Question 1. VCG [25 MARKS]

Suppose we want to sell two goods a and b, and there are 3 agents with the following values for different bundles (subsets of goods). Run the VCG auction on this instance.

Bundles	Agent 1	Agent 2	Agent 3
$\emptyset$	0	0	0
$\{a\}$	9	6	3
$\{b\}$	6	2	5
$\{a, b\}$	11	10	13

Part (a) [10 MARKS]

What allocation would VCG choose? What is the social welfare under this allocation? No justification is needed.

Solution: The allocation is to give a to agent 1 and b to agent 3. This generates a social welfare of 14.

Part (b) [15 MARKS]

What payments would be charged to the three agents by VCG? Show your calculation by identifying, for each agent, the allocation that would be implemented and the welfare it would yield to the other agents in the absence of the agent.

Solution:

Agent 1:
Allocation in her absence: give both a and b to agent 3
Welfare to others in her absence: 13
Welfare to others in her presence: 5
Payment: $p_1 = 13 - 5 = 8$.

Agent 2:
Agent 2 clearly pays $p_2 = 0$ because she does not receive any goods. In other words, the allocation and welfare to others in her absence are the same as those in her presence.

Agent 3:
Allocation in her absence: give a to agent 2 and b to agent 1
Welfare to others in her absence: 12
Welfare to others in her presence: 9
Payment: $p_3 = 12 - 9 = 3$.

Question 2. Bayes-Nash Equilibria [25 MARKS]

Two companies are fighting to obtain mining rights near Mountain Plentiful. One side of the mountain contains v_1 thousand tonnes of copper, while the other side contains v_2 thousand tonnes of zinc. Company 1 is more interested in copper than zinc, so it values the mining rights at $v_1 + v_2/2$. Company 2 is more interested in zinc than in copper, so it values the mining rights at $v_1/2 + v_2$. Only one company can win the mining rights.

Both companies know that v_1 and v_2 are drawn iid from the uniform distribution $U[0, 1]$. The state holds a second-price auction. Each company $i \in \{1, 2\}$ will bid an amount b_i, and the company with the higher bid will win the mining rights and pay the second highest bid.

Since company 1 is more interested in copper, it invests effort to determine the precise value of v_1, but still knows nothing about v_2 other than that it is drawn from $U[0, 1]$. Similarly, company 2 has figured out the exact value of v_2, but knows nothing about v_1 other than that it is drawn from $U[0, 1]$.

Show that each bidder bidding $b_i = 1.5v_i$ is a Bayes-Nash equilibrium in this auction. Due to symmetry, you only need to show that company 1 would want to bid $b_1 = 1.5v_1$ if it assumes that company 2 will bid $b_2 = 1.5v_2$, whatever the value of v_2 is. So in the questions below, assume that company 2 bids $b_2 = 1.5v_2$, and derive the optimal bid b_1 of company 1.

Part (a) [5 MARKS]

Argue why the optimal bid b_1 must be at most 1.5.

Solution: Company 2 is bidding $b_2 = 1.5v_2 \leqslant 1.5$. Hence, if company 1 is bidding $b_1 > 1.5$, it can reduce its bid to 1.5. It is still guaranteed to win the auction and does not pay more, which can only increase its expected utility.

Part (b) [5 MARKS]

Derive the probability of company 1 winning the auction if it bids b_1.

Solution: Company 1 wins the auction when it outbids company 2. Hence, this probability is

$$\Pr[b_1 \geqslant b_2] = \Pr[b_1 \geqslant 1.5v_2] = \Pr[v_2 \leqslant b_1/1.5] = b_1/1.5,$$

where the last step holds because $b_1/1.5 \leqslant 1$ due to part (a).

Part (c) [5 MARKS]

Derive the expected value that company 1 will derive conditioned on it winning the auction.

Solution: When company 1 wins the auction, its value is $v_1 + v_2/2$. It wins the auction when $b_1 \geqslant b_2$, i.e., $v_2 \leqslant b_1/1.5$. Hence, the answer is

$$\mathbb{E}[v_1 + v_2/2 | v_2 \leqslant b_1/1.5] = v_1 + \frac{1}{2} \cdot \mathbb{E}[v_2 | v_2 \leqslant b_1/1.5] = v_1 + \frac{1}{2} \cdot \frac{b_1}{3} = v_1 + b_1/6.$$

Part (d) [5 MARKS]

Derive the expected payment that company 1 will have to pay conditioned on it winning the auction.

Solution: When company 1 wins the auction (i.e. $b_1 \geqslant b_2$ or $v_2 \leqslant b_1/1.5$), it pays the second highest bid b_2. Hence, the answer is

$$\mathbb{E}[b_2 | v_2 \leqslant b_1/1.5] = \mathbb{E}[1.5v_2 | v_2 \leqslant b_1/1.5] = 1.5 \cdot \frac{b_1}{3} = b_1/2.$$

Part (e) [5 MARKS]

Using the calculations in parts (b), (c), and (d), show that the optimal bid for company 1 is $b_1 = 1.5v_1$ by maximizing its expected utility.

Solution: Combining the previous answers, the expected utility to company 1 when bidding b_1 is

$$\Pr[\text{win}] \cdot \mathbb{E}[\text{value} - \text{payment} | \text{win}] = \frac{b_1}{1.5} \cdot \left(v_1 + \frac{b_1}{6} - \frac{b_1}{2} \right) = \frac{b_1}{1.5} \cdot \left(v_1 - \frac{b_1}{3} \right).$$

To maximize this with respect to b_1, we set the derivative w.r.t. b_1 to zero, and get

$$\frac{1}{1.5} \cdot \left(v_1 - \frac{b_1}{3} \right) - \frac{b_1}{1.5} \cdot \frac{1}{3} = 0 \iff v_1 - \frac{2b_1}{3} = 0 \iff b_1 = 1.5v_1.$$

Note that the second derivative is the constant $-1/4.5 - 1/4.5 < 0$. Hence, the point $b_1 = 1.5v_1$ where the derivative vanishes must be a maximum.