(PRINT) Name: _____ Student No.: _____

Signature: _____ Marks: _____ / 70

CSC 304 — Algorithmic Game Theory and Mechanism Design

Midterm 2, November 16, 2018

This test has seven (7) pages. Make sure that you have a complete paper.
Answer all questions on this paper. You may use the back side of any page as scrap paper. You are allowed to use one, two-sided 8.5" x 11" sheet of *handwritten* notes. No other materials or aids of any type are permitted. Unless mentioned otherwise, justify your answer.

Time: 50 minutes; Total Marks: 70

Q1. VCG Auction (15 pts)

Consider an auction in which three houses $\{h_1, h_2, h_3\}$ are to be sold to three agents $\{X, Y, Z\}$. The values of the agents for the houses are as follows.

	h_1	h_2	h_3
X	1	3	3
Y	0	5	7
Z	1	0	4

An agent only wants a single house; so her value for receiving multiple houses is the maximum of her values of the houses she receives. An agent that receives no house has zero value.

(a) [3 points] Derive the allocation under the VCG auction. You only need to state the allocation and the welfare achieved; you do not need to further justify your answer.

Allocation: h_1 goes to Z, h_2 goes to X, and h_3 goes to Y.

Welfare: $7 + 3 + 1 = 11$.

(b) [12 points] Derive the payment charged to each agent under the VCG auction.

Payment charged to X:
Maximum welfare to others without X = 9
Welfare to others with X = 8
Payment charged to X = 1

Payment charged to Y:
Maximum welfare to others without Y = 7
Welfare to others with Y = 4
Payment charged to Y = 3

Payment charged to Z:
Maximum welfare to others without Z = 10
Welfare to others with Z = 10
Payment charged to Z = 0

Q2. Bayes-Nash Equilibria (25 pts)

Suppose there is an auction for a two-tiered cake, in which two bidders participate. The value of tier 1 is v_1 and the value of tier 2 is v_2. The total value of the cake is $v = v_1 + v_2$. The bidder who wins the cake will derive value v. It is common information that each v_i is drawn from the uniform distribution over the unit interval $U([0, 1])$.

Before bidding, the bidders request to have a look at the cake. But the auctioneer decides that bidder 1 will only get to see tier 1 and bidder 2 will only get to see tier 2 (this is also known to both bidders). Thus, after the viewing, each bidder $i \in \{1, 2\}$ learns the exact value v_i of tier i, but still knows nothing about the value of the other tier except that it is drawn from $U([0, 1])$.

The bidders now need to bid on the entire cake. The auctioneer holds a second-price auction in which the highest bidder wins the cake and pays the second highest bid. In this question, you will show that each bidder i bidding $b_i = 2v_i$ is a Bayes-Nash equilibrium. Due to symmetry, you will only need to show that bidder 1 wants to bid $2v_1$ if she assumes bidder 2 will bid $2v_2$. In the analysis below, assume that a bidder will never want to bid more than 2 (i.e., assume the optimal bid $b_i \leq 2$ for each i).

(a) [5 points] Derive the probability of bidder 1 winning the cake if she bids b_1.

Probability of bidder 1 winning the cake is $\Pr[b_2 \leq b_1] = \Pr[2v_2 \leq b_1] = \Pr[v_2 \leq b_1/2] = b_1/2$.

(b) [5 points] Derive the expected payment that bidder 1 will pay conditioned on her winning the cake.

If bidder 1 wins, her bid b_1 is higher than b_2, and she pays b_2. So her expected payment conditioned on her winning the cake is $\mathbb{E}[b_2|b_2 \leq b_1] = \mathbb{E}[2v_2|2v_2 \leq b_1] = 2 \cdot \mathbb{E}[v_2|v_2 \leq b_1/2] = 2 \cdot (b_1/4) = b_1/2$, where we used the fact that the expected value of a number from $U([0, 1])$ conditioned on being at most x is $x/2$.

(c) [5 points] Derive the expected value of v_2 conditioned on bidder 1 winning the cake.

This is $\mathbb{E}[v_2|b_2 \leq b_1] = \mathbb{E}[v_2|2v_2 \leq b_1] = \mathbb{E}[v_2|v_2 \leq b_1/2] = b_1/4$.

(d) [10 points] Using the three calculations above, show that the optimal bid for bidder 1 is $b_1 = 2v_1$.

The expected utility of bidder 1 in bidding b_1 is $\Pr[\text{win}] \cdot \mathbb{E}[\text{utility}|\text{win}] + \Pr[\text{lose}] \cdot \mathbb{E}[\text{utility}|\text{lose}]$. Since the bidder gains no value and pays nothing if she loses, $\mathbb{E}[\text{utility}|\text{lose}]$ is zero.

Hence, the expected utility is $\Pr[b_2 \leq b_1] \cdot \mathbb{E}[v_1 + v_2 - b_2|b_2 \leq b_1] = \frac{b_1}{2} \cdot \left(v_1 + \frac{b_1}{4} - \frac{b_1}{2}\right) = \frac{b_1 \cdot (4v_1 - b_1)}{8}$.

To maximize this quantity, we take derivative w.r.t. b_1 and set it equal to zero. We get $4v_1 - 2b_1 = 0 \Rightarrow b_1 = 2v_1$. To ensure that this is a maximum, we check that the second derivative is $-1/4$, i.e., negative. Hence, the optimal bid for bidder 1 is $b_1 = 2v_1$.

Q3. Sponsored Search Auctions (30 pts)

Consider the following sponsored search auction with three slots s_1, s_2, s_3, and four bidders A, B, C, D. The *clickthrough rate* of each slot and the *value per click* for each bidder are as follows. Recall that in sponsored search, a bidder with value per click v that is assigned a slot with clickthrough rate c and charged payment *per click* p has expected value $v \cdot c$, expected payment $p \cdot c$, and expected utility $(v - p) \cdot c$.

Values per Click	Bidders	Slots	Clickthrough Rates
6	A	s_1	0.8
5	B	s_2	0.3
2	C	s_3	0.1
1	D		

(a) [10 points] Compute the allocation of slots to bidders and the *expected payment* charged to each bidder under the VCG auction.

Both VCG and GSP match k^{th} highest bidder to the k^{th} best slot. Hence, the allocation is as follows.

Allocation: A gets s_1, B gets s_2, C gets s_3, and D gets no slot.

Payments:
Bidder A pays: $5 \cdot (0.8 - 0.3) + 2 \cdot (0.3 - 0.1) + 1 \cdot (0.1 - 0) = 2.5 + 0.4 + 0.1 = 3$
Bidder B pays: $2 \cdot (0.3 - 0.1) + 1 \cdot (0.1 - 0) = 0.4 + 0.1 = 0.5$
Bidder C pays: $1 \cdot (0.1 - 0) = 0.1 = 0.1$
Bidder D pays nothing.

(b) [10 points] Compute the allocation of slots to bidders and the *expected payment* charged to each bidder under the GSP auction, assuming that bidders bid truthfully.

Allocation: (Same as that of VCG) A gets s_1, B gets s_2, C gets s_3, and D gets no slot.

Payments:
Bidder A pays: $5 \cdot 0.8 = 4$
Bidder B pays: $2 \cdot 0.3 = 0.6$
Bidder C pays: $1 \cdot 0.1 = 0.1$
Bidder D pays nothing.

(c) [10 points] Suppose bidder A wants to lie about her value per click under the GSP auction. Find a manipulation that gives bidder A the highest expected utility, assuming that the other bidders bid truthfully. State the manipulated value, and derive the gain in the expected utility to bidder A due to the manipulation.

Let us consider the different ranges of reports for bidder A, and the corresponding expected utility.
Any report in $(5, \infty)$ results in utility $(6-5) \cdot 0.8 = 0.8$
Any report in $(2, 5)$ results in utility $(6-2) \cdot 0.3 = 1.2$
Any report in $(1, 2)$ results in utility $(6-1) \cdot 0.1 = 0.5$
Any report in $(0, 1)$ results in utility $(6-0) \cdot 0 = 0$

Hence, bidding any value in $(2, 5)$ (for example, 4) is an optimal bid for bidder A. Bidding truthfully (a report in $(5, \infty)$) yields utility 0.8, so the gain in expected utility due to the manipulation is $1.2 - 0.8 = 0.4$.