CSC304 Fall'26
Assignment 2 Solutions

Q1 [20 Points] VCG in a Continuous Space

The government decides to help out three companies by building a single warehouse that they can all share. The warehouse is to be built at a location $x \in [0, 10]$. Each company $i \in \{1, 2, 3\}$ is located at y_i, where $y_1 = 1$, $y_2 = 5$, and $y_3 = 9$. A warehouse at location x provides value $100 - (y_i - x)^2$ to each company i simultaneously.

Under the VCG mechanism, decide where the warehouse should be built, and how much payment each company should be charged. Show your calculation.

Solution to Q1

Throughout the solution, we use the fact that maximizing $\sum_i (100 - (y_i - x)^2)$ is equivalent to minimizing $\sum_i (y_i - x)^2$, which is achieved at $x = \text{avg}_i\ y_i$ (i.e., at the average of the y-s). We can either check this by taking the derivative w.r.t. x or by rewriting

$$\sum_{i=1}^{n} (y_i - x)^2 = \sum_{i=1}^{n} y_i^2 - 2x \sum_{i=1}^{n} y_i + nx^2 = n \cdot \left(x - \frac{\sum_{i=1}^n y_i}{n} \right)^2 + \left(\sum_{i=1}^{n} y_i^2 - \frac{(\sum_{i=1}^n y_i)^2}{n} \right).$$

Since only the term in the first bracket depends on x and it is a perfect square, the expression is minimized when it is zero, i.e., when $x = (\sum_{i=1}^n y_i)/n$.

Outcome: VCG chooses an outcome maximizing the overall welfare. This is $x = (1+5+9)/3 = 5$. Under this outcome, the agents have values $v_1 = 100 - 4^2 = 84$, $v_2 = 100$, $v_3 = 100 - 4^2 = 84$.

Payment charged to company 1: With agent 1 present, the other two agents get 184. Without agent 1, VCG chooses $x' = (5 + 9)/2 = 7$, under which the two agents have values $v_2' = v_3' = 100 - 2^2 = 96$. Hence, the payment charged to agent 1 is $2 * 96 - 184 = 8$.

Payment charged to company 2: Without agent 2, VCG still chooses $x' = (1 + 9)/2 = 5$. So the values to the other two agents do not change. Hence, agent 2 is charged 0.

Payment charged to company 3: With agent 3 present, the other two agents get 184. Without agent 3, VCG chooses $x' = (1 + 5)/2 = 3$, under which the two agents have values $v_1' = v_2' = 100 - 2^2 = 96$. Hence, the payment charged to agent 3 is same as payment charged to agent 1: $2 * 96 - 184 = 8$.

Solution summary: outcome = 5, payments = (8, 0, 8).

Q2 [20 Points] VCG for Public Project

Suppose a city with n residents wants to decide whether to build a new bridge. The cost of the project would be C. There are only two possible outcomes: a) the bridge is built, and each resident i derives a private value $v_i \geq 0$, or b) it is not built, and each resident derives zero value. The government wants to build the bridge only if $\sum_i v_i \geq C$, and if the bridge is built, it can charge each resident i a potentially different payment p_i.

Your goal is to run the VCG auction while adding the city as agent $n+1$, which derives value $-C$ if the bridge is built and zero if it is not. If both outcomes are optimal for VCG (i.e., if $\sum_i v_i = C$), assume that VCG builds the bridge.

(a) [2.5 Points] Show that VCG would indeed choose to build the bridge only if $\sum_i v_i \geq C$.

(b) [2.5 Points] Show that resident i is charged a non-zero payment only if she is pivotal, i.e., if $\sum_j v_j \geq C > \sum_{j \neq i} v_j$.

(c) [5 Points] What is the payment charged to a pivotal resident i?

(d) [10 Points] Show that the total payment charged to the residents is at most C (thus, the city can never make profit from building the bridge).

Solution to Q2

Recall that VCG chooses an outcome a^* maximizing the total value of all $n+1$ agents (the n residents and the city), and charges agent i the payment $p_i = \max_a \sum_{j \neq i} v_j(a) - \sum_{j \neq i} v_j(a^*)$, where the sums are over all agents other than i (so, for a resident, they include the city). Since VCG is strategyproof, we assume that the residents report their true values. For a resident i, let $w_i = \sum_{j \neq i} v_j$ denote the total value of the other residents (the city is not included).

(a) If the bridge is built, the total value of the $n+1$ agents is $\sum_i v_i - C$, whereas if it is not built, the total value is 0. VCG chooses an outcome with the higher total value. Hence, if VCG builds the bridge, we must have $\sum_i v_i - C \geq 0$, i.e., $\sum_i v_i \geq C$. (Conversely, VCG builds the bridge whenever $\sum_i v_i \geq C$: if $\sum_i v_i > C$, building the bridge is the unique optimal outcome, and if $\sum_i v_i = C$, both outcomes are optimal and we assumed that VCG builds the bridge.)

(b) Fix a resident i. The agents other than i (the other residents and the city) have total value $w_i - C$ if the bridge is built and 0 if it is not. Hence, the first term in the payment is $\max(w_i - C, 0)$, irrespective of the outcome chosen. We consider two cases.

The bridge is not built: By part (a), $\sum_j v_j - C \leq 0$, so $w_i \leq w_i + v_i \leq C$ (as $v_i \geq 0$). Hence, the first term is $\max(w_i - C, 0) = 0$. The second term (the total value of the others under the chosen outcome) is also 0. Hence, $p_i = 0$.

The bridge is built: The total value of the others under the chosen outcome is $w_i - C$. Hence, $p_i = \max(w_i - C, 0) - (w_i - C) = \max(C - w_i, 0)$. This is non-zero only if $C > w_i = \sum_{j \neq i} v_j$. Since the bridge is built, part (a) also gives $\sum_j v_j \geq C$. Hence, i is pivotal.

In both cases, a non-zero payment implies that i is pivotal.

(c) Let i be a pivotal resident. As $\sum_j v_j \geq C$, the bridge is built (by part (a) and the tie convention in the problem statement), and by part (b), the payment charged to i is

$$p_i = C - \sum_{j \neq i} v_j.$$

Note that $0 < p_i \leq v_i$ because $\sum_j v_j \geq C > \sum_{j \neq i} v_j$.

(d) If the bridge is not built, then by part (b) all payments are zero. Suppose the bridge is built. If no resident is pivotal, then by part (b) all payments are zero, which is at most C. Otherwise, fix

a pivotal resident k. By part (c), $p_k = C - w_k$. Every other resident $j \neq k$ is charged either 0 (if she is not pivotal) or $p_j \leq v_j$ (if she is pivotal, by part (c)), so $p_j \leq v_j$ in either case since $v_j \geq 0$. Hence,

$$\sum_{j=1}^{n} p_j = p_k + \sum_{j \neq k} p_j \leq (C - w_k) + \sum_{j \neq k} v_j = (C - w_k) + w_k = C.$$

Q3 [20 Points] Sponsored Search

Consider the following sponsored search auction with three slots s_1, s_2, s_3, and four bidders A, B, C, D. The *clickthrough rate* of each slot and the *value per click* for each bidder are as follows.

Recall that in sponsored search, a bidder with value per click v that is assigned a slot with clickthrough rate c and charged payment *per click* p has expected value $v \cdot c$, expected payment $p \cdot c$, and expected utility $(v - p) \cdot c$.

Values per Click	Bidders	Slots	Clickthrough Rates
4	A	s_1	0.3
8	B	s_2	0.5
5	C	s_3	0.2
2	D		

(a) [5 Points] Under VCG, determine the allocation of slots to bidders and the payments charged to bidders *per click*. What is the total expected revenue under this mechanism? Show your calculation for full credit.

(b) [5 Points] Under GSP, determine the allocation of slots to bidders and the payments charged to bidders *per click*, assuming bidders bid truthfully. What is the total expected revenue under this mechanism? Show your calculation for full credit.

(c) [10 Points] Note that if bidders bid truthfully, GSP obtains at least as much expected revenue as VCG in this example. Show that this is true generally for all sponsored search auctions, not just the one drawn above. For this part, you can use the notation from class: there are k slots and n bidders, bidder j has value per click v_j, and slot j has clickthrough rate c_j. Without loss of generality, assume that $v_1 \geq \ldots \geq v_n$ and $c_1 \geq \ldots \geq c_k$, and, for convenience, let $v_j = 0$ for $j > n$ and $c_{k+1} = 0$.

Solution to Q3

(a) VCG:

Allocation: $B \to s_2, C \to s_1, A \to s_3$.

Expected Payments:

$p_B = 5 \cdot (0.5 - 0.3) + 4 \cdot (0.3 - 0.2) + 2 \cdot (0.2 - 0) = 1.8$

$p_C = 4 \cdot (0.3 - 0.2) + 2 \cdot (0.2 - 0) = 0.8$

$p_A = 2 \cdot (0.2 - 0) = 0.4$

$p_D = 0$

Total expected revenue: 3

Payments per click: $B \to \frac{1.8}{0.5} = 3.6, C \to \frac{0.8}{0.3} = 2.6667, A \to \frac{0.4}{0.2} = 2, D \to 0.$

(b) GSP:

Allocation: Same as VCG: $B \to s_2, C \to s_1, A \to s_3$.

Payments per click: $B \to 5, C \to 4, A \to 2, D \to 0.$
Expected Payments:

$p_B = 5 \cdot 0.5 = 2.5, p_C = 4 \cdot 0.3 = 1.2, p_A = 2 \cdot 0.2 = 0.4, p_D = 0$

Total expected revenue: 4.1

(c) Recall that the expected payment under GSP is $\sum_{i=1}^{k} v_{i+1} \cdot c_i$, whereas that under VCG is $\sum_{i=1}^{k} \sum_{j=i+1}^{k+1} v_j \cdot (c_{j-1} - c_j)$, using the notation and conventions from the problem statement. Now, we note that

$$\begin{aligned}
\sum_{i=1}^{k} \sum_{j=i+1}^{k+1} v_j \cdot (c_{j-1} - c_j) &\leq \sum_{i=1}^{k} \sum_{j=i+1}^{k+1} v_{i+1} \cdot (c_{j-1} - c_j) \\
&= \sum_{i=1}^{k} v_{i+1} \sum_{j=i+1}^{k+1} (c_{j-1} - c_j) \\
&= \sum_{i=1}^{k} v_{i+1} \cdot (c_i - c_{k+1}) \\
&= \sum_{i=1}^{k} v_{i+1} \cdot c_i.
\end{aligned}$$

Q4 [20 Points] Bayes-Nash Equilibria

Consider an auction between a buyer and a seller. Seller s wants to sell a good for which his own value is v_s. Buyer b has value v_b for the good. Both v_s and v_b are drawn independently from the uniform distribution $U[0, 1]$. Each player knows his own value, but only knows the distribution of the other player's value.

Here's how the auction goes down. The seller and the buyer independently write down some prices (say p_s and p_b, respectively) on different pieces of paper. Then, they compare their prices. If it turns out that $p_s \leq p_b$, then the item is sold at price p_s, the seller gets utility $p_s - v_s$, and the buyer

gets utility $v_b - p_s$. On the other hand, if $p_s > p_b$, the item remains unsold and both players get zero utility.

Is there a Bayes-Nash equilibrium in which the buyer honestly writes down her value (i.e. $p_b = v_b$)? If so, what is the seller's price p_s in this equilibrium? Show your calculation.

[Hint: Assume $p_b = v_b$, and derive the optimal price p_s for the seller to write. Then, check whether given this strategy of the seller, $p_b = v_b$ is indeed the optimal price for the buyer to write.]

Solution to Q4

Let us assume that there is a Bayes-Nash equilibrium (p_s, p_b) in which $p_b = v_b$.

Seller's optimal strategy: Given p_b, the choice of p_s must maximize seller's expected utility.

First, note that the optimal p_s is at most 1 because $p_b = v_b \leq 1$, so any $p_s > 1$ will yield zero utility. Thus, we assume $0 \leq p_s \leq 1$.

The seller's expected utility is

$$\begin{aligned}&\Pr[p_s \leq p_b] \cdot \mathbb{E}[p_s - v_s | p_s \leq p_b] + \Pr[p_s \geq p_b] \cdot 0 \\ &= \Pr[v_b \geq p_s] \cdot (p_s - v_s) = (1 - p_s) \cdot (p_s - v_s).\end{aligned}$$

This is a concave quadratic function of p_s. To find the choice of p_s that maximizes it, we set the derivative to 0 and obtain:

$$1 - p_s = p_s - v_s \Rightarrow p_s = \frac{1 + v_s}{2}.$$

Note that this lies in $[1/2, 1] \subseteq [0, 1]$.

Hence, our candidate BNE is (p_s, p_b) where $p_s = (1 + v_s)/2$ and $p_b = v_b$.

Buyer's optimal strategy: To establish that this is indeed a BNE, we need to verify that $p_b = v_b$ is optimal for the buyer assuming $p_s = (1 + v_s)/2$. Note that p_s is uniformly distributed over $[1/2, 1]$. We consider three ranges of the buyer's price p_b.

If $p_b < 1/2$: Since $p_s \geq 1/2 > p_b$, the item is never sold, and the buyer's utility is 0.

If $p_b > 1$: Since $p_s \leq 1 < p_b$, the item is always sold at price p_s, which does not depend on p_b. Hence, this gives the same utility as $p_b = 1$, which is covered by the next case.

If $1/2 \leq p_b \leq 1$: The buyer's expected utility is

$$\begin{aligned}u(p_b) &= \Pr[p_s \leq p_b] \cdot \mathbb{E}[v_b - p_s | p_s \leq p_b] + \Pr[p_s \geq p_b] \cdot 0 \\ &= \Pr\left[\frac{1 + v_s}{2} \leq p_b\right] \cdot \left(v_b - \mathbb{E}\left[\frac{1 + v_s}{2} \,\middle|\, \frac{1 + v_s}{2} \leq p_b\right]\right) \\ &= \Pr[v_s \leq 2p_b - 1] \cdot \left(v_b - \frac{1 + \mathbb{E}[v_s | v_s \leq 2p_b - 1]}{2}\right) \\ &= (2p_b - 1) \cdot \left(v_b - \frac{1 + p_b - 1/2}{2}\right) = (2p_b - 1) \cdot \left(v_b - \frac{p_b}{2} - \frac{1}{4}\right),\end{aligned}$$

where we used $\Pr[X \le z] = z$ and $\mathbb{E}[X|X \le z] = z/2$ for $X \sim U[0,1]$ and $z = 2p_b - 1 \in [0,1]$. Taking the derivative,

$$u'(p_b) = 2 \cdot \left(v_b - \frac{p_b}{2} - \frac{1}{4} \right) - \frac{1}{2} \cdot (2p_b - 1) = 2(v_b - p_b),$$

and $u''(p_b) = -2 < 0$. Hence, u is strictly concave, and it is maximized over the reals at $p_b = v_b$.

- If $v_b \ge 1/2$, then $p_b = v_b$ lies in the range $[1/2, 1]$, so it maximizes u over this range. Further, $u(v_b) = (2v_b - 1)^2/4 \ge 0$, which is at least the utility of 0 obtained from any $p_b < 1/2$.
- If $v_b < 1/2$, then $u'(p_b) < 0$ for all $p_b \in [1/2, 1]$, so u is decreasing on this range, and $u(1/2) = 0$. Hence, $u(p_b) \le 0$ for all $p_b \in [1/2, 1]$, so the best the buyer can do is to obtain utility 0, and $p_b = v_b < 1/2$ achieves it.

Hence, $p_b = v_b$ is optimal for the buyer for every value $v_b \in [0,1]$, and there is a BNE with $p_s = (1 + v_s)/2$ and $p_b = v_b$.

Q5 [20 Points] Second-Price Auction with a Twist

In class, we proved that the second-price auction is strategyproof. Consider using the second-price auction for selling a single item, but with a twist. After all the bids are submitted, the auctioneer is supposed to sell the item to the highest bidder at price equal to the second-highest bid. The auctioneer usually does that, but with some probability p, makes an error and sells the item to the *second-highest bidder* at price equal to the second-highest bid (i.e., the winner's own bid).

You show up to such an auction and see that there is only one bidder other than yourself. Your value for the item is $v \in [0,1]$. You know that this other bidder has a value v' drawn from the uniform distribution $U[0,1]$ and that the other bidder, unaware of the fact that the auctioneer has this tendency to make an error, is going to truthfully bid their value v'. What is the optimal bid that you should submit? Your answer can depend on your own value v and the probability of error p.

Solution to Q5

Suppose your bid is $b \ge 0$.

Any bid $b > 1$ exceeds $v' \le 1$ with probability 1, and hence results in exactly the same outcome (with probability 1) as the bid $b = 1$. Thus, we assume $b \le 1$.

With probability $1 - p$, you win the item when you are the highest bidder ($b \ge v'$) and pay the second-highest bid (v'). With the remaining probability p, you win the item when you are the second-highest bidder ($b \le v'$) and pay the second-highest bid (b).

Hence, your expected utility with bid b is

$$\begin{aligned} U(b) =& (1-p) \cdot \Pr[b \ge v'] \cdot \mathbb{E}[v - v' | b \ge v'] & + & \quad p \cdot \Pr[b \le v'] \cdot (v - b) \\ =& \quad (1-p) \cdot b \cdot (v - b/2) & + & \quad p \cdot (1-b) \cdot (v - b). \end{aligned}$$

To find the optimal value of b, we take the derivative with respect to b:

$$U'(b) = (1-p)\cdot(v-b) - p\cdot(1+v-2b) = \big((1-2p)\cdot v - p\big) - (1-3p)\cdot b,$$

and $U''(b) = 3p-1$. Setting $U'(b)=0$ gives $b = b^*$, where

$$b^* = \frac{(1-2p)\cdot v - p}{1-3p}.$$

We take two cases.

Case 1: $\mathbf{p < 1/3}$. In this case, $U'(b) = (1-3p)\cdot(b^*-b)$, so the utility is increasing when $b \le b^*$ and decreasing when $b \ge b^*$. Further, $b^* \le 1$ because $(1-2p)\cdot v - p \le 1-3p$ is equivalent to $(1-2p)\cdot v \le 1-2p$, which holds since $v \le 1$ and $1-2p > 0$. Hence, the optimal bid is as follows:

- If $b^* \ge 0$ (i.e., $v \ge p/(1-2p)$), submit b^*.
- If $b^* < 0$, submit 0 (this lowest bid has the highest utility on $[0,1]$).

Case 2: $\mathbf{p \ge 1/3}$. In this case, $U''(b) \ge 0$, so U is convex on $[0,1]$. That is, $b = b^*$ (if it is in the range) actually yields the lowest (not the highest) utility. The highest utility is at one of the two endpoints, 0 or 1. The expected utility at $b=0$ is $p\cdot v$ and the expected utility at $b=1$ is $(1-p)\cdot(v-1/2)$. Their difference is

$$U(0) - U(1) = (2p-1)\cdot v + \frac{1-p}{2},$$

which is linear in v. At $v=0$, it equals $(1-p)/2 \ge 0$, and at $v=1$, it equals $(3p-1)/2 \ge 0$. Hence, it is non-negative for every $v \in [0,1]$, and the optimal bid is 0.

Solution summary: the optimal bid is $\max\left(0, \frac{(1-2p)v-p}{1-3p}\right)$ if $p < 1/3$, and 0 if $p \ge 1/3$.

Bonus Question

Q6 [20 Points] VCG for Housing Reassignment

Consider the following housing reassignment setting. There are n agents. Each agent i owns a house that she values at $v_{i,i} \geq 0$, and values the house of agent j at $v_{i,j} \geq 0$. When a set of agents S participate in the mechanism, the set of possible outcomes consist of all bijections $\sigma : S \to S$, where $\sigma(i) = j$ denotes that agent i is given the house of agent j; $\sigma(i) = i$ would indicate that agent i retains her own house. Note that in this setting, the set of outcomes changes if one of the agents is absent because that agent's house would then no longer be available for reassignment. The value derived by a participating agent i under an outcome σ is $v_{i,\sigma(i)}$, and if she is asked to pay p_i, her net utility would be $v_{i,\sigma(i)} - p_i$.

Suppose all agents participate in the VCG mechanism and p_i is the payment charged to agent i. Note that $p_i < 0$ and $p_i > 0$ are both possible in this setting. Either prove or disprove by providing a counter-example: in every housing reassignment instance, $\sum_i p_i \leq 0$ (i.e., VCG will never make overall profit).

[Hint: For a set S of agents, let $W(S)$ denote the maximum total value of the agents in S over all bijections $\sigma : S \to S$. Express $\sum_i p_i$ in terms of $W(\{1,\ldots,n\})$ and $W(\{1,\ldots,n\} \setminus \{i\})$ for $i \in \{1,\ldots,n\}$. Then, think of each outcome as a perfect matching between agents and houses in a bipartite graph, and consider the union of the optimal outcomes for the sets $\{1,\ldots,n\} \setminus \{i\}$ over all i (choosing one optimal outcome for each i). What can you say about the structure of this graph?]

Solution to Q6

The claim is true. For a set S of agents, let $W(S) = \max_\sigma \sum_{i \in S} v_{i,\sigma(i)}$, where the maximum is over all bijections $\sigma : S \to S$ (and $W(\emptyset) = 0$). Let $[n] = \{1,\ldots,n\}$, let σ^* be the outcome chosen by VCG, i.e., a bijection maximizing the welfare of all n agents, and let $W = W([n])$.

Step 1: Reduction. When agent i is absent, the outcomes available to the others are the bijections of $[n] \setminus \{i\}$, so the maximum welfare of the others is $W([n] \setminus \{i\})$. Under σ^*, the welfare of the others is $W - v_{i,\sigma^*(i)}$. Hence, $p_i = W([n] \setminus \{i\}) - (W - v_{i,\sigma^*(i)})$. Summing over i and using $\sum_i v_{i,\sigma^*(i)} = W$,

$$\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} W([n] \setminus \{i\}) - nW + W = \sum_{i=1}^{n} W([n] \setminus \{i\}) - (n-1) \cdot W.$$

Thus, it suffices to show that $\sum_{i=1}^{n} W([n] \setminus \{i\}) \leq (n-1) \cdot W$. For $n = 1$, both sides are 0, so assume $n \geq 2$.

Step 2: A regular bipartite multigraph. For each $i \in [n]$, let τ_i be a bijection of $[n] \setminus \{i\}$ with $\sum_{j \neq i} v_{j,\tau_i(j)} = W([n] \setminus \{i\})$. Consider the bipartite multigraph G with the agents $[n]$ on the left and the houses $[n]$ on the right, which has, for every $i \in [n]$ and every $j \neq i$, an edge between agent j and house $\tau_i(j)$ with weight $v_{j,\tau_i(j)}$ (parallel edges are allowed). Then the total weight of the edges of G is exactly $\sum_{i=1}^{n} W([n] \setminus \{i\})$.

Every agent j has exactly one edge for each $i \neq j$, so she has degree $n-1$. Every house h is assigned by τ_i to exactly one agent for each $i \neq h$ (as τ_i is a bijection of $[n] \setminus \{i\}$, which contains h), so she has degree $n-1$ as well. Hence, G is $(n-1)$-regular.

Step 3: Decomposition into perfect matchings. We claim that every k-regular bipartite multigraph with $k \geq 1$ and n vertices on each side has a perfect matching. Let X be a set of left vertices. The $k|X|$ edges incident to X all end in the neighborhood $N(X)$, which can absorb at most $k|N(X)|$ edges. Hence, $|N(X)| \geq |X|$, and Hall's theorem gives a perfect matching.

Removing a perfect matching from a k-regular bipartite multigraph leaves a $(k-1)$-regular bipartite multigraph. Applying this repeatedly to G, we partition the edges of G into $n-1$ perfect matchings $M_1, \ldots, M_{n-1}$. Each M_t matches every agent to a distinct house, so it corresponds to a bijection $\sigma_t : [n] \to [n]$, whose welfare (the total weight of M_t) is at most W by the definition of W.

Step 4: Conclusion. Since the matchings partition the edges of G,

$$\sum_{i=1}^{n} W([n] \setminus \{i\}) = \sum_{t=1}^{n-1} \sum_{i=1}^{n} v_{i,\sigma_t(i)} \leq (n-1) \cdot W.$$

By Step 1, $\sum_i p_i \leq 0$, i.e., VCG never makes an overall profit.

Remark. Individual payments can have either sign. For $n = 2$ with $v_{1,1} = 3, v_{1,2} = 2, v_{2,1} = 3, v_{2,2} = 1$, VCG swaps the houses ($W = 5$), and $p_1 = W(\{2\}) - (W - v_{1,2}) = 1 - 3 = -2$ and $p_2 = W(\{1\}) - (W - v_{2,1}) = 3 - 2 = 1$.