CSC304 Lecture 3

Game Theory
(More examples, PoA, PoS)

Recap

- Normal form games
- Domination among strategies
  - Weak/strict domination
- Hope 1: Find a weakly/strictly dominant strategy
- Hope 2: Iterated elimination of dominated strategies
- Guarantee 3: Nash equilibria
  - Pure – may be none, unique, or multiple
    - Identified using best response diagrams
  - Mixed – at least one!
    - Identified using the indifference principle

Recap: Nash Equilibrium (NE)

- Nash Equilibrium
  - A strategy profile $\vec{s}$ is in Nash equilibrium if s_i is the best action for player i given that other players are playing $\vec{s}_{-i}$

$$u_i(s_i, \vec{s}_{-i}) \geq u_i(s_i', \vec{s}_{-i}), \forall i, s_i'$$

No quantifier on $\vec{s}_{-i}$

  - Each player's strategy is only best *given* the strategies of others, and not *regardless*.

Pure vs Mixed Nash Equilibria

- A pure strategy s_i is deterministic
  - That is, player i plays a single action w.p. 1
- A mixed strategy s_i can *possibly* randomize over actions
  - In a fully-mixed strategy, every action is played with a positive probability
- A strategy profile $\vec{s}$ is pure if each s_i is pure
  - These are the "cells" in the normal form representation
- A pure Nash equilibrium (PNE) is a pure strategy profile that is a Nash equilibrium

Pure Nash Equilibria

- Best response
  - The best response of player i to others' strategies $\vec{s}_{-i}$ is the highest reward action:

$$s_i^* \in \operatorname{argmax}_{s_i} u_i(s_i, \vec{s}_{-i})$$

- Best-response diagram:
  - From each cell $\vec{s}$, for each player i, draw an arrow to $(s_i^*, \vec{s}_{-i})$, where s_i^* = player i's best response to $\vec{s}_{-i}$
    - unless s_i is already a best response

- Pure Nash equilibria (PNE)
  - Each player is already playing their best response
  - No outgoing arrows

Example Games

- Stag Hunt: (Stag , Stag) and (Hare , Hare) are PNE

Hunter 1 \ Hunter 2	Stag	Hare
Stag	(4 , 4)	(0 , 2)
Hare	(2 , 0)	(1 , 1)

- Rock-Paper-Scissor : No PNE! Why?

P1 \ P2	Rock	Paper	Scissor
Rock	(0 , 0)	(-1 , 1)	(1 , -1)
Paper	(1 , -1)	(0 , 0)	(-1 , 1)
Scissor	(-1 , 1)	(1 , -1)	(0 , 0)

Nash's Beautiful Result

- Nash's Theorem:
  - Every normal form game has at least one (possibly mixed) Nash equilibrium.
  - Proof? We'll prove a special case later.

- We identify pure NE using best-response diagrams.
  - How do we find mixed NE?

- The Indifference Principle
  - *If $(s_i, \vec{s}_{-i})$ is a Nash equilibrium and s_i randomizes over a set of actions T_i, then each action in T_i must be the best action best given $\vec{s}_{-i}$.*

Revisiting Stag-Hunt

Hunter 1 \ Hunter 2	Stag	Hare
Stag	**(4 , 4)**	**(0 , 2)**
Hare	**(2 , 0)**	**(1 , 1)**

- Symmetric: $s_1 = s_2 = \{\text{Stag w.p. } p, \text{Hare w.p. } 1-p\}$
- Indifference principle:
  - Equal expected reward for Stag and Hare given the other hunter's strategy
  - $\mathbb{E}[\text{Stag}] = p * 4 + (1-p) * 0$
  - $\mathbb{E}[\text{Hare}] = p * 2 + (1-p) * 1$
  - $4p = 2p + (1-p) \;\Rightarrow\; p = 1/3$

Revisiting Rock-Paper-Scissor

- Blackboard derivation of a special case:
  - "Fully mixed"
    - Each player uses all actions with some probability
  - Symmetric

- Exercise:
  - Check if other cases provide any mixed NE

P1 \ P2	Rock	Paper	Scissor
Rock	(0 , 0)	(-1 , 1)	(1 , -1)
Paper	(1 , -1)	(0 , 0)	(-1 , 1)
Scissor	(-1 , 1)	(1 , -1)	(0 , 0)

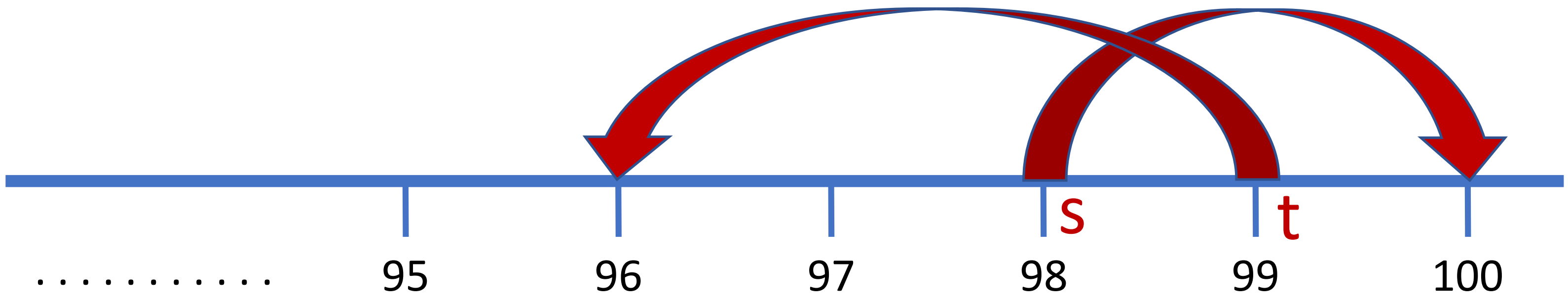
Extra Fun 1: Inspect Or Not

Inspector / Driver	Inspect	Don't Inspect
Pay Fare	(-10 , -1)	(-10 , 0)
Don't Pay Fare	(-90 , 29)	(0 , -30)

- Game:
  - Fare = 10
  - Cost of inspection = 1
  - Fine if fare not paid = 30
  - Total cost to driver if caught = 90
- Nash equilibrium?

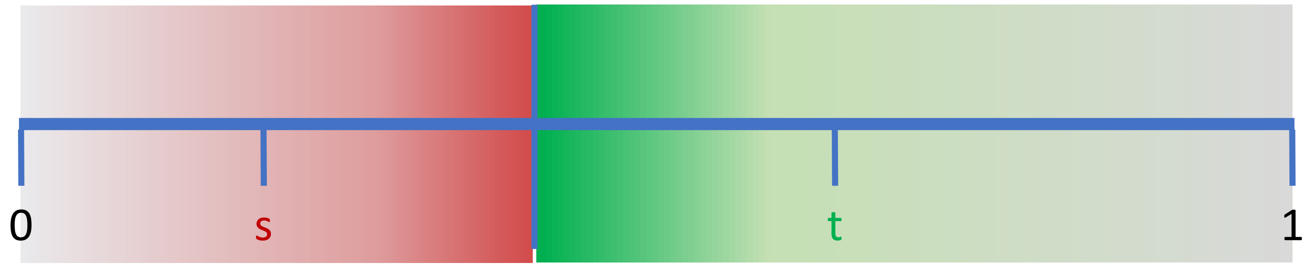
Extra Fun 2: Cunning Airlines

- Two travelers lose their luggage.
- Airline agrees to refund up to $100 to each.
- Policy: Both travelers would submit a number between 2 and 99 (inclusive).
  - If both report the same number, each gets this value.
  - If one reports a lower number (s) than the other (t), the former gets s+2, the latter gets s-2.



Extra Fun 3: Ice Cream Shop

- Two brothers, each wants to set up an ice cream shop on the beach ([0,1]).
- If the shops are at s, t (with $s \leq t$)
  - The brother at s gets $\left[0, \frac{s+t}{2}\right]$, the other gets $\left[\frac{s+t}{2}, 1\right]$



Computational Complexity

- Pure Nash equilibria
  - Existence: Checking the existence of a pure Nash equilibrium can be NP-hard.
  - Computation: Computing a pure NE can be PLS-complete, even in games in which a pure NE is guaranteed to exist.

- Mixed Nash equilibria
  - Existence: Always exist due to Nash's theorem
  - Computation: Computing a mixed NE is PPAD-complete.

Nash Equilibria: Critique

- Noncooperative game theory provides a framework for analyzing rational behavior.

- But it relies on many assumptions that are often violated in the real world.

- Due to this, human actors are observed to play Nash equilibria in some settings, but play something far different in other settings.

Nash Equilibria: Critique

- Assumptions:
  - Rationality is common knowledge.
    - All players are rational.
    - All players know that all players are rational.
    - All players know that all players know that all players are rational.
    - … [Aumann, 1976]
    - Behavioral economics
  - Rationality is perfect = "infinite wisdom"
    - Computationally bounded agents
  - Full information about what other players are doing.
    - Bayes-Nash equilibria

Nash Equilibria: Critique

- Assumptions:
  - No binding contracts.
    - Cooperative game theory
  - No player can commit first.
    - Stackelberg games (will study this in a few lectures)
  - No external help.
    - Correlated equilibria
  - Humans reason about randomization using expectations.
    - Prospect theory

Nash Equilibria: Critique

- Also, there are often multiple equilibria, and no clear way of "choosing" one over another.

- For many classes of games, finding even a single Nash equilibrium is provably hard.
  - Cannot expect humans to find it if your computer cannot.

Nash Equilibria: Critique

- Conclusion:
  - For human agents, take it with a grain of salt.
  - For AI agents playing against AI agents, perfect!