Q1. VCG (10 pts)

The government decides to help out three companies by building a single warehouse that they can all share. The warehouse is to be built at a location $x \in [0, 10]$. Each company $i \in \{1, 2, 3\}$ is located at y_i, where $y_1 = 1$, $y_2 = 5$, and $y_3 = 9$. A warehouse at location x provides value $100 - (y_i - x)^2$ to each company i simultaneously.

Under the VCG mechanism, decide where the warehouse should be built, and how much payment each company should be charged. Show your calculation.

[Note: In this case, the space of possible outcomes is infinite, but VCG still applies.]

Notes to graders:

- No marks should be cut if the students leave the answer in an expression form (e.g. $2 * (100 - 4) - (100 + 84)$ instead of 8).
- No marks should be cut if the students use derivatives to find optimal solutions but do not check against endpoints.
- Cut only 1 mark if the students miscalculated the answer (numerical error) but had the correct expression.

Solution: Throughout the solution, we use the fact that maximizing $\sum_i (100 - (y_i - x)^2)$ is equivalent to minimizing $\sum_i (y_i - x)^2$, which is achieved at $x = \text{avg}_i(y_i)$.

Outcome: VCG chooses an outcome maximizing the overall welfare. This is $x = (1+5+9)/3 = 5$. Under this outcome, the agents have values $v_1 = 100 - 4^2 = 84$, $v_2 = 100$, $v_3 = 100 - 4^2 = 84$.

Payment charged to company 1: With agent 1 present, the other two agents get 184. Without agent 1, VCG chooses $x' = (5 + 9)/2 = 7$, under which the two agents have values $v_2' = v_3' = 100 - 2^2 = 96$. Hence, the payment charged to agent 1 is $2 * 96 - 184 = 8$.

Payment charged to company 2: Without agent 2, VCG still chooses $x' = (1 + 9)/2 = 5$. So the values to the other two agents does not change. Hence, agent 2 is charged 0.

Payment charged to company 3: With agent 3 present, the other two agents get 184. Without agent 3, VCG chooses $x' = (1 + 5)/2 = 3$, under which the two agents have values $v_1' = v_2' = 100 - 2^2 = 96$. Hence, the payment charged to agent 3 is same as payment charged to agent 1: $2 * 96 - 184 = 8$.

Solution summary: outcome = 5, payments = $(8, 0, 8)$.

Q2. Bayes-Nash Equilibria (10 pts)

Consider an auction between a buyer and a seller. Seller s wants to sell a good for which his own value is v_s. Buyer b has value v_b for the good. Both v_s and v_b are drawn independently from the uniform distribution $U[0,1]$. Each player knows his own value, but only knows the distribution of the other player's value.

Here's how the auction goes down. The seller and the buyer independently write down some prices (say p_s and p_b, respectively) on different pieces of paper. Then, they compare their prices. If it turns out that $p_s \leq p_b$, then the item is sold at price p_s, the seller gets utility $p_s - v_s$, and the buyer gets utility $v_b - p_s$. On the other hand, if $p_s > p_b$, the item remains unsold and both players get zero utility.

Is there a Bayes-Nash equilibrium in which the buyer honestly writes down her value (i.e. $p_b = v_b$)? If so, what is the seller's price p_s in this equilibrium? Show your calculation.

[Hint: Assume $p_b = v_b$, and derive the optimal price p_s for the seller to write. Then, check whether given this strategy of the seller, $p_b = v_b$ is indeed the optimal price for the buyer to write.]

Notes to graders:

- No marks should be cut if the students do not "normalize" optimal bids (i.e. show that they do not exceed 1) and directly use say $\Pr[v_b \geq p_s] = 1 - p_s$ (as I did below).
- No marks should be cut if the student did not check the critical point against endpoints.
- 5 marks for deriving the optimal strategy for the seller, 5 marks for showing that under that strategy buyer's strategy is also optimal.
- Cut 1 mark every time the student took a wrong derivative.

Solution: Let us assume that there is a Bayes-Nash equilibrium (p_s, p_b) in which $p_b = v_b$.

Seller's optimal strategy: Given p_b, p_s must maximize seller's expected utility, which is

$$\begin{aligned}&\Pr[p_s \leq p_b] \cdot \mathbb{E}[p_s - v_s | p_s \leq p_b] + \Pr[p_s \geq p_b] \cdot 0 \\ &= \Pr[v_b \geq p_s] \cdot (p_s - v_s) = (1 - p_s) \cdot (p_s - v_s).\end{aligned}$$

To find the p_s that maximizes this, we set the derivative to 0 and obtain:

$$1 - p_s = p_s - v_s \Rightarrow p_s = \frac{1 + v_s}{2}.$$

Hence, our candidate BNE is (p_s, p_b) where $p_s = (1 + v_s)/2$ and $p_b = v_b$.

Buyer's optimal strategy: To establish that this is indeed a BNE, we need to verify that $p_b = v_b$ is optimal for the buyer assuming $p_s = (1 + v_s)/2$. Note that p_s is uniformly distributed over $[1/2, 1]$. If $p_b < 1/2$, then $p_s > p_b$ always, so the item is never sold and the buyer's utility is 0. If $p_b > 1$, the item is always sold at price p_s, which does not depend on p_b, so the utility equals that for $p_b = 1$. Hence, we focus on $1/2 \leq p_b \leq 1$, for which the buyer's expected utility is

$$\begin{aligned}&\Pr[p_s \leq p_b] \cdot \mathbb{E}[v_b - p_s | p_s \leq p_b] + \Pr[p_s \geq p_b] \cdot 0 \\ &= \Pr[(1 + v_s)/2 \leq p_b] \cdot (v_b - \mathbb{E}[(1 + v_s)/2 | (1 + v_s)/2 \leq p_b]) \\ &= \Pr[v_s \leq 2p_b - 1] \cdot \left(v_b - \frac{1 + \mathbb{E}[v_s | v_s \leq 2p_b - 1]}{2} \right) \\ &= (2p_b - 1) \cdot \left(v_b - \frac{1 + p_b - 1/2}{2} \right) = (2p_b - 1) \cdot \left(v_b - \frac{p_b}{2} - \frac{1}{4} \right).\end{aligned}$$

Again, to find the optimal p_b, we set the derivative to 0 and obtain:

$$2 \cdot \left(v_b - \frac{p_b}{2} - \frac{1}{4} \right) = \frac{1}{2} \cdot (2p_b - 1) \Rightarrow 2v_b - p_b - 1/2 = p_b - 1/2 \Rightarrow p_b = v_b.$$

Since $u''(p_b) = -2 < 0$, the function $u(p_b) = (2p_b - 1) \cdot \left(v_b - \frac{p_b}{2} - \frac{1}{4}\right)$ is strictly concave, so it is maximized over the reals at $p_b = v_b$. We now consider two cases.

- If $v_b \geq 1/2$, then $p_b = v_b$ lies in $[1/2, 1]$, so it maximizes u over this range. Further, $u(v_b) = (2v_b - 1)^2/4 \geq 0$, which is at least the utility of 0 obtained from any $p_b < 1/2$.
- If $v_b < 1/2$, then u is decreasing on $[1/2, 1]$ (because the maximizer v_b lies to its left), and $u(1/2) = 0$. Hence, $u(p_b) \leq 0$ for all $p_b \in [1/2, 1]$, so the best the buyer can do is obtain utility 0, which $p_b = v_b < 1/2$ achieves.

Hence, $p_b = v_b$ is optimal for the buyer for every value $v_b \in [0, 1]$, and there is a BNE with $p_s = (1 + v_s)/2$ and $p_b = v_b$.

Q3. Sponsored Search (10 pts)

Consider the following sponsored search auction with three slots A, B and C, and four bidders W, X, Y and Z. The *clickthrough rate* for each slot and the *value per click* for each bidder are as follows:

Values per Click	Bidders	Slots	Clickthrough Rates
10	W	A	0.6
5	X	B	0.4
3	Y	C	0.3
1	Z		

(a) [5 points] Suppose we run the generalized second price (GSP) auction and the bidders bid their true values. Compute the assignment of slots to bidders and the *expected payments* charged to bidders under GSP. Show your calculation.

Notes to graders:

- Cut no marks if they leave an expression like $5 \cdot 0.6$ without evaluating it to 3.
- Cut 1 mark if they calculated payments per clicks rather than expected payments.
- Cut 1 mark for numerical errors, as long as their general idea was right.

Solution:

Allocation: Under GSP, the k-th highest bidder gets the k-th best slot. So the assignment is: $W \to A$, $X \to B$, $Y \to C$, and Z gets nothing.

Payments: Under GSP, the payment per click charged to the k-th highest bidder is the value per click of the $k+1$-th highest bidder. This is multiplied by clickthrough rates. So we get:

- $p_W = 5 \cdot 0.6 = 3$
- $p_X = 3 \cdot 0.4 = 1.2$
- $p_Y = 1 \cdot 0.3 = 0.3$
- $p_Z = 0$

(b) [5 points] Find a bidder who has an incentive to misreport his value under GSP, assuming the other bidders bid truthfully. Identify the bidder, state the beneficial manipulation, and derive the gain in utility to the bidder due to the manipulation. Show your calculation.

Notes to graders:

- I think I have set it up such that only one bidder (X) can gain by misreporting in only one way (by changing from being 2nd highest to being 3rd highest).
- Whether the student identifies the manipulation as "anything in $(1, 3)$", or "anything in $[1, 3]$", or "2" etc, it is valid.
- Cut 1 mark for numerical errors, as long as their general idea was right.

Solution: Bidder X has an incentive to misreport.

Current expected utility: $(5 - 3) \cdot 0.4 = 0.8$

Expected utility if she reports a value in $(1, 3)$: $(5 - 1) \cdot 0.3 = 1.2$.

Gain in utility: $1.2 - 0.8 = 0.4$.