# CSC304 Lectures 17-18

## Voting 3: Axiomatic, Statistical, and Utilitarian Approaches to Voting

# Recap

- We introduced a plethora of voting rules
  - Plurality
  - Borda
  - Veto
  - $k$-Approval
  - STV
  - Plurality with runoff
  - Kemeny
  - Copeland
  - Maximin
- Which is the right way to aggregate preferences?
  - GS Theorem: There is no good strategyproof voting rule.
  - For now, let us forget about incentives. Let us focus on how to aggregate given truthful votes.

# Recap

- Set of voters $N = \{1, \dots, n\}$
- Set of alternatives $A$, $|A| = m$
- Voter $i$ has a preference ranking $\succ_i$ over the alternatives

| 1 | 2 | 3 |
|---|---|---|
| a | c | b |
| b | a | a |
| c | b | c |

- Preference profile $\vec{\succ}$ = collection of all voter rankings
- Voting rule (social choice function) $f$
  - Takes as input a preference profile $\vec{\succ}$
  - Returns an alternative $a \in A$

# Axiomatic Approach

- Goal: Define a set of reasonable desiderata, and find voting rules satisfying them
  - Ultimate hope: a unique voting rule satisfies the axioms we are interested in!

- Sadly, it's often the opposite case.
  - Many combinations of reasonable axioms cannot be satisfied by any voting rule.
  - GS theorem: nondictatorship + ontoness + strategyproofness = $\emptyset$
  - Arrow's theorem: we'll see
  - …

# Axiomatic Approach

- **Unanimity:** If all voters have the same top choice, that alternative is the winner.

$$(top(\succ_i) = a \;\; \forall i \in N) \Rightarrow f(\overrightarrow{\succ}) = a$$

  - I used $top(\succ_i) = a$ to denote $a \succ_i b \; \forall b \neq a$

- **Pareto optimality:** If all voters prefer $a$ to $b$, then $b$ is not the winner.

$$(a \succ_i b \; \forall i \in N) \Rightarrow f(\overrightarrow{\succ}) \neq b$$

- **Q:** *What is the relation between these axioms?*
  - *Pareto optimality ⇒ Unanimity*

# Axiomatic Approach

- Anonymity: Permuting votes does not change the winner (i.e., voter identities don't matter).
  - E.g., these two profiles must have the same winner:
    {voter 1: $a \succ b \succ c$, voter 2: $b \succ c \succ a$}
    {voter 1: $b \succ c \succ a$, voter 2: $a \succ b \succ c$}
- Neutrality: Permuting the alternative names permutes the winner accordingly.
  - E.g., say $a$ wins on {voter 1: $a \succ b \succ c$, voter 2: $b \succ c \succ a$}
  - We permute all names: $a \rightarrow b$, $b \rightarrow c$, and $c \rightarrow a$
  - New profile: {voter 1: $b \succ c \succ a$, voter 2: $c \succ a \succ b$}
  - Then, the new winner must be $b$.

# Axiomatic Approach

- Neutrality is tricky
  - As we defined it, it is inconsistent with anonymity!
    - Imagine {voter 1: $a \succ b$, voter 2: $b \succ a$}
    - Without loss of generality, say $a$ wins
    - Imagine a different profile: {voter 1: $b \succ a$, voter 2: $a \succ b$}
      - Neutrality: We just exchanged $a \leftrightarrow b$, so winner is $b$.
      - Anonymity: We just exchanged the votes, so winner stays $a$.
  - Typically, we only require neutrality for…
    - Randomized rules: E.g., a rule could satisfy both by choosing $a$ and $b$ as the winner with probability ½ each, on both profiles
    - Deterministic rules allowed to return ties: E.g., a rule could return $\{a, b\}$ as tied winners on both profiles.

# Axiomatic Approach

- Majority consistency: If a majority of voters have the same top choice, that alternative wins.

$$\left( |\{ i : top(\succ_i) = a \}| > \frac{n}{2} \right) \Rightarrow f(\overrightarrow{\succ}) = a$$

  - ➢ Satisfied by plurality, but not by Borda count

- Condorcet consistency: If $a$ defeats every other alternative in a pairwise election, $a$ wins.

$$\left( |\{ i : a \succ_i b \}| > \frac{n}{2}, \forall b \neq a \right) \Rightarrow f(\overrightarrow{\succ}) = a$$

  - ➢ Condorcet consistency ⇒ Majority consistency
  - ➢ Violated by both plurality and Borda count

# Axiomatic Approach

- Is even the weaker axiom majority consistency a reasonable one to expect?

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| a | a | a | b | b |
| b | b | b | | |
| | | | | |
| | | | | |
| | | | | |
| | | | | |
| | | | | |
| | | | | |
| | | | a | a |

# Axiomatic Approach

- Consistency: If $a$ is the winner on two profiles, it must be the winner on their union.

$$f(\overrightarrow{\succ}_1) = a \wedge f(\overrightarrow{\succ}_2) = a \Rightarrow f(\overrightarrow{\succ}_1 + \overrightarrow{\succ}_2) = a$$

  - Example: $\overrightarrow{\succ}_1 = \{a \succ b \succ c\}$, $\overrightarrow{\succ}_2 = \{a \succ c \succ b, b \succ c \succ a\}$
  - Then, $\overrightarrow{\succ}_1 + \overrightarrow{\succ}_2 = \{a \succ b \succ c, a \succ c \succ b, b \succ c \succ a\}$

- Is this reasonable?
  - Young [1975] showed that subject to mild requirements, a voting rule is consistent if and only if it is a positional scoring rule!
  - Thus, plurality with runoff, STV, Kemeny, Copeland, Maximin, etc are *not* consistent.

# Axiomatic Approach

- Weak monotonicity: If $a$ is the winner, and $a$ is "pushed up" in some votes, $a$ remains the winner.
  - $f(\overrightarrow{\succ}) = a \Rightarrow f(\overrightarrow{\succ}') = a$ if
    1. $b \succ_i c \Leftrightarrow b \succ_i' c, \forall i \in N,\ b, c \in A \backslash \{a\}$
       "Order among other alternatives preserved in all votes"
    2. $a \succ_i b \Rightarrow a \succ_i' b, \forall i \in N,\ b \in A \backslash \{a\}$ ($a$ only improves)
       "In every vote, $a$ still defeats all the alternatives it defeated"
- Contrast: strong monotonicity requires $f(\overrightarrow{\succ}') = a$ even if $\overrightarrow{\succ}'$ only satisfies the 2nd condition
  - It is thus too strong. Equivalent to strategyproofness!
  - Only satisfied by dictatorial/non-onto rules [GS theorem]

# Axiomatic Approach

- Weak monotonicity: If $a$ is the winner, and $a$ is "pushed up" in some votes, $a$ remains the winner.
  - $f(\overrightarrow{\succ}) = a \Rightarrow f(\overrightarrow{\succ}') = a$, where
    - $b \succ_i c \Leftrightarrow b \succ_i' c, \forall i \in N,\ b, c \in A \backslash \{a\}$ (Order of others preserved)
    - $a \succ_i b \Rightarrow a \succ_i' b, \forall i \in N,\ b \in A \backslash \{a\}$ ($a$ only improves)

- Weak monotonicity is satisfied by most voting rules
  - Only exceptions (among rules we saw): STV and plurality with runoff
  - But this helps STV be hard to manipulate
    - [Conitzer & Sandholm 2006]: "Every weakly monotonic voting rule is easy to manipulate on average."

# Axiomatic Approach

- STV violates weak monotonicity

| 7 voters | 5 voters | 2 voters | 6 voters |
|---|---|---|---|
| a | b | b | c |
| b | c | c | a |
| c | a | a | b |

| 7 voters | 5 voters | 2 voters | 6 voters |
|---|---|---|---|
| a | b | a | c |
| b | c | b | a |
| c | a | c | b |

- First $c$, then $b$ eliminated
- Winner: $a$

- First $b$, then $a$ eliminated
- Winner: $c$

# Axiomatic Approach

NOT IN SYLLABUS

- For social welfare functions that output a ranking:
- Independence of Irrelevant Alternatives (IIA):
  - If the preferences of all voters between $a$ and $b$ are unchanged, then the social preference between $a$ and $b$ should not change.
- Arrow's Impossibility Theorem
  - No voting rule satisfies IIA, Pareto optimality, and nondictatorship.
  - Proof omitted.
  - Foundations of the axiomatic approach to voting

# Statistical Approach

NOT IN SYLLABUS

- Assume that there is a "true" ranking of alternatives
  - Unknown to us apriori

- Votes $\{\succ_i\}$ are generated i.i.d. from a distribution parametrized by a ranking $\sigma^*$
  - $\Pr[\succ|\sigma^*]$ denotes the probability of drawing a vote $\succ$ given that the ground truth is $\sigma^*$

- Maximum likelihood estimate (MLE):
  - Given $\vec{\succ}$, return $\text{argmax}_\sigma\left(\Pr[\vec{\succ}|\sigma] = \prod_{i=1}^n \Pr[\succ_i|\sigma]\right)$

# Statistical Approach

NOT IN SYLLABUS

- Example: Mallows' model
  - Recall Kendall-tau distance $d$ between two rankings: #pairs of alternatives on which they disagree
  - Malllows' model: $\Pr[\succ|\sigma^*] \propto \varphi^{d(\succ,\sigma^*)}$, where
    - $\varphi \in (0,1]$ is the "noise parameter"
    - $\varphi \to 0 : \Pr[\sigma^*|\sigma^*] \to 1$
    - $\varphi = 1$ : uniform distribution
    - Normalization constant $Z_\varphi = \sum_\succ \varphi^{d(\succ,\sigma^*)}$ does not depend on $\sigma^*$
  - The greater the distance from the ground truth, the smaller the probability

# Statistical Approach

NOT IN SYLLABUS

- Example: Mallows' model
  - What is the MLE ranking for Mallows' model?

$$\max_{\sigma} \prod_{i=1}^{n} \Pr[\succ_i \mid \sigma^*] = \max_{\sigma} \prod_{i=1}^{n} \frac{\varphi^{d(\succ_i,\sigma^*)}}{Z_\varphi} = \max_{\sigma} \frac{\varphi^{\sum_{i=1}^{n} d(\succ_i,\sigma^*)}}{Z_\varphi}$$

  - The MLE ranking $\sigma^*$ minimizes $\sum_{i=1}^{n} d(\succ_i, \sigma^*)$
  - This is precisely the Kemeny ranking!
- Statistical approach yields a unique rule, but is specific to the assumed distribution of votes

# Utilitarian Approach

- Each voter $i$ still submits a ranking $\succ_i$
  - But the voter has "implicit" numerical utilities $\{v_i(a) \geq 0\}$

$$\Sigma_a \, v_i(a) = 1$$
$$a \succ_i b \Rightarrow v_i(a) \geq v_i(b)$$

- Goal:
  - Select $a^*$ with the maximum social welfare $\sum_i v_i(a^*)$
    - Cannot always find this given only rankings from voters
  - Refined goal: Select $a^*$ that gives the best worst-case approximation of welfare

# Distortion

- The distortion of a voting rule $f$ is its approximation ratio of social welfare, on the worst preference profile.

$$dist(f) = \sup_{valid\ \{v_i\}} \frac{\max_b \sum_i v_i\,(b)}{\sum_i v_i\left(f\left(\overrightarrow{\succ}\right)\right)}$$

  - where each $v_i$ is valid if $\Sigma_a\, v_i(a) = 1$
  - $\overrightarrow{\succ} = (\succ_1, \dots, \succ_n)$ where $\succ_i$ represents the ranking of alternatives according to $v_i$

# Example

- Suppose there are 2 voters and 3 alternatives
- Suppose our $f$ returns $c$ on this profile

Rankings

| 1 | 2 |
|---|---|
| a | c |
| b | a |
| c | b |

Utilities

| 1 | 2 |
|---|---|
| a : 1.0 | c : 0.5 |
| b : 0.0 | a : 0.5 |
| c : 0.0 | b : 0.0 |

Social welfare
$a$ = 1.5 (optimal)
$c$ = 0.5
$dist(f) \geq 3$

Utilities

| 1 | 2 |
|---|---|
| a : 0.4 | c : 0.7 |
| b : 0.3 | a : 0.2 |
| c : 0.3 | b : 0.1 |

Social welfare
$c$ = 1.0 (optimal)

$dist(f) \geq 1$

…

$dist(f)$ is the largest such number you can find by constructing consistent utility profiles

# Optimal Deterministic Rules

- Theorem [Caragiannis et al. '17]:
  Plurality achieves $O(m^2)$ distortion.
- Proof:
  - The winner is the top choice of at least $n/m$ voters.
  - Each voter must have utility at least $1/m$ for her top choice. (WHY?)
  - Plurality achieves social welfare at least $\frac{n}{m} \cdot \frac{1}{m} = \frac{n}{m^2}$
  - No alternative can achieve social welfare more than $n$ (WHY?)
  - QED!

# Optimal Deterministic Rules

- Theorem [Caragiannis et al. '17]:
  Every deterministic voting rule has $\Omega(m^2)$ distortion.
- Proof:
  - $n$ voters divided into $m-1$ blocks of equal size
  - Preference profile:
    - voters in block $i$ put $a_i$ first, $a_m$ next, and the rest arbitrarily
  - If output = $a_m \Rightarrow \infty$ distortion (WHY?)
  - If output $\in \{a_1, \dots, a_{m-1}\} \Rightarrow \Omega(m^2)$ distortion
    - Derivation on the board!

$$
\left.\begin{matrix} n/(m-1) \\ \text{times} \end{matrix}\right\{ \begin{array}{l} a_1 \succ a_m \succ \cdots \\ a_2 \succ a_m \succ \cdots \\ a_3 \succ a_m \succ \cdots \\ \vdots \\ a_{m-1} \succ a_m \succ \cdots \end{array}
$$

# Optimal Randomized Rules

- Theorem [Boutilier et al. ‘15]:
  There is a randomized rule with $O(\sqrt{m \cdot \log m})$ distortion.
- Proof:
  - Given profile $\vec{\succ}$, define the harmonic score $\text{sc}(a, \vec{\succ})$:
    - Each voter gives $1/k$ points to her $k^{th}$ most preferred alternative
    - $\text{sc}(a, \vec{\succ})$ = sum of points received by $a$ from all voters
  - Want to compare $\text{sc}\left(a, \vec{\succ}\right)$ to social welfare $\text{sw}(a, \vec{v})$
    - $\text{sw}(a, \vec{v}) \le \text{sc}(a, \vec{\succ})$ (WHY?)
    - $\sum_a sc(a, \vec{\succ}) = n \cdot \sum_{k=1}^{m} 1/k \le n \cdot (\ln m + 1)$

# Optimal Voting Rules

- Proof (continued):
  - Golden voting rule:
    - Rule 1: Choose every $a$ w.p. proportional to $\mathrm{sc}(a, \vec{\succ})$
    - Rule 2: Choose every $a$ w.p. $1/m$ (uniformly at random)
    - Execute rule 1 and rule 2 with probability ½ each
  - Distortion $\le 2\sqrt{m \cdot (\ln m + 1)}$ (proof on the board!)
  - Trick: Take optimal alternative $a^* \in \mathrm{argmax}_{a \in A}\, \mathrm{sw}(a, \vec{v})$
    - If $\mathrm{sc}\left(a^*, \vec{\succ}\right) \ge n\sqrt{(\ln m + 1)/m}$ :
      - Rule 1 picks $a^*$ with enough probability
    - Otherwise, we know $\mathrm{sw}(a^*, \vec{v}) \le \mathrm{sc}\left(a^*, \vec{\succ}\right) \le n\sqrt{(\ln m + 1)/m}$ :
      - Rule 2 generates enough social welfare ($n/m$).

# Optimal Randomized Rules

- Theorem [Boutilier et al. '15]:
  No randomized rule has distortion better than $\sqrt{m}/3$.
- Proof:
  - Pick $\sqrt{m}$ special alternatives: $a_1, \ldots, a_{\sqrt{m}}$
  - $n$ voters divided into $\sqrt{m}$ equal-size blocks
  - Preference profile:
    - For $i \in \{1, \ldots, \sqrt{m}\}$, voters in block $i$ put $a_i$ first, and others arbitrarily
  - Pigeonhole principle:
    - $\exists a_i \in \{a_1, \ldots, a_{\sqrt{m}}\}$ that the voting rule picks with probability at most $1/\sqrt{m}$
    - Construct worst-case valuation to make $a_i$ look as good as possible in hindsight to derive $\sqrt{m}/3$ distortion bound (proof on the board!)

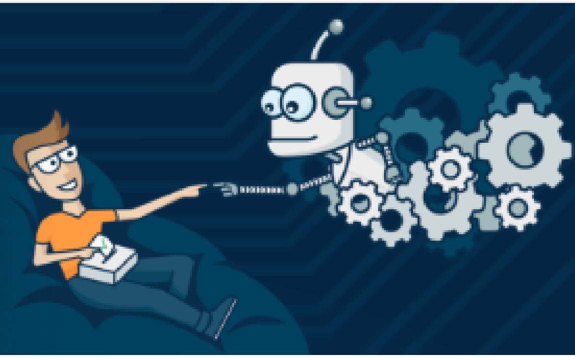
# Utilitarian Approach

- Pros: Uses minimal assumptions and yields a uniquely optimal voting rule
- Cons: The optimal rule is difficult to compute and unintuitive to humans

- This approach is currently deployed on RoboVote.org
  - It has been extended to select a set of alternatives, select a ranking, select public projects subject to a budget constraint, etc.

# AI-Driven Decisions

RoboVote is a free service that helps users combine their preferences or opinions into optimal decisions. To do so, RoboVote employs state-of-the-art voting methods developed in artificial intelligence research. Learn More

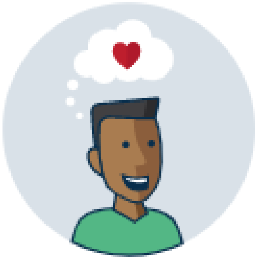


## Poll Types

RoboVote offers two types of polls, which are tailored to different scenarios; it is up to users to indicate to RoboVote which scenario best fits the problem at hand.

### Objective Opinions

In this scenario, some alternatives are objectively better than others, and the opinion of a participant reflects an attempt to estimate the correct order. RoboVote's proposed outcome is guaranteed to be as close as possible — based on the available information — to the best outcome. Examples include deciding which product prototype to develop, or which company to invest in, based on a metric such as projected revenue or market share. Try the demo.



### Subjective Preferences

In this scenario participants' preferences reflect their subjective taste; RoboVote proposes an outcome that mathematically makes participants as happy as possible overall. Common examples include deciding which restaurant or movie to go to as a group, which destination to choose for a family vacation, or whom to elect as class president. Try the demo.

## Ready to get started?

CREATE A POLL