices
  - ➢ Call n the "size" of the problem

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} * \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

  - ➢ Naively, this requires 8 multiplications of size $n/2$
    - $A_{11} * B_{11}, A_{12} * B_{21}, A_{11} * B_{12}, A_{12} * B_{22}, \dots$
  - ➢ Strassen's insight: replace 8 multiplications by 7
    - Running time: $T(n) = 7\,T(n/2) + O(n^2) \Rightarrow T(n) = O\left(n^{\log_2 7}\right)$

Strassen's Algorithm

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} * \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

STRASSEN(n, A, B)

IF $(n = 1)$ RETURN $A \times B$.

assume n is a power of 2

Partition A and B into 2-by-2 block matrices.

keep track of indices of submatrices (don't copy matrix entries)

$P_1 \leftarrow$ STRASSEN$(n / 2, A_{11}, (B_{12} - B_{22}))$.

$P_2 \leftarrow$ STRASSEN$(n / 2, (A_{11} + A_{12}), B_{22})$.

$P_3 \leftarrow$ STRASSEN$(n / 2, (A_{21} + A_{22}), B_{11})$.

$P_4 \leftarrow$ STRASSEN$(n / 2, A_{22}, (B_{21} - B_{11}))$.

$P_5 \leftarrow$ STRASSEN$(n / 2, (A_{11} + A_{22}) \times (B_{11} + B_{22}))$.

$P_6 \leftarrow$ STRASSEN$(n / 2, (A_{12} - A_{22}) \times (B_{21} + B_{22}))$.

$P_7 \leftarrow$ STRASSEN$(n / 2, (A_{11} - A_{21}) \times (B_{11} + B_{12}))$.

$C_{11} = P_5 + P_4 - P_2 + P_6$.

$C_{12} = P_1 + P_2$.

$C_{21} = P_3 + P_4$.

$C_{22} = P_1 + P_5 - P_3 - P_7$.

RETURN C.

Median & Selection

- Selection:
  - Given array A of n comparable elements, find kth smallest
  - $k = 1$ is min, $k = n$ is max, $k = \lfloor (n+1)/2 \rfloor$ is median
  - $O(n)$ is easy for min/max
- What about k-selection?
  - $O(nk)$ by modifying bubble sort
  - $O(n \log n)$ by sorting
  - $O(n + k \log n)$ using min-heap
  - $O(k + n \log k)$ using max-heap
- Q: What about just $O(n)$?
- A: Yes! Selection is easier than sorting.

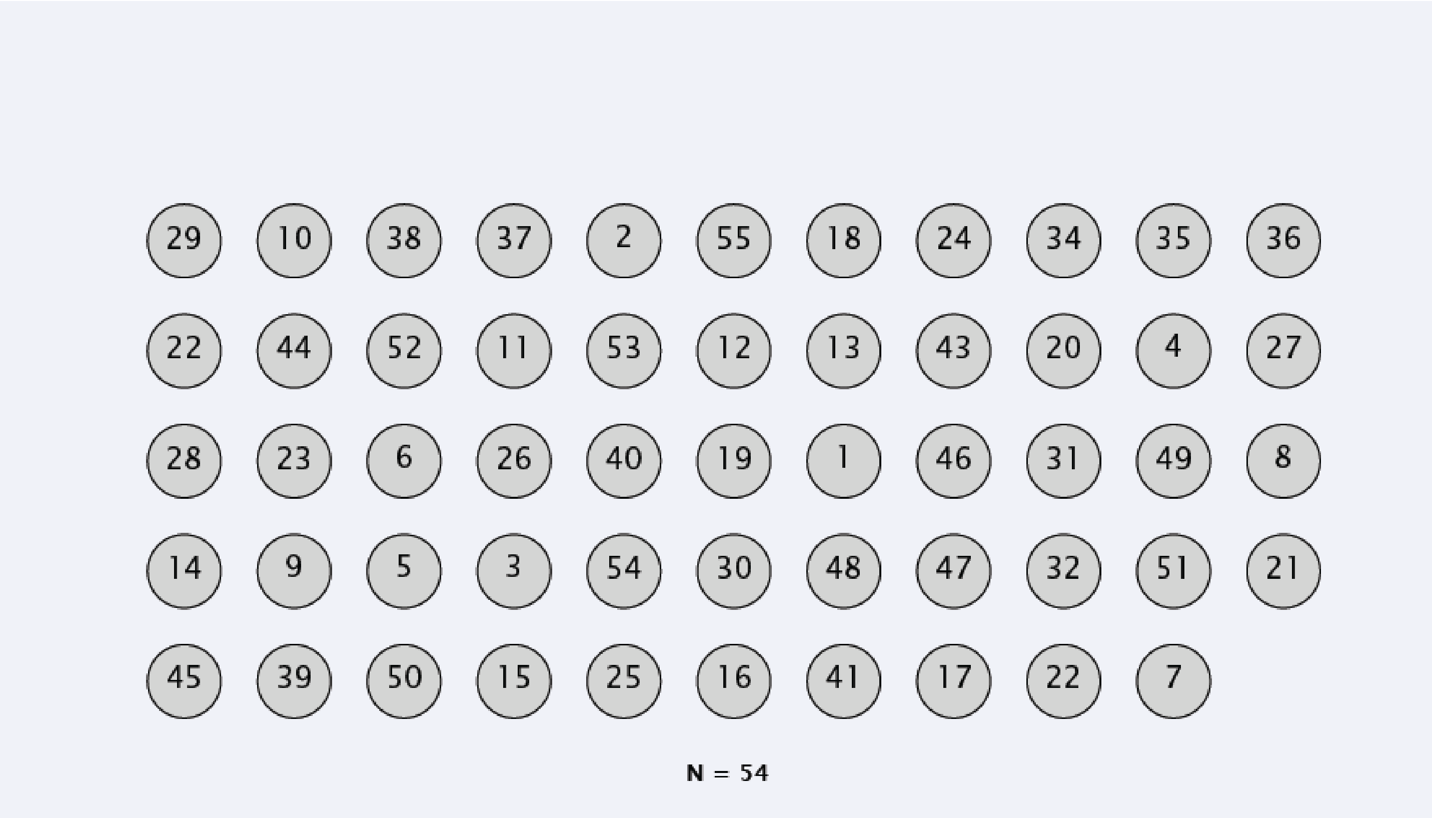
QuickSelect

- Find a pivot p
- Divide A into two sub-arrays
  - A_{less} = elements $\leq p$, A_{more} = elements $> p$
  - If $|A_{less}| \geq k$, return kth smallest in A_{less}, otherwise return $(k - |A_{less}|)$th smallest in A_{more}

- Problem?
  - If pivot is close to the min or the max, then we basically get $T(n) \leq T(n-1) + O(n)$, which only gives $T(n) = O(n^2)$
  - Want to reduce $n - 1$ to a fraction of n (like $n/2$, $5n/6$, etc)

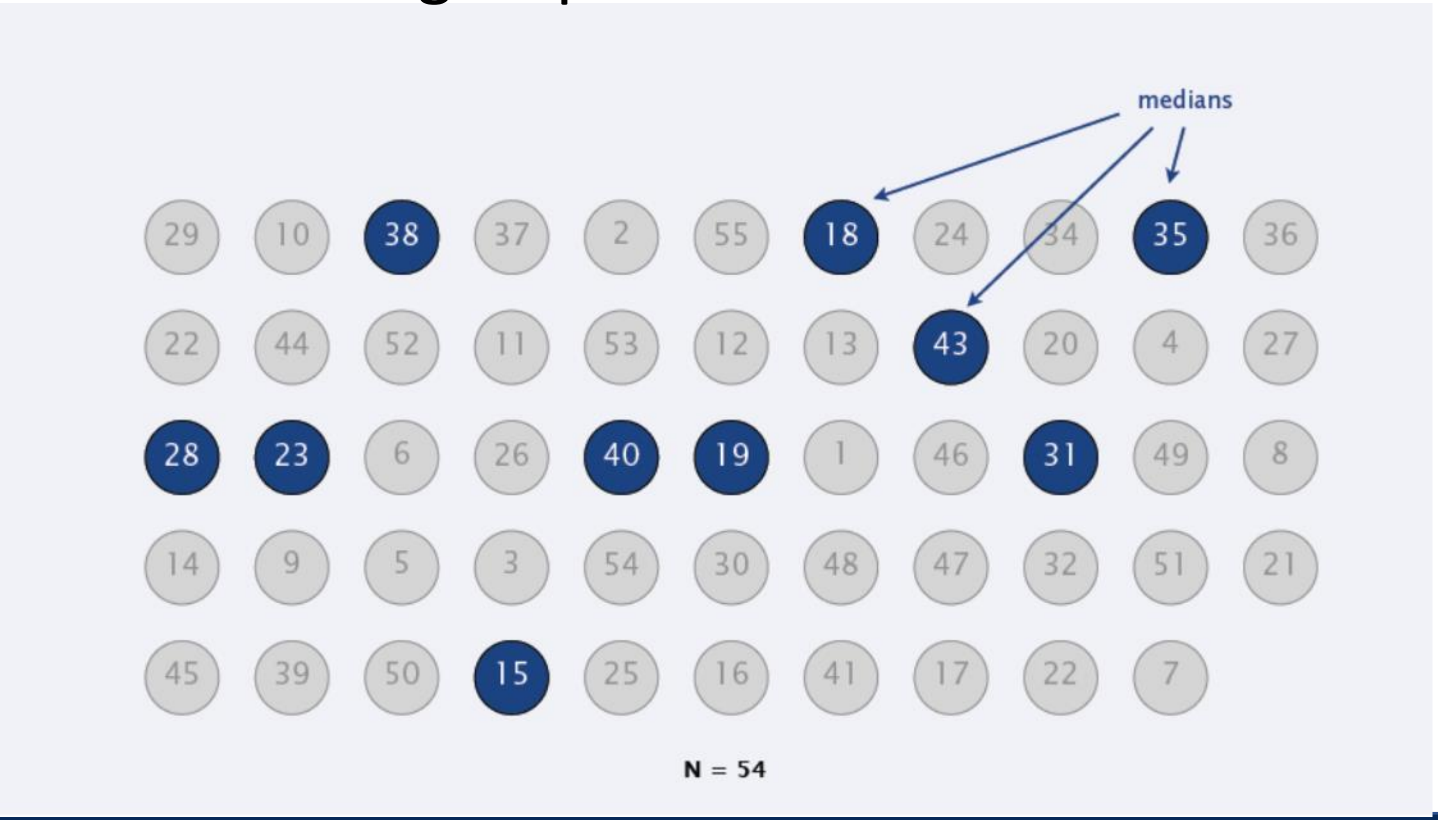
Finding a Good Pivot

- Divide n elements into $^n/_5$ groups of 5 each



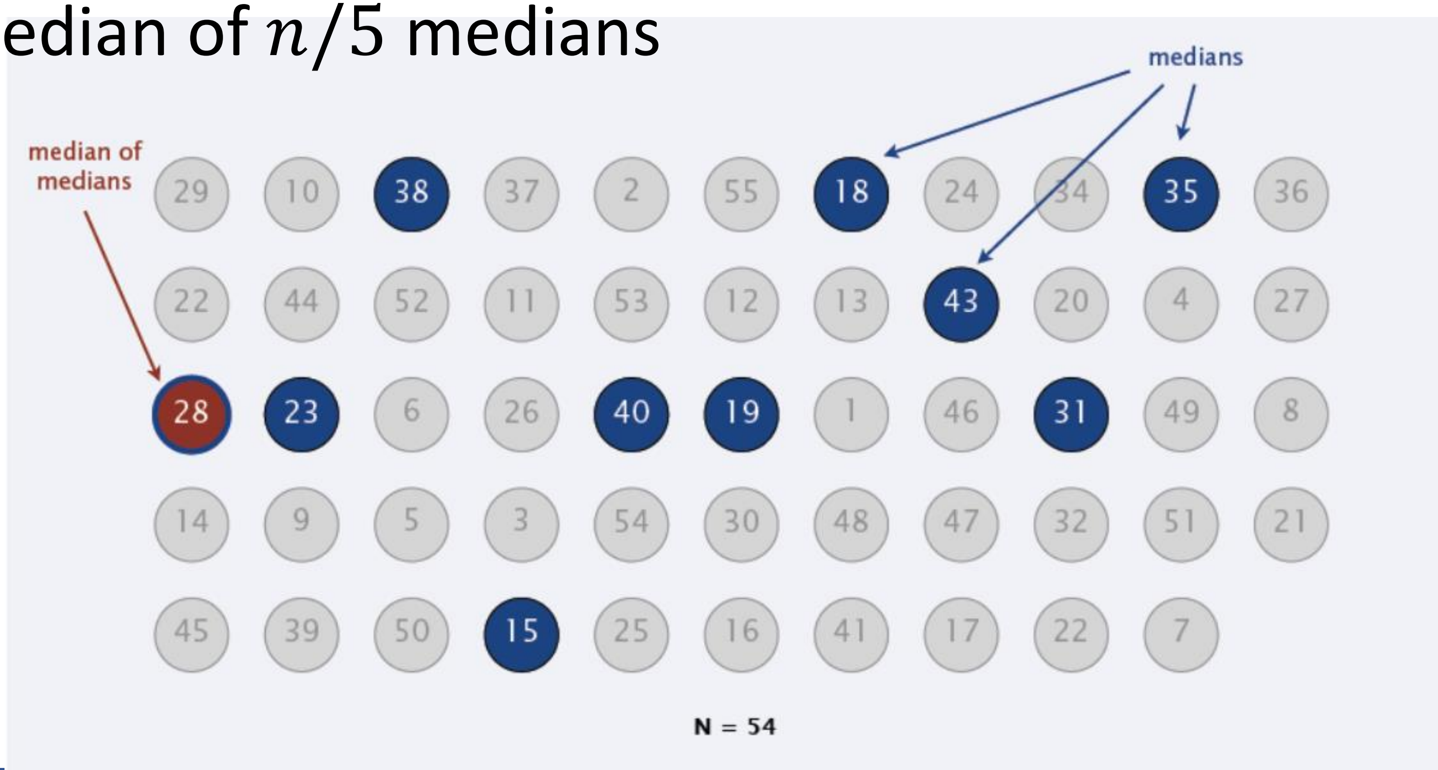
Finding a Good Pivot

- Divide n elements into $n/_5$ groups of 5 each
- Find the median of each group



Finding a Good Pivot

- Divide n elements into ${}^{n}/_{5}$ groups of 5 each
- Find the median of each group
- Find the median of $n/5$ medians



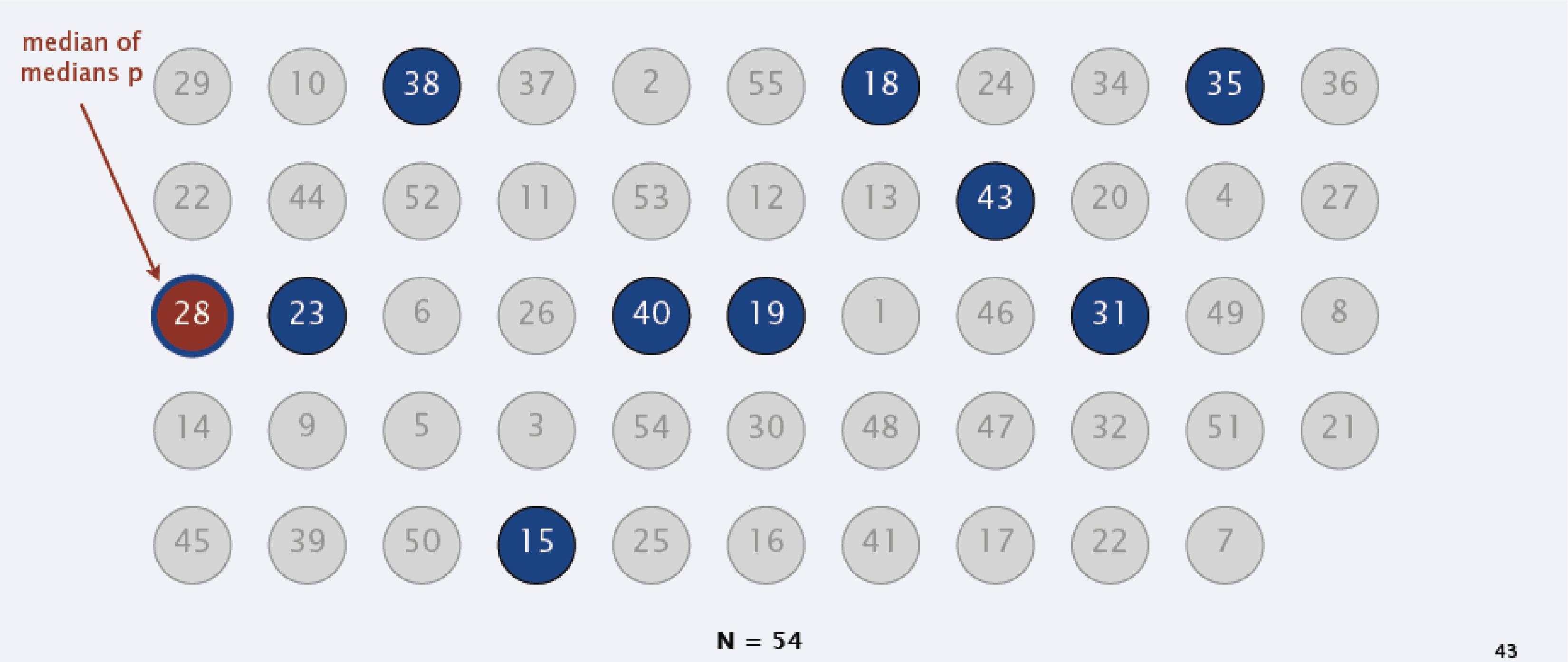
Finding a Good Pivot

- Divide n elements into ${}^{n}/_{5}$ groups of 5 each
- Find the median of each group
- Find the median of $n/5$ medians
- Use this median of medians as the pivot in quickselect

- Q: Why does this work?

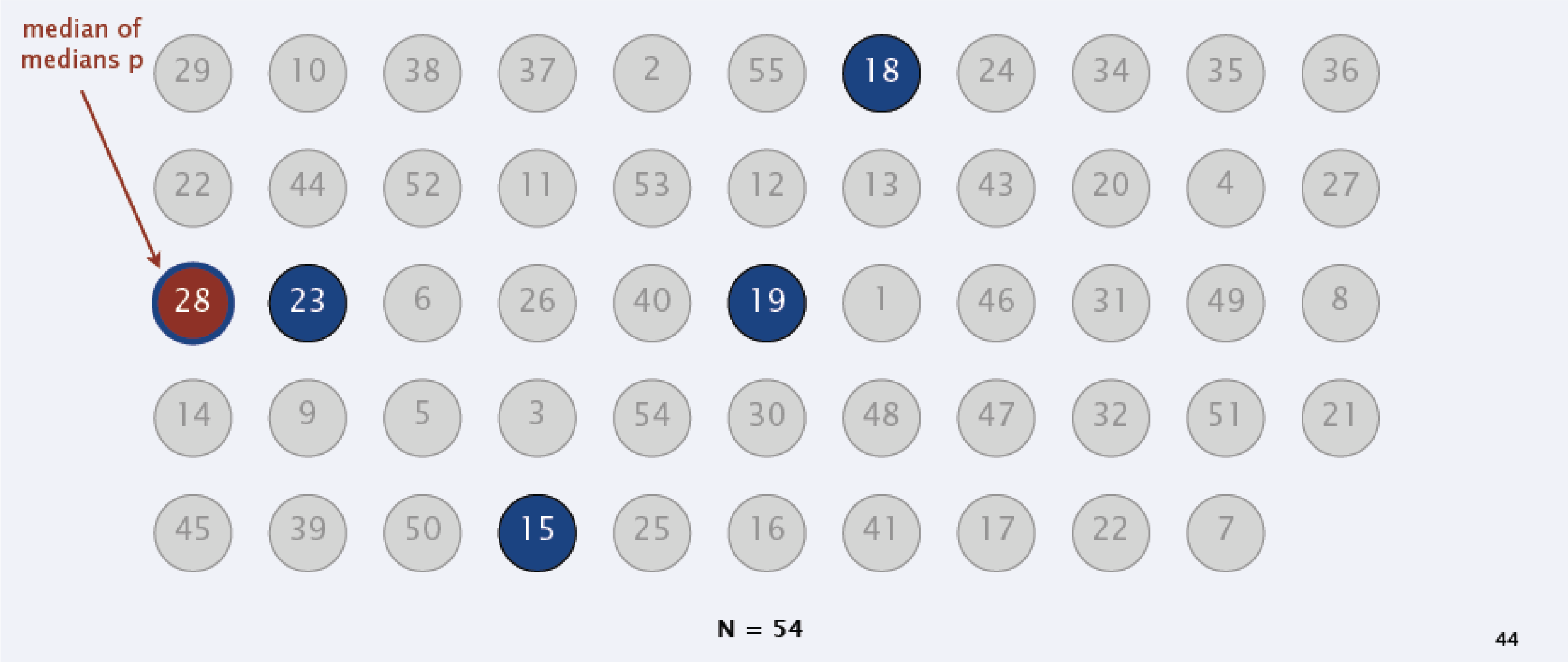
Analysis

- How many elements can be $\leq p^*$?
  - Out of $n/5$ medians, $n/10$ are $> p^*$



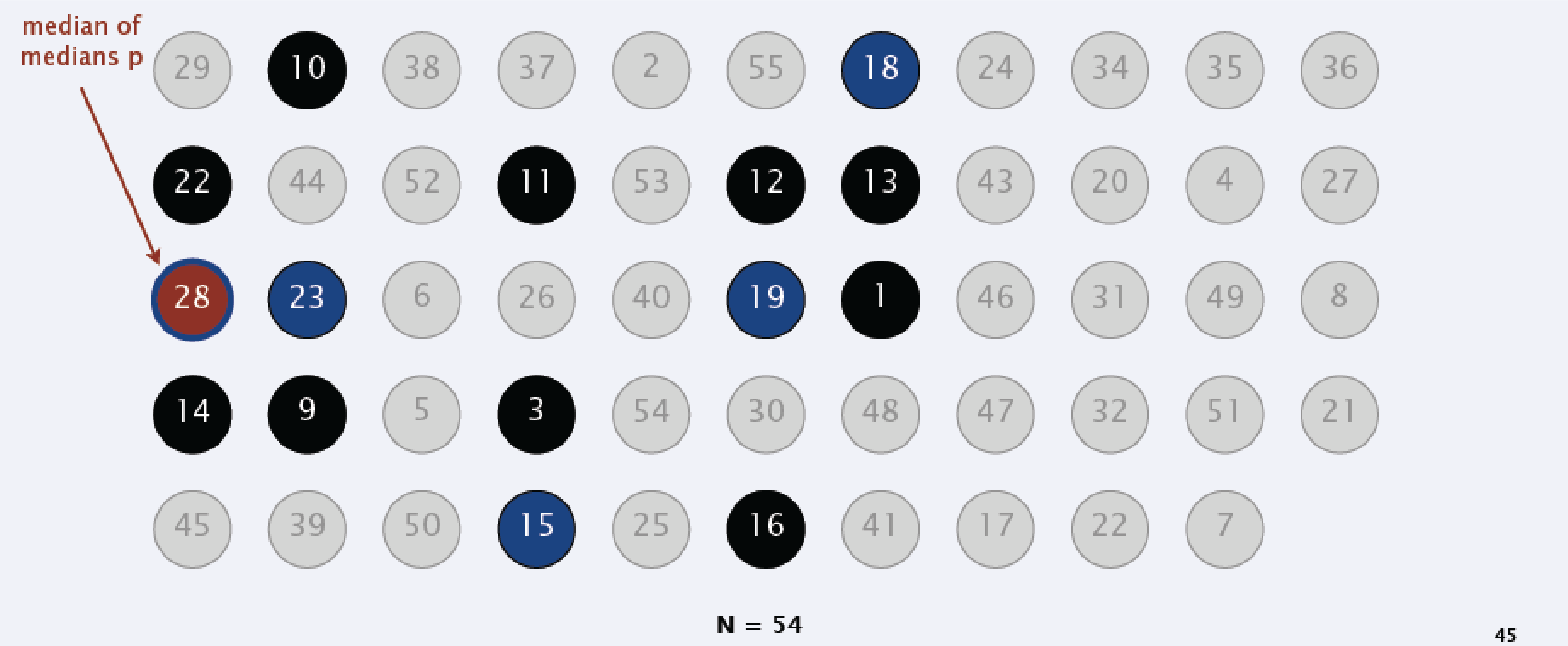
Analysis

- How many elements can be $\leq p^*$?
  - Out of $n/5$ medians, $n/10$ are $> p^*$



Analysis

- $n/_{10}$ of the $n/_5$ medians are $\leq p^*$
  - For each such median, there are 3 elements $\leq p^*$
  - So there can be at most $^{7n}/_{10}$ elements that can be $> p^*$



Analysis

- Thus, $|A_{more}| \leq {}^{7n}/_{10}$
  - Similarly, $|A_{less}| \leq {}^{7n}/_{10}$
  - (These are rough calculations…)

- How does this factor into overall algorithm analysis?

Analysis

- Divide n elements into $^n/_5$ groups of 5 each
- Find the median of each group
- Find p^* = median of $^n/_5$ medians
- Create A_{less} and A_{more} according to p^*
- Run selection on *one of* A_{less} or A_{more}

$O(n)$

$T(n/5)$

$O(n)$

$T(7n/10)$

- $T(n) \leq T(^n/_5) + T(^{7n}/_{10}) + O(n)$
- Note: $^n/_5 + {}^{7n}/_{10} = {}^{9n}/_{10}$
  - ➢ Only a fraction of n, so by the Master theorem, $T(n) = O(n)$

Residual Notes

- Best algorithm for a problem?
  - Typically hard to determine
  - We still don't know best algorithms for multiplying two n-digit integers or two $n \times n$ matrices
    - Integer multiplication
      - Breakthrough in March 2019: first $O(n \log n)$ time algorithm
      - It is conjectured that this is asymptotically optimal
    - Matrix multiplication
      - 1969 (Strassen): $O(n^{2.807})$
      - 1990: $O(n^{2.376})$
      - 2013: $O(n^{2.3729})$
      - 2014: $O(n^{2.3728639})$

Residual Notes

- Best algorithm for a problem?
  - Usually, we design an algorithm and then analyze its running time

  - Sometimes we can do the reverse:
    - E.g., if you know you want an $O(n^2 \log n)$ algorithm
    - Master theorem suggests that you can get it by
$$T(n) = 4\,T\left({}^{n}/_{2}\right) + O(n^2)$$
    - So maybe you want to break your problem into 4 problems of size $n/2$ each, and then do $O(n^2)$ computation to combine

Residual Notes

- Access to input
  - For much of this analysis, we are assuming random access to elements of input
  - So we're ignoring underlying data structures (e.g. doubly linked list, binary tree, etc.)

- Machine operations
  - We're only counting the number of comparison or arithmetic operations
  - So we're ignoring issues like how real numbers are stored in the closest pair problem
  - When we get to P vs NP, representation will matter

Residual Notes

- Size of the problem
  - Can be any reasonable parameter of the problem
  - E.g., for matrix multiplication, we used n as the size
  - But an input consists of two matrices with n^2 entries
  - It doesn't matter whether we call n or n^2 the size of the problem
  - The actual running time of the algorithm won't change