CSC304 Lecture 12

Mechanism Design w/ Money: Revenue maximization Myerson's Auction

Revenue Maximization

Welfare vs Revenue

- In welfare maximization, we want to maximize $\sum_i v_i(a)$
  - VCG = strategyproof + maximizes welfare on every single instance
  - Beautiful!

- In revenue maximization, we want to maximize $\sum_i p_i$
  - We can still use strategyproof mechanisms (revelation principle).
  - BUT…

Welfare vs Revenue

- Different strategyproof mechanisms are better for different instances.

- Example: 1 item, 1 bidder, unknown value v
  - strategyproof = fix a price r, let the agent decide to "take it" ($v \geq r$) or "leave it" ($v < r$)
  - Maximize welfare → set $r = 0$
    - Must allocate item as long as the agent has a positive value
  - Maximize revenue → $r = ?$
    - Different r are better for different v

Welfare vs Revenue

- We cannot optimize revenue on every instance
  - Need to optimize the *expected* revenue in the Bayesian framework

- If we want to achieve higher expected revenue than VCG, we cannot always allocate the item
  - Revenue equivalence principle!

Single Item + Single Bidder

- Value v is drawn from distribution with CDF F
- Goal: post the optimal price r on the item

- Revenue from price $r = r \cdot \big(1 - F(r)\big)$ (Why?)

- Optimal $r^* = \text{argmax}_r \, r \cdot \big(1 - F(r)\big)$
  - "Monopoly price"
  - Note: r^* depends on F, but not on v, so the mechanism is strategyproof.

Example

- Suppose F is the CDF of the uniform distribution over $[0,1]$ (denote by $U[0,1]$).
  - CDF is given by $F(x) = x$ for all $x \in [0,1]$.
- Recall: E[Revenue] from price r is $r \cdot \left(1 - F(r)\right)$
  - Q: What is the optimal posted price?
  - Q: What is the corresponding optimal revenue?
- Compare this to the revenue of VCG, which is 0
  - This is because if the value is less than r^*, we are willing to risk not selling the item.

Single Item + Two Bidders

- $v_1, v_2 \sim U[0,1]$
- VCG revenue = 2nd highest bid = $\min(v_1, v_2)$
  - $E[\min(v_1, v_2)] = 1/3$ (Exercise!)
- Improvement: "VCG with reserve price"
  - Reserve price r
  - Highest bidder gets the item if bid more than r
  - Pays $\max(r, 2^{\text{nd}}\text{ highest bid})$
    - "Critical payment" : Pay the least value you could have bid and still won the item

Single Item + Two Bidders

- Reserve prices are ubiquitous
  - E.g., opening bids in eBay auctions
  - Guarantee a certain revenue to auctioneer if item is sold
- $E[\text{revenue}] = E[\max(r, \min(v_1, v_2))]$
  - Maximize over r? Hard to think about.
- What about a strategyproof mechanism that is not VCG + reserve price?
  - What about just BNIC mechanisms?

Single-Parameter Environments

- Roger B. Myerson solved revenue optimal auctions in "single-parameter environments"
- Proposed a simple auction that maximizes expected revenue

Single-Parameter Environments

- Each agent i…
  - has a private value v_i drawn from a distribution with CDF F_i and PDF f_i
  - is "satisfied" at some level $x_i \in [0,1]$, which gives the agent value $x_i \cdot v_i$
  - is asked to pay p_i

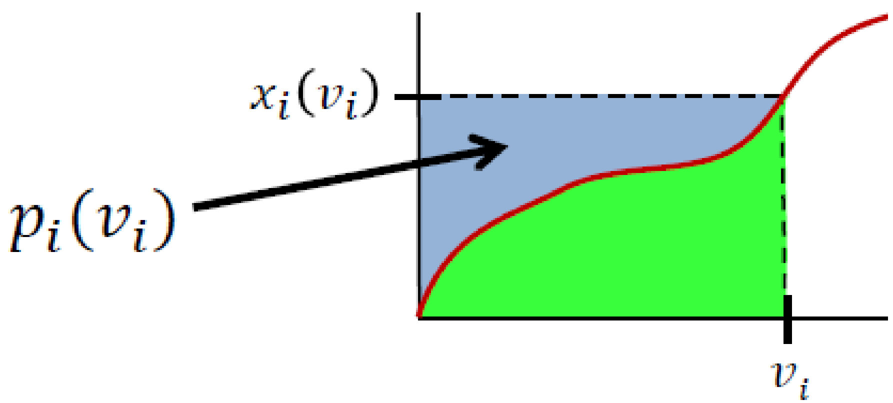
- Examples
  - Single divisible item
  - Single indivisible item ($x_i \in \{0,1\}$ – this is okay too!)
  - Many items, single-minded bidders (again $x_i \in \{0,1\}$)

Myerson's Lemma

- Myerson's Lemma:
  For a single-parameter environment, a mechanism is strategyproof if and only if for all i
  1. x_i is monotone non-decreasing in v_i
  2. $p_i = v_i \cdot x_i(v_i) - \int_0^{v_i} x_i(z)dz + p_i(0)$

  (typically, $p_i(0) = 0$)

- Generalizes critical payments
  - For every "δ" allocation, pay the lowest value that would have won it



Myerson's Lemma

- Note: allocation determines unique payments

$$p_i = v_i \cdot x_i(v_i) - \int_0^{v_i} x_i(z)dz + p_i(0)$$

- A corollary: revenue equivalence
  - If two mechanisms use the same allocation x_i, they "essentially" have the same expected revenue

- Another corollary: optimal revenue auctions
  - Optimizing revenue = optimizing some function of allocation (easier to analyze)

Myerson's Theorem

- "Expected Revenue = Expected Virtual Welfare"
  - Recall: $p_i = v_i \cdot x_i(v_i) - \int_0^{v_i} x_i(z) dz + p_i(0)$
  - Take expectation over draw of valuations + lots of calculus

$$E_{\{v_i \sim F_i\}}[\Sigma_i \, p_i] = E_{\{v_i \sim F_i\}}[\Sigma_i \, \varphi_i \cdot x_i \,]$$

- $\varphi_i = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ = virtual value of bidder i

- $\sum_i \varphi_i \cdot x_i$ = virtual welfare

Myerson's Theorem

- Myerson's auction:
  - A strategyproof auction maximizes the (expected) revenue if its allocation rule maximizes the virtual welfare subject to monotonicity and it charges critical payments.

- Charging critical payments is easy.
- But maximizing virtual welfare *subject to monotonicity* is tricky.
  - Let's get rid of the monotonicity requirement!

Myerson's Theorem Simplified

- Regular Distributions
  - A distribution F is regular if its virtual value function $\varphi(v) = v - (1 - F(v))/f(v)$ is non-decreasing in v.
  - Many important distributions are regular, e.g., uniform, exponential, Gaussian, power-law, …
- Lemma
  - If all F_i's are regular, the allocation rule maximizing virtual welfare is already monotone.
- Myerson's Corollary:
  - When all F_i's are regular, the strategyproof auction maximizes virtual welfare and charges critical payments.

Single Item + Single Bidder

- Setup:
  - Single indivisible item, single bidder, value v drawn from a regular distribution with CDF F and PDF f
- Goal:
  - Maximize $\varphi \cdot x$, where $\varphi = v - \frac{1-F(v)}{f(v)}$ and $x \in \{0,1\}$
- Optimal auction:
  - $x = 1$ iff $\varphi \geq 0 \iff v \geq \frac{1-F(v)}{f(v)} \iff v \geq v^*$ where $v^* = \frac{1-F(v^*)}{f(v^*)}$
  - Critical payment: v^*
  - This is VCG with a reserve price of $\varphi^{-1}(0)$!

Example

- Optimal auction:
  - $x = 1$ iff $\varphi \geq 0 \Leftrightarrow v \geq \frac{1-F(v)}{f(v)}$
  - Critical payment: v^* such that $v^* = \frac{1-F(v^*)}{f(v^*)}$

- Distribution is $U[0,1]$:
  - $x = 1$ iff $v \geq \frac{1-v}{1} \Leftrightarrow v \geq \frac{1}{2}$
  - Critical payment = $\frac{1}{2}$
  - That is, we post the optimal price of 0.5

Single Item + n Bidders

- Setup:
  - Single indivisible item, each bidder i has value v_i drawn from a regular distribution with CDF F_i and PDF f_i

- Goal:
  - Maximize $\sum_i \varphi_i \cdot x_i$ where $\varphi_i = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ and $x_i \in \{0,1\}$ such that $\sum_i x_i \leq 1$

Single Item + n Bidders

- Optimal auction:
  - If all $\varphi_i < 0$:
    - Nobody gets the item, nobody pays anything
    - For all i, $x_i = p_i = 0$
  - If some $\varphi_i \geq 0$:
    - Agent with highest φ_i wins the item, pays critical payment
    - $i^* \in argmax_i\ \varphi_i(v_i), x_{i^*} = 1,\ x_i = 0\ \forall i \neq i^*$
    - $p_{i^*} = \varphi_{i^*}^{-1}\left(\max\left(0, \max_{j \neq i^*} \varphi_j(v_j)\right)\right)$

- Note: The item doesn't necessarily go to the highest value agent!

Special Case: iid Values

- Suppose all distributions are identical (say CDF F and PDF f)

- Check that the auction simplifies to the following
  - Allocation: item goes to bidder i^* with highest value if her value $v_{i^*} \geq \varphi^{-1}(0)$
  - Payment charged = $\max\left(\varphi^{-1}(0), \max_{j \neq i^*} v_j\right)$

- This is again VCG with a reserve price of $\varphi^{-1}(0)$

Example

- Two bidders, both drawing iid values from $U[0,1]$
  - $\varphi(v) = v - \frac{1-v}{1} = 2v - 1$
  - $\varphi^{-1}(0) = 1/2$

- Auction:
  - Give the item to the highest bidder if their value is at least ½
  - Charge them $\max(½, 2^{\text{nd}}\text{ highest bid})$

Example

- Two bidders, one with value from $U[0,1]$, one with value from $U[3,5]$
  - $\varphi_1(v_1) = 2v_1 - 1$
  - $\varphi_2(v_2) = v_2 - \frac{1-F_2(v_2)}{f_2(v_2)} = v_2 - \frac{1-\frac{v_2-3}{2}}{1/2} = 2v_2 - 5$

- Auction:
  - If $v_1 < ½$ and $v_2 < 5/2$, the item remains unallocated.
  - Otherwise…
    - If $2v_1 - 1 > 2v_2 - 5$, agent 1 gets it and pays $\max(½, v_2 - 2)$
    - If $2v_1 - 1 < 2v_2 - 5$, agent 2 gets it and pays $\max(5/2, v_1 + 2)$

Extensions

- Irregular distributions:
  - E.g., multi-modal or extremely heavy tail distributions
  - Need to add the monotonicity constraint
  - Turns out, we can “iron” irregular distributions to make them regular and then use Myerson’s framework
- Relaxing DSIC to BNIC
  - Myerson’s mechanism has optimal revenue among all DSIC mechanisms
  - Turns out, it also has optimal revenue among the much larger class of BNIC mechanisms!

Approx. Optimal Auctions

- Optimal auctions become unintuitive and difficult to understand with unequal distributions, even if they are regular
  - Simpler auctions preferred in practice
  - We still want approximately optimal revenue

- Theorem [Hartline & Roughgarden, 2009]:
  - For iid values from regular distributions, VCG with bidder-specific reserve prices gives a 2-approximation of the optimal revenue.

Approximately Optimal

- Still relies on knowing bidders' distributions
- Theorem [Bulow and Klemperer, 1996]:
  - For i.i.d. values, E[Revenue of VCG with $n + 1$ bidders] $\geq E$[Optimal revenue with n bidders]
- "Spend that effort in getting one more bidder than in figuring out the optimal auction"

Simple proof

- One can show that VCG with $n + 1$ bidders has the max revenue among all $n + 1$ bidder strategyproof auctions *that always allocate the item*
  - Via revenue equivalence

- Consider the auction: "Run n-bidder Myerson on the first n bidders. If the item is unallocated, give it to agent $n + 1$ for free."
  - $n + 1$ bidder DSIC auction
  - As much revenue as n-bidder Myerson auction

Optimizing Revenue is Hard

- Slow progress beyond single-parameter setting
  - Even with just two items and one bidder with i.i.d. values for both items, the optimal auction DOES NOT run Myerson's auction on individual items!
  - "Take-it-or-leave-it" offers for the two items bundled might increase revenue

- But nowadays, the focus is on simple, approximately optimal auctions instead of complicated, optimal auctions.