CSC304 Lecture 9

Mechanism Design with Money: More VCG examples; greedy approximation of VCG

VCG Recap

- $f(\tilde{v}) = a^* = \text{argmax}_{a \in A} \sum_i \tilde{v}_i(a)$
  - Choose the allocation maximizing *reported* welfare
- $p_i(\tilde{v}) = \left[\max_a \sum_{j \neq i} \tilde{v}_j(a)\right] - \left[\sum_{j \neq i} \tilde{v}_j(a^*)\right]$
  - Each agent pays the loss to others due to her presence
- Four properties
  - Strategyproofness
  - Individual rationality (IR)
  - No payments to agents
  - Welfare maximization

Seller as Agent

- Seller (S) wants to sell his car (c) to buyer (B)

- Seller has a value for his own car: $v_S(c)$
  - Individual rationality for the seller mandates that seller must get revenue at least $v_S(c)$

- Idea: Add seller as another agent, and make his values part of the welfare calculations!

Seller as Agent

$v_S(c) = 3$ $\qquad$ $v_B(c) = 5$

- What if…
  - ➢ We give the car to buyer when $v_B(c) > v_S(c)$ and
  - ➢ Buyer pays seller $v_B(c)$: Not strategyproof for buyer!
  - ➢ Buyer pays seller $v_S(c)$: Not strategyproof for seller!

What would VCG do?

$v_S(c) = 3$

$v_B(c) = 5$

- Allocation?
  - Buyer gets the car (welfare = 5)
- Payment?
  - Buyer pays: $3 - 0 = 3$
  - Seller pays: $0 - 5 = -5$

Mechanism takes \$3 from buyer, and gives \$5 to the seller!

- Need external subsidy

Problems with VCG

- Difficult to understand
  - Need to reason about what welfare maximizing allocation in agent i's absence
- Does not care about revenue
  - Although we can lower bound its revenue
- With sellers as agents, need subsidy
  - With no subsidy, cannot get the other three properties
- Might be NP-hard to compute

Single-Minded Bidders

- Combinatorial auction for a set of m items S

- Each agent i has two private values (v_i, S_i)
  - $S_i \subseteq S$ is the set of desired items
  - When given a bundle of items A_i, agent has value v_i if $S_i \subseteq A_i$ and 0 otherwise
  - "Single-minded"

- Welfare-maximizing allocation
  - Agent i either gets S_i or nothing
  - Find a subset of players with the highest total value such that their desired sets are disjoint

Single-Minded Bidders

- Weighted Independent Set (WIS) problem
  - Given a graph with weights on nodes, find an independent set of nodes with the maximum weight
  - Known to be NP-hard
- Easy to reduce our problem to WIS
  - Not even $O(m^{0.5-\epsilon})$ approximation of welfare unless $NP \subseteq ZPP$
- Luckily, there's a simple, $\sqrt{m}$-approximation greedy algorithm

Greedy Algorithm

- Input: (v_i, S_i) for each agent i
- Output: Agents with mutually independent S_i

- Greedy Algorithm:
  - Sort the agents in a specific order (we'll see).
  - Relabel them as $1,2, \ldots, n$ in this order.
  - $W \leftarrow \emptyset$
  - For $i = 1, \ldots, n$:
    - If $S_i \cap S_j = \emptyset$ for every $j \in W$, then $W \leftarrow W \cup \{i\}$
  - Give agents in W their desired items.

Greedy Algorithm

- Sort by what?
- We want to satisfy agents with higher values.
  - $v_1 \geq v_2 \geq \cdots \geq v_n \Rightarrow m$-approximation ☹
- But we don't want to exhaust too many items.
  - $\frac{v_1}{|S_1|} \geq \frac{v_2}{|S_2|} \geq \cdots \frac{v_n}{|S_n|} \Rightarrow m$-approximation ☹
- $\sqrt{m}$-approximation : $\frac{v_1}{\sqrt{|S_1|}} \geq \frac{v_2}{\sqrt{|S_2|}} \geq \cdots \frac{v_n}{\sqrt{|S_n|}}$?

[Lehmann et al. 2011]

Proof of Approximation

- Definitions
  - OPT = Agents satisfied by the optimal algorithm
  - W = Agents satisfied by the greedy algorithm
  - For $i \in W$,
$$OPT_i = \{j \in OPT, j \geq i : S_i \cap S_j \neq \emptyset\}$$
- Claim 1: $OPT \subseteq \cup_{i \in W} OPT_i$
- Claim 2: It is enough to show that $\forall i \in W$
$$\sqrt{m} \cdot v_i \geq \Sigma_{j \in OPT_i} v_j$$
- Observation: For $j \in OPT_i$, $v_j \leq v_i \cdot \frac{\sqrt{|S_j|}}{\sqrt{|S_i|}}$

Proof of Approximation

- Summing over all $j \in OPT_i$:

$$\Sigma_{j \in OPT_i} v_j \leq \frac{v_i}{\sqrt{|S_i|}} \cdot \Sigma_{j \in OPT_i} \sqrt{|S_j|}$$

- Using Cauchy-Schwarz ($\Sigma_i x_i y_i \leq \sqrt{\Sigma_i x_i^2} \cdot \sqrt{\Sigma_i y_i^2}$)

$$\Sigma_{j \in OPT_i} \sqrt{|S_j| \cdot 1} \leq \sqrt{|OPT_i|} \cdot \sqrt{\Sigma_{j \in OPT_i} |S_j|}$$

$$\leq \sqrt{|S_i|} \cdot \sqrt{m}$$

Strategyproofness

- Agent i pays $p_i = v_{j^*} \cdot \sqrt{\frac{|S_i|}{|S_{j^*}|}}$
  - j^* is the smallest index j such that j is currently not selected by greedy but would be selected if we remove (v_i, S_i) from the system
  - Exercise: Show that we must have $j^* > i$
  - Exercise: Show that $S_i \cap S_{j^*} \neq \emptyset$
  - Another interpretation: p_i = lowest value i can report and still win

Strategyproofness

- Critical payment
  - Charge each agent the lowest value they can report and still win
- Monotonic allocation
  - If agent i wins when reporting (v_i, S_i), she must win when reporting $v_i' \geq v_i$ and $S_i' \subseteq S_i$.
  - Greedy allocation rule satisfies this.
- Theorem: Critical payment + monotonic allocation rule imply strategyproofness.

Moral

- VCG can sometimes be too difficult to implement
  - May look into approximately maximizing welfare
  - As long as the allocation rule is monotone, we can charge critical payments to achieve strategyproofness
  - Note: approximation is needed for computational reasons
- Later in mechanism design without money…
  - We will not be able to use payments to achieve strategyproofness
  - Hence, we will need to approximate welfare just to get strategyproofness, even without any computational restrictions