CSC304 Lecture 6

Game Theory : Zero-Sum Games, The Minimax Theorem

Zero-Sum Games

- Special case of games
  - Total reward to all players is constant in every outcome
  - Without loss of generality, sum of rewards = 0
  - Inspired terms like "zero-sum thinking" and "zero-sum situation"

- Focus on two-player zero-sum games (2p-zs)
  - "The more I win, the more you lose"

Zero-Sum Games

Zero-sum game: Rock-Paper-Scissor

P2 / P1	Rock	Paper	Scissor
Rock	(0 , 0)	(-1 , 1)	(1 , -1)
Paper	(1 , -1)	(0 , 0)	(-1 , 1)
Scissor	(-1 , 1)	(1 , -1)	(0 , 0)

Non-zero-sum game: Prisoner's dilemma

John / Sam	Stay Silent	Betray
Stay Silent	(-1 , -1)	(-3 , 0)
Betray	(0 , -3)	(-2 , -2)

Zero-Sum Games

- Why are they interesting?
  - Many physical games we play are zero-sum: chess, tic-tac-toe, rock-paper-scissor, …
  - (win, lose), (lose, win), (draw, draw)
  - (1, -1), (-1, 1), (0, 0)

- Why are they technically interesting?
  - We'll see.

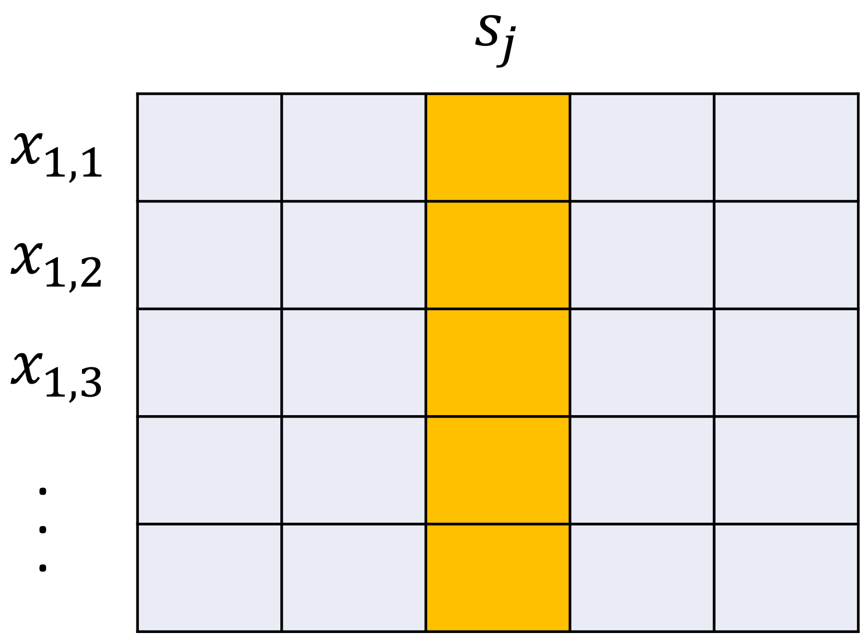
Zero-Sum Games

- Reward for P2 = - Reward for P1
  - Only need to write a single entry in each cell (say reward of P1)
  - Hence, we get a matrix A
  - P1 wants to maximize the value, P2 wants to minimize it

P1 \ P2	Rock	Paper	Scissor
Rock	**0**	**-1**	**1**
Paper	**1**	**0**	**-1**
Scissor	**-1**	**1**	**0**

Rewards in Matrix Form

- Say P1 uses mixed strategy $x_1 = (x_{1,1}, x_{1,2}, \dots)$
  - What are the rewards of P1 for different actions chosen by P2?

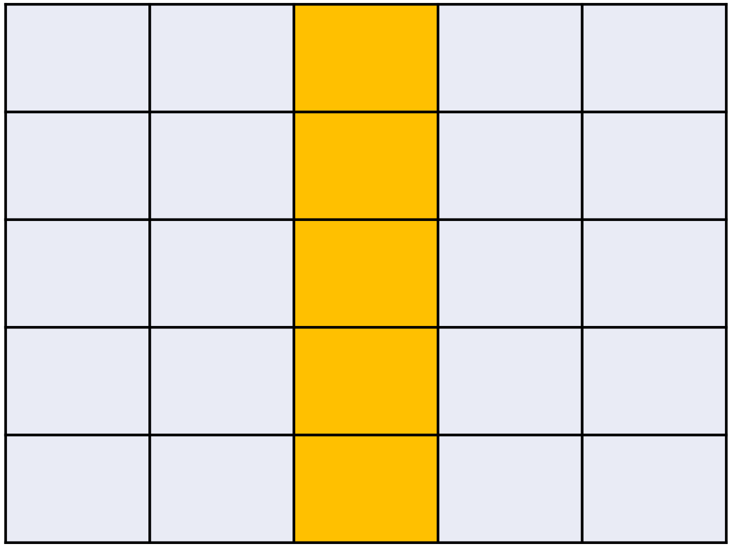


Rewards in Matrix Form

- Say P1 uses mixed strategy $x_1 = (x_{1,1}, x_{1,2}, \ldots)$
  - What are the rewards for P1 corresponding to different possible actions of P2?

$$[x_{1,1}, x_{1,2}, x_{1,3}, \ldots] \quad *$$

s_j



- Reward of P1 when P2 chooses $s_j = \left(x_1^T * A\right)_j$

Rewards in Matrix Form

- Reward for P1 when…
  - P1 uses a mixed strategy x_1
  - P2 uses a mixed strategy x_2

$$\left[\left(x_1^T * A\right)_1, \left(x_1^T * A\right)_2, \left(x_1^T * A\right)_3 \dots\right] \quad * \quad \begin{bmatrix} x_{2,1} \\ x_{2,2} \\ x_{2,3} \\ \vdots \end{bmatrix}$$

$$= \quad x_1^T * A * x_2$$

How would the two players act in this zero-sum game?

John von Neumann, 1928

Maximin Strategy

- Worst-case thinking by P1…
  - Suppose I don't know anything about what P2 would do.
  - If I choose a mixed strategy x_1, in the worst case, P2 chooses an x_2 that minimizes my reward (i.e., maximizes his reward)
  - Let me choose x_1 to maximize this "worst-case reward"

$$V_1^* = \max_{x_1} \min_{x_2} x_1^T * A * x_2$$

Maximin Strategy

$$V_1^* = \max_{x_1} \min_{x_2} x_1^T * A * x_2$$

- V_1^* : maximin value of P1
- x_1^* (maximizer) : maximin strategy of P1
- "By playing x_1^*, I guarantee myself at least V_1^*"

- P2 can similarly think of her worst case.

Maximin vs Minimax

Player 1

Choose x_1 to maximize my reward in the worst case over P2's strategy

$$V_1^* = \max_{x_1} \min_{x_2} x_1^T * A * x_2$$

x_1^*

Player 2

Choose x_2 to minimize P1's reward in the worst case over P1's strategy

$$V_2^* = \min_{x_2} \max_{x_1} x_1^T * A * x_2$$

x_2^*

Question: Relation between V_1^* and V_2^*?

Maximin vs Minimax

$$V_1^* = \max_{x_1} \min_{x_2} x_1^T * A * x_2 \qquad V_2^* = \min_{x_2} \max_{x_1} x_1^T * A * x_2$$

x_1^* x_2^*

- What if (P1,P2) play (x_1^*, x_2^*) simultaneously?
  - P1's guarantee: P1 must get reward at least V_1^*
  - P2's guarantee: P1 must get reward at most V_2^*
  - $V_1^* \leq V_2^*$

Maximin vs Minimax

$$V_1^* = \max_{x_1} \; \min_{x_2} \; x_1^T * A * x_2 \qquad\qquad V_2^* = \min_{x_2} \; \max_{x_1} \; x_1^T * A * x_2$$

x_1^* $\qquad\qquad$ x_2^*

- Another way to see this:

$$V_1^* = \min_{x_2} \; (x_1^*)^T * A * x_2 \; \le \; (x_1^*)^T * A * x_2^*$$

$$\le \; \max_{x_1} \; x_1^T * A * x_2^* = \; V_2^*$$

The Minimax Theorem

- Jon von Neumann [1928]

- Theorem: For any 2p-zs game,
  - $V_1^* = V_2^* = V^*$ (called the minimax value of the game)
  - Set of Nash equilibria =

    $\{(\mathrm{x}_1^*, \mathrm{x}_2^*) : \mathrm{x}_1^* = \text{maximin for P1}, \mathrm{x}_2^* = \text{minimax for P2}\}$

- Corollary: x_1^* is best response to x_2^* and vice-versa.

The Minimax Theorem

- An alternative interpretation of maximin strategies
  - x_1^* is the strategy P1 would choose if she were to commit to her strategy *first,* and P2 were to choose her strategy after observing P1's strategy
  - Similarly, x_2^* is the strategy P2 would choose if P2 were to commit first
  - However, x_1^* and x_2^* are best responses to each other.
  - Hence, in zero-sum games, it doesn't matter which player commits first (or if both players commit together).

The Minimax Theorem

- Jon von Neumann [1928]

  "As far as I can see, there could be no theory of games … without that theorem …

  I thought there was nothing worth publishing until the Minimax Theorem was proved"

Proof of the Minimax Theorem

- Simpler proof using Nash's theorem
  - But predates Nash's theorem
- Suppose $(\tilde{x}_1, \tilde{x}_2)$ is a NE
  - Note: A Nash equilibrium exists due to Nash's theorem
- P1 gets value $\tilde{v} = (\tilde{x}_1)^T A\, \tilde{x}_2$
- $\tilde{x}_1$ is best response for P1 : $\tilde{v} = \max_{x_1} (x_1)^T A\, \tilde{x}_2$
- $\tilde{x}_2$ is best response for P2 : $\tilde{v} = \min_{x_2} (\tilde{x}_1)^T A\, x_2$

Proof of the Minimax Theorem

$$V_2^* = \min_{x_2} \max_{x_1} x_1^T * A * x_2 \leq$$

$$\max_{x_1} (x_1)^T A\, \tilde{x}_2 = \tilde{v} = \min_{x_2} (\tilde{x}_1)^T A\, x_2$$

$$\leq \max_{x_1} \min_{x_2} x_1^T * A * x_2 = V_1^*$$

- But we already saw $V_1^* \leq V_2^*$
  - $V_1^* = V_2^*$

Proof of the Minimax Theorem

$$V_2^* = \min_{x_2} \max_{x_1} x_1^T * A * x_2 =$$

$$\max_{x_1} (x_1)^T A \, \tilde{x}_2 = \tilde{v} = \min_{x_2} (\tilde{x}_1)^T A \, x_2$$

$$= \max_{x_1} \min_{x_2} x_1^T * A * x_2 = V_1^*$$

- When $(\tilde{x}_1, \tilde{x}_2)$ is a NE, $\tilde{x}_1$ and $\tilde{x}_2$ must be maximin and minimax strategies for P1 and P2, respectively.
- The reverse direction is also easy to prove.

Computing Nash Equilibria

- Recall that in general games, computing a Nash equilibrium is hard even with two players.

- For 2p-zs games, a Nash equilibrium can be computed in polynomial time.
  - Polynomial in #actions of the two players: m_1 and m_2
  - Exploits the fact that Nash equilibrium is simply composed of maximin strategies, which can be computed using linear programming

Computing Nash Equilibria

Maximize v

Subject to

$$\left(x_1^T A\right)_j \geq v,\ j \in \{1, \dots, m_2\}$$

$$x_1(1) + \cdots + x_1(m_1) = 1$$

$$x_1(i) \geq 0, i \in \{1, \dots, m_1\}$$

Limitation of Minimax Theorem

- It only makes sense to play your maximin strategy x_1^* if you know the other player is rational enough to choose the best response x_2^*
- If the other player is choosing a suboptimal strategy x_2, the best response to x_2 might be different
- This is what computer programs playing Chess exploit when they play against human players

Minimax Theorem in Real Life?

Kicker \ Goalie	L	R
L	0.58	0.95
R	0.93	0.70

Kicker

Maximize v

Subject to

$0.58p_L + 0.93p_R \geq v$

$0.95p_L + 0.70p_R \geq v$

$p_L + p_R = 1$

$p_L \geq 0, p_R \geq 0$

Goalie

Minimize v

Subject to

$0.58q_L + 0.95q_R \leq v$

$0.93q_L + 0.70q_R \leq v$

$q_L + q_R = 1$

$q_L \geq 0, q_R \geq 0$

Minimax Theorem in Real Life?

Kicker \ Goalie	L	R
L	0.58	0.95
R	0.93	0.70

Kicker

Maximin:

$p_L = 0.38, p_R = 0.62$

Reality:

$p_L = 0.40, p_R = 0.60$

Goalie

Maximin:

$q_L = 0.42, q_R = 0.58$

Reality:

$p_L = 0.423, q_R = 0.577$

Some evidence that people may play minimax strategies.

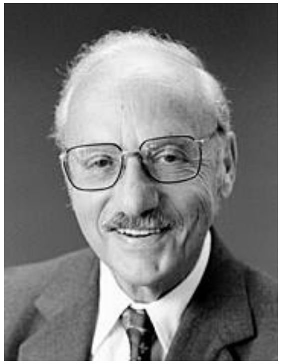
Minimax Theorem

- We proved it using Nash's theorem
  - Cheating. Typically, Nash's theorem (for the special case of 2p-zs games) is proved using the minimax theorem.

- Useful for proving Yao's principle, which provides lower bound for randomized algorithms

- Equivalent to linear programming duality

John von Neumann



George Dantzig

von Neumann and Dantzig

George Dantzig loves to tell the story of his meeting with John von Neumann on October 3, 1947 at the Institute for Advanced Study at Princeton. Dantzig went to that meeting with the express purpose of describing the linear programming problem to von Neumann and asking him to suggest a computational procedure. He was actually looking for methods to benchmark the simplex method. Instead, he got a 90-minute lecture on Farkas Lemma and Duality (Dantzig's notes of this session formed the source of the modern perspective on linear programming duality). Not wanting Dantzig to be completely amazed, von Neumann admitted:

"I don't want you to think that I am pulling all this out of my sleeve like a magician. I have recently completed a book with Morgenstern on the theory of games. What I am doing is conjecturing that the two problems are equivalent. The theory that I am outlining is an analogue to the one we have developed for games.“

- (Chandru & Rao, 1999)