CSC304 Fall'26
Solutions to Tutorial 6

Q1 VCG Mechanism for Auction

Suppose we want to sell three goods, a, b, and c. There are three agents, 1, 2, and 3.

- Agent 1 either wants $\{a, b\}$ at \$4 or $\{c\}$ at \$3, or $\{a, b, c\}$ at \$7, and has zero value for other combinations.
- Agent 2 values a, b, and c at \$3, \$2, and \$2, respectively, and has additive values for bundles of goods (i.e., her value for a set of goods is the sum of her values for the individual goods in the set). So, for instance, agent 2's value for $\{a, b\}$ would be $3 + 2 = 5$.
- Agent 3 values a, b, and c at \$2, \$3, and \$2, respectively, and also has additive values for bundles of goods.

Determine how the goods would be allocated and how much the three agents would have to pay under VCG.

Solution to Q1

Since agent 1 has only a few possible values, it's easier to break cases based on agent 1's value.

- **If agent** 1 **gets** $\{a, b, c\}$, then agents 2 and 3 get nothing. Hence, the maximum social welfare is $7 + 0 + 0 = 7$.
- **If agent** 1 **gets** $\{a, b\}$, then c can be given to either of agents 2 and 3, and the social welfare is $4 + 2 + 0 = 6$.
- **If agent** 1 **gets** $\{c\}$, then the maximum social welfare is generated by giving a to agent 2 and b to agent 3, which generates social welfare $3 + 3 + 3 = 9$.
- **If agent** 1 **gets** $\emptyset$, then the maximum social welfare is generated by giving $\{a, c\}$ to agent 2 and $\{b\}$ to agent 3, generating social welfare $0 + 5 + 3 = 8$.

Since VCG implements the outcome with the highest social welfare, it will choose:

$$\begin{aligned} \{c\} \to 1, \quad & v_1(\{c\}) = 3 \\ \{a\} \to 2, \quad & v_2(\{a\}) = 3 \\ \{b\} \to 3, \quad & v_3(\{b\}) = 3 \end{aligned}$$

Payments charged to the agents:

- **Agent** 1**:** Highest social welfare of agents 2 and 3...
  - In the absence of agent 1: generated by giving $\{a, c\}$ to agent 2 and $\{b\}$ to agent 3, which has welfare $5 + 3 = 8$.

- In the presence of agent 1: $3 + 3 = 6$.
- Payment charged to agent $1 = 8 - 6 = 2$.

- **Agent 2:** Highest social welfare of agents 1 and 3...
  - In the absence of agent 2: generated by giving $\{c\}$ to agent 1 and $\{a, b\}$ to agent 3, which has welfare $3 + 5 = 8$.
  - In the presence of agent 2: $3 + 3 = 6$.
  - Payment charged to agent $2 = 8 - 6 = 2$.

- **Agent 3:** Highest social welfare of agents 1 and 2...
  - In the absence of agent 3: generated by giving $\{c\}$ to agent 1 and $\{a, b\}$ to agent 2, which has welfare $3 + 5 = 8$.
  - In the presence of agent 3: $3 + 3 = 6$.
  - Payment charged to agent $3 = 8 - 6 = 2$.

Final outcome:

$$\begin{aligned} \{c\} \to 1, \quad v_1(\{c\}) = 3, \quad p_1 = 2 \\ \{a\} \to 2, \quad v_2(\{a\}) = 3, \quad p_2 = 2 \\ \{b\} \to 3, \quad v_3(\{b\}) = 3, \quad p_3 = 2 \end{aligned}$$

Q2 VCG for Public Policy

Suppose a government wants to choose a tax policy from three proposed alternatives A, B, and C. The government asks three lobbying groups ($L1$, $L2$, and $L3$) to submit their values for the three proposals, and wants to run a VCG auction.

If the following bids are submitted, which proposal would be selected, and how much would each lobbying group be charged under VCG?

	A	B	C
$L1$	12	9	2
$L2$	5	8	10
$L3$	4	5	6

Solution to Q2

The welfare under the three outcomes are:

$$\begin{aligned} A &\to 12 + 5 + 4 = 21 \\ B &\to 9 + 8 + 5 = 22 \\ C &\to 2 + 10 + 6 = 18 \end{aligned}$$

Since VCG chooses the outcome with the highest welfare, it will implement B, which gives values $(9, 8, 5)$ to $L1$, $L2$, and $L3$, respectively.

Payments charged to the agents:

- **Agent** 1**:** Highest social welfare of agents 2 and 3...
  - In the absence of agent 1: $10 + 6 = 16$ by implementing C
  - In the presence of agent 1: $8 + 5 = 13$.
  - Payment charged to agent $1 = 16 - 13 = 3$.
- **Agent** 2**:** Highest social welfare of agents 1 and 3...
  - In the absence of agent 2: $12 + 4 = 16$ by implementing A
  - In the presence of agent 2: $9 + 5 = 14$.
  - Payment charged to agent $2 = 16 - 14 = 2$.
- **Agent** 3**:** Highest social welfare of agents 1 and 2...
  - In the absence of agent 3: $12 + 5 = 9 + 8 = 17$ by implementing A or B
  - In the presence of agent 3: $9 + 8 = 17$.
  - Payment charged to agent $3 = 17 - 17 = 0$.

Hence, $L1$, $L2$, and $L3$ will be charged 3, 2, and 0, respectively.

Q3 Sponsored Search

Consider the following sponsored search auction with three slots A, B and C, and four bidders W, X, Y and Z. The *clickthrough rate* of each slot and the *value per click* of each bidder is as follows:

Values per Click	Bidders	Slots	Clickthrough Rates
10	W	A	0.6
5	X	B	0.4
3	Y	C	0.3
1	Z		

(a) Suppose we run the VCG auction and the bidders bid their true values. Compute the assignment of slots to bidders, the payment per click charged to each bidder, and the expected payment charged to each bidder. Show your calculation.

(b) Suppose we run the generalized second-price (GSP) auction and the bidders bid their true values. Compute the assignment of slots to bidders, the payment per click charged to each bidder, and the expected payment charged to each bidder. Show your calculation.

(c) Suppose we run the GSP auction. Is it a Nash equilibrium for all bidders to bid their true values? If not, find a bidder who can strictly increase her expected utility by changing only her

own bid (while all other bidders still bid their true values), and compute her expected utility before and after the change. Show your calculation.

[Hint: For each bidder, consider the slots she could get by changing her bid, and the payment per click she would be charged for each of them.]

Solution to Q3

(a) Allocation: Under VCG, the k-th highest bidder gets the k-th best slot. So the assignment is: $W \to A$, $X \to B$, $Y \to C$, and Z gets nothing.

Payments: Under VCG, the expected payment charged to the k-th highest bidder is $\sum_{i=k+1}^{4} v_i \cdot (c_{i-1} - c_i)$, and payment per click is obtained by dividing this by c_k.

Agent	Expected payment	Payment per click
W	$5 \cdot (0.6 - 0.4) + 3 \cdot (0.4 - 0.3) + 1 \cdot 0.3 = 1.6$	$\frac{1.6}{0.6}$
X	$3 \cdot (0.4 - 0.3) + 1 \cdot 0.3 = 0.6$	$\frac{0.6}{0.4}$
Y	$1 \cdot 0.3 = 0.3$	$\frac{0.3}{0.3}$
Z	0	0

(b) Allocation: Under GSP, the k-th highest bidder gets the k-th best slot. So the assignment is: $W \to A$, $X \to B$, $Y \to C$, and Z gets nothing.

Payments: Under GSP, the payment per click charged to the k-th highest bidder is the value per click of the $(k+1)$-st highest bidder. Expected payments are obtained by multiplying these by the clickthrough rates of the assigned slots.

Agent	Payment per click	Expected payment
W	5	$5 \cdot 0.6 = 3$
X	3	$3 \cdot 0.4 = 1.2$
Y	1	$1 \cdot 0.3 = 0.3$
Z	0	0

(c) No, it is not a Nash equilibrium for all bidders to bid their true values under GSP.

Under truthful bidding, the expected utility of a bidder is her value per click minus her payment per click, multiplied by the clickthrough rate of her slot. Using the payments from part (b), W gets $(10 - 5) \cdot 0.6 = 3$, X gets $(5 - 3) \cdot 0.4 = 0.8$, Y gets $(3 - 1) \cdot 0.3 = 0.6$, and Z gets 0.

Now suppose X changes her bid from 5 to 2, while the other bidders still bid truthfully. The bids in decreasing order are then W (10), Y (3), X (2), and Z (1), so X moves from slot B to slot C, and is charged the bid of the next bidder Z, i.e., 1 per click. Her expected utility becomes $(5 - 1) \cdot 0.3 = 1.2 > 0.8$. Hence, X strictly benefits from deviating, so truthful bidding is not a Nash equilibrium.

PS: X is the only bidder with a profitable deviation here. Bidder W would get $(10-3)\cdot 0.4 = 2.8$ in slot B and $(10-1)\cdot 0.3 = 2.7$ in slot C, both less than 3. Bidders Y and Z cannot do better either, as getting any better slot would require them to pay more than their value per click (Y would pay at least 5 per click, and Z would pay at least 3 per click). Note that this is in contrast to VCG, where truthful bidding is a dominant strategy for every bidder.

Q4 Sealed-Bid First-Price Auction

In class, we studied the sealed-bid first-price auction for a single item, in which each bidder privately submits a bid, and the one with the highest bid wins the item and pays her bid.

We proved that with $n = 2$ bidders drawing their values from $U[0,1]$ iid, it is a Bayes-Nash equilibrium for each bidder i to bid $b(v_i) = v_i/2$.

Your goal is to generalize this to n bidders: If each bidder i bidding $b(v_i) = c \cdot v_i$ is a (symmetric) Bayes-Nash equilibrium, then $c = \frac{n-1}{n}$.

Solution to Q4

Suppose each bidder i bidding $b(v_i) = c \cdot v_i$ is a BNE for some c. Due to symmetry, we only need to verify that this is the optimal strategy for bidder 1 given the other bidders' strategies.

Let the optimal bid for bidder 1 be x. First, notice that no other bidder is bidding more than $c \cdot 1$. Hence, the optimal bid of bidder 1 cannot be higher than c, i.e., $x \leq c$. The expected utility of bidder 1 is $\Pr[win] \cdot \mathbb{E}[utility \mid win] + \Pr[lose] \cdot \mathbb{E}[utility \mid lose]$, where win and lose indicate the events of receiving and not receiving the item. Since $\mathbb{E}[utility \mid lose] = 0$, we can discard the second term.

For the first term,

$$\Pr[win] = \Pr[x \geq b(v_i), \forall i \in \{2, \ldots, n\}] = \prod_{i=2}^{n} \Pr[v_i \leq x/c] = (x/c)^{n-1},$$

where we use the fact that $b(v_i) = c \cdot v_i$ for all bidders $i \in \{2, \ldots, n\}$, their values are drawn iid from $U[0,1]$, and $x/c \leq 1$. Note that conditioned on winning the item, the utility of bidder 1 is precisely $v_1 - x$. Hence, the expected utility of the bidder is $(x/c)^{n-1} \cdot (v_1 - x)$.

To find the optimal x, let us set the derivative w.r.t. x as 0.

$$(n-1)(x/c)^{n-2}(v_1 - x) \cdot (1/c) = (x/c)^{n-1} \Leftrightarrow (n-1)(v_1 - x) = x \Leftrightarrow x = \frac{n-1}{n} \cdot v_1.$$

However, we want the optimal bid of bidder 1 to be equal to $x = c \cdot v_1$ in our BNE. Hence, it must be that $c = \frac{n-1}{n}$. Note that since our derivation was done using "if and only if" relation, $c = \frac{n-1}{n}$ does in fact form a BNE.

Q5 Lobbying

The government is considering n possible tax policies ($\{1, 2, \ldots, n\}$) and there are n companies ($\{1, 2, \ldots, n\}$). Each company i has a private value v_i drawn from $U[0, 1]$, independently of all other companies, such that the company gets value v_i if policy i is implemented and value 0 if any other policy is implemented. The company lobbies the government to implement policy i by giving a donation of d_i. The greedy government announces that it will implement the policy recommended by its biggest donor, but obviously, it will keep all the donations it receives.

Suppose that this game has a symmetric Bayes-Nash equilibrium in which each company donates $d_i = c \cdot v_i^\alpha$, for some $c, \alpha > 0$. Find the values of c and α.

Solution to Q5

Same as in Q4, let us verify this from the perspective of company 1. Let the optimal donation by company 1 be x. First, notice that no other company is donating more than $c \cdot 1^\alpha$. Hence, the optimal donation of company 1 cannot be higher than c, i.e., $x \leq c$. The expected utility of company 1 is $\Pr[win] \cdot \mathbb{E}[utility \mid win] + \Pr[lose] \cdot \mathbb{E}[utility \mid lose]$, where win and lose indicate the events of the government implementing policy i or not. Note that in this case, $\mathbb{E}[utility \mid lose] \neq 0$ because the company loses its donation in this case.

For the first term,

$$\Pr[win] = \Pr[x \geq d_i, \forall i \in \{2, \ldots, n\}] = \prod_{i=2}^{n} \Pr[v_i \leq (x/c)^{1/\alpha}] = (x/c)^{(n-1)/\alpha},$$

where we use the fact that $d_i = c \cdot v_i^\alpha$ for all companies $i \in \{2, \ldots, n\}$, their values are drawn iid from $U[0, 1]$, and $x/c \leq 1$. Note that conditioned on winning the item, the utility of company 1 is precisely $v_1 - x$.

For the second term, $\Pr[lose] = 1 - \Pr[win] = 1 - (x/c)^{(n-1)/\alpha}$ and the expected utility conditioned on losing is $-x$ (since the donation is not returned). Hence, the overall expected utility of company 1 is

$$(x/c)^{(n-1)/\alpha} \cdot (v_1 - x) + (1 - (x/c)^{(n-1)/\alpha}) \cdot (-x) = (x/c)^{(n-1)/\alpha} \cdot v_1 - x.$$

To find the optimal x, let us set the derivative w.r.t. x as 0. We get

$$\frac{n-1}{\alpha} \left(\frac{x}{c}\right)^{\frac{n-1}{\alpha}-1} \cdot v_1 \cdot \frac{1}{c} = 1.$$

Since we want $x = c \cdot v_1^\alpha$ to be the solution of this, we need

$$\frac{n-1}{\alpha} v_1^{n-1-\alpha} \cdot v_1 \cdot \frac{1}{c} = 1.$$

For this to always be true, we need the exponent of v_1 to be 0 on the left hand side, so $\alpha = n$. Then, we also need

$$\frac{n-1}{n} \cdot \frac{1}{c} = 1,$$

so $c = \frac{n-1}{n}$. Hence, the BNE is $d_i = \frac{n-1}{n} \cdot v_i^n$ for each company i.