CSC304 Lecture 19

Fair Division 1: Cake-Cutting

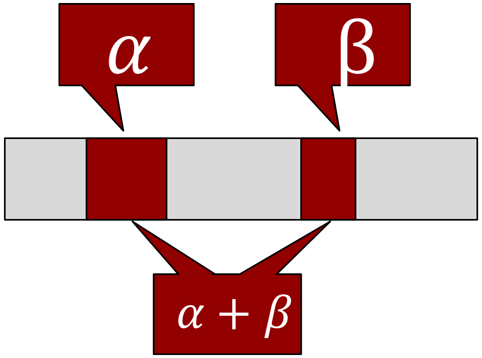
[Image and Illustration Credit: Ariel Procaccia]

Cake-Cutting

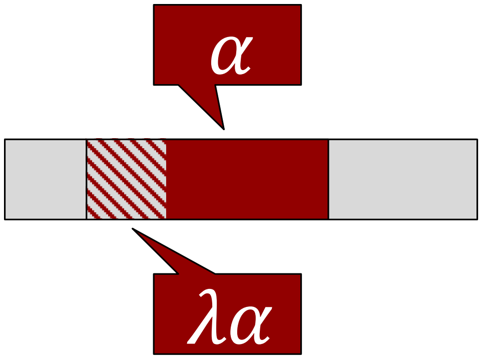
- A heterogeneous, divisible good
  - Heterogeneous: it may be valued differently by different individuals
  - Divisible: we can share/divide it between individuals
- Represented as $[0,1]$
  - Almost without loss of generality
- Set of players $N = \{1, \dots, n\}$
- Piece of cake $X \subseteq [0,1]$
  - A finite union of disjoint intervals

Agent Valuations

- Each player i has a valuation V_i that is very much like a probability distribution over $[0,1]$



- Additive: For $X \cap Y = \emptyset$, $V_i(X) + V_i(Y) = V_i(X \cup Y)$
- Normalized: $V_i([0,1]) = 1$
- Divisible: $\forall \lambda \in [0,1]$ and X, $\exists Y \subseteq X$ s.t. $V_i(Y) = \lambda V_i(X)$



Fairness Goals

- Allocation: disjoint partition $A = (A_1, \dots, A_n)$
  - A_i = piece of the cake given to player i

- Desired fairness properties:
  - Proportionality (Prop):

$$\forall i \in N:\ V_i(A_i) \geq \frac{1}{n}$$

  - Envy-Freeness (EF):

$$\forall i, j \in N: V_i(A_i) \geq V_i(A_j)$$

Fairness Goals

- Prop: $\forall i \in N: V_i(A_i) \geq 1/n$
- EF: $\forall i, j \in N: V_i(A_i) \geq V_i(A_j)$

- Question: What is the relation between proportionality and EF?
  1. Prop $\Rightarrow$ EF
  2. EF $\Rightarrow$ Prop
  3. Equivalent
  4. Incomparable

CUT-AND-CHOOSE

- Algorithm for $n = 2$ players

- Player 1 divides the cake into two pieces X, Y s.t.
$$V_1(X) = V_1(Y) = 1/2$$

- Player 2 chooses the piece she prefers.

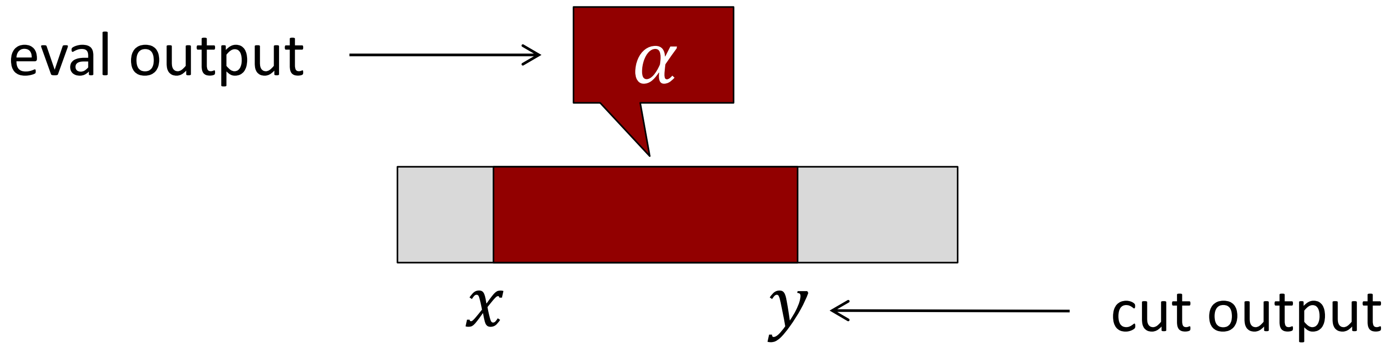
- This is envy-free and therefore proportional.
  - Why?

Input Model

- How do we measure the "time complexity" of a cake-cutting algorithm for n players?
- Typically, time complexity is a function of the length of input encoded as binary.
- Our input consists of functions V_i, which require infinite bits to encode.
- We want running time as a function of n.

Robertson-Webb Model

- We restrict access to valuation V_i through two types of queries:
  - $\text{Eval}_i(x, y)$ returns $\alpha = V_i([x, y])$
  - $\text{Cut}_i(x, \alpha)$ returns any y such that $V_i([x, y]) = \alpha$
    - If $V_i([x, 1]) < \alpha$, return 1.



Robertson-Webb Model

- Two types of queries:
  - $\text{Eval}_i(x, y) = V_i([x, y])$
  - $\text{Cut}_i(x, \alpha) = y$ s.t. $V_i([x, y]) = \alpha$

- Question: How many queries are needed to find an EF allocation when $n = 2$?

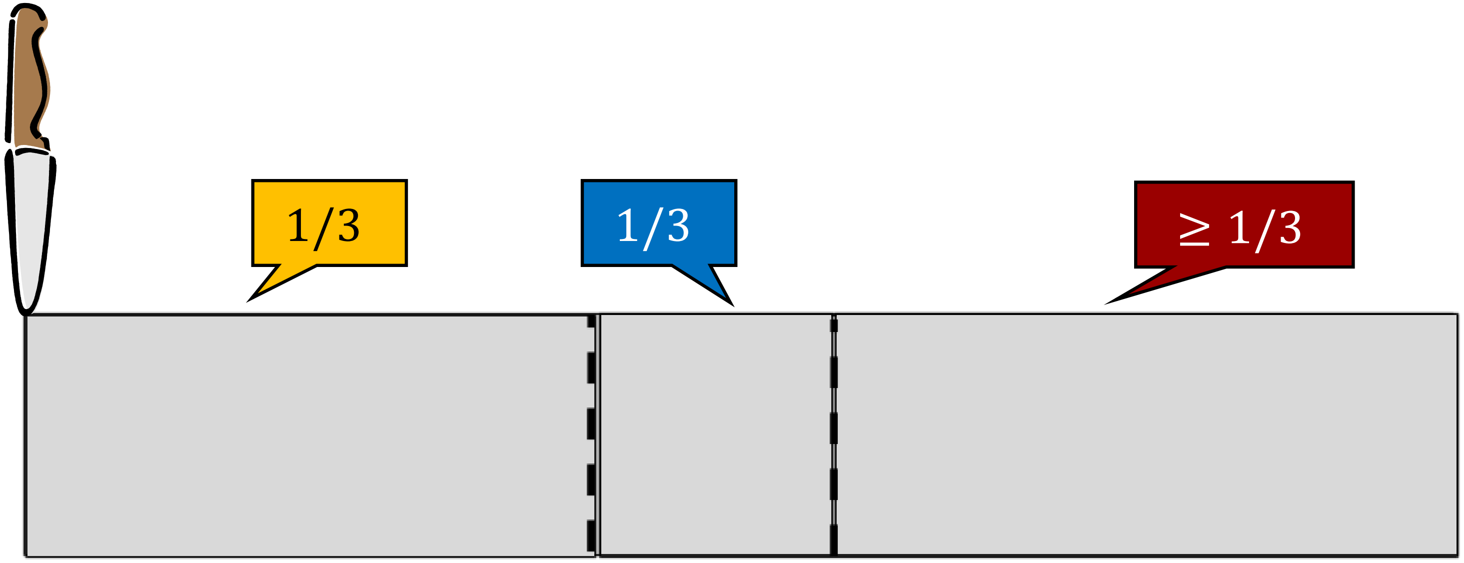
- Answer: 2

Dubins-Spanier

- Protocol for finding a proportional allocation for n players

- Referee starts at 0, and moves a knife to the right.
- Repeat: When the piece to the left of the knife is worth $1/n$ to some player, the player shouts "stop", gets that piece, and exits.
- The last player gets the remaining piece.

Dubins-Spanier

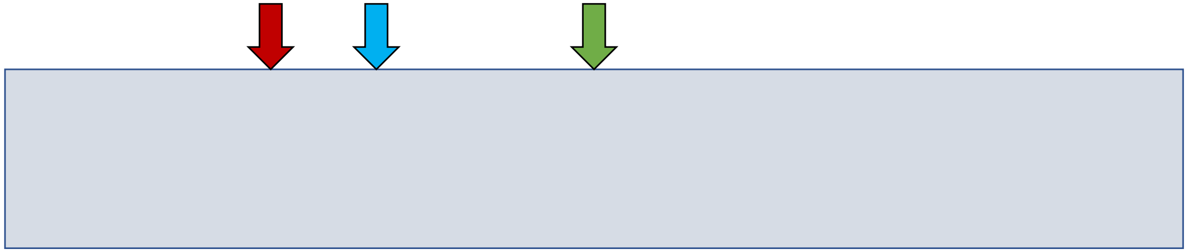


Dubins-Spanier

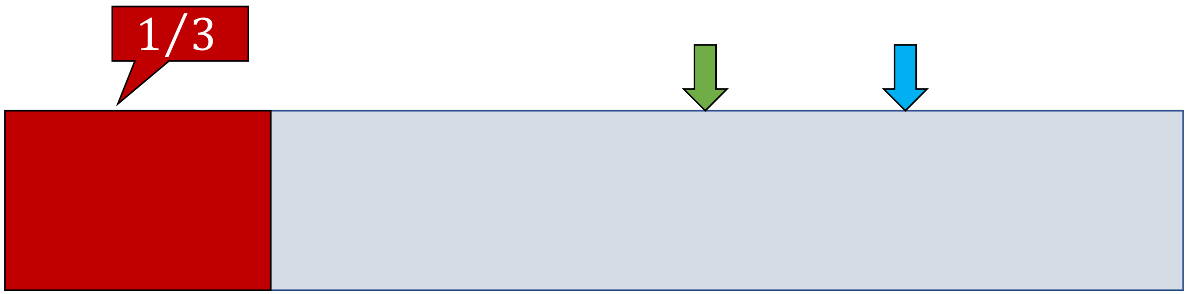
- Robertson-Webb model? Cut-Eval queries?
  - Moving knife is not really needed.

- At each stage, we want to find the remaining player that has value $1/n$ from the smallest next piece.
  - Ask each remaining player a cut query to mark a point where her value is $1/n$ from the current point.
  - Directly move the knife to the leftmost mark, and give that piece to that player.

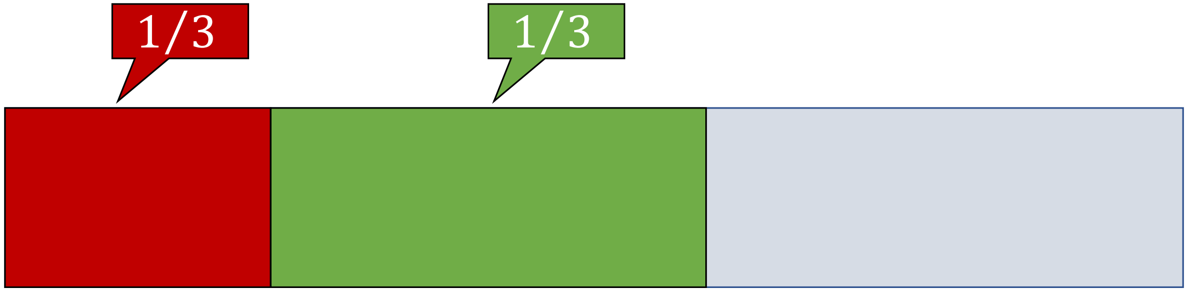
VISUAL PROOF OF PROPORTIONALITY



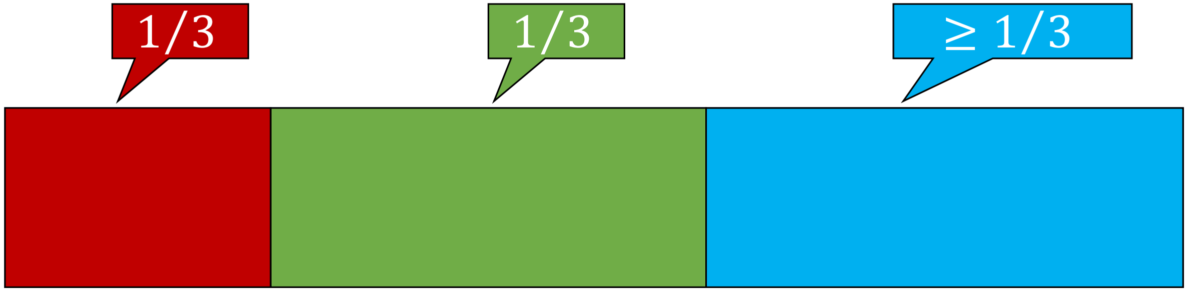
VISUAL PROOF OF PROPORTIONALITY



Visual Proof of Proportionality



VISUAL PROOF OF PROPORTIONALITY



Dubins-Spanier

- Question: What is the complexity of the Dubins-Spanier protocol in the Robertson-Webb model?

1. $\Theta(n)$
2. $\Theta(n \log n)$
3. $\Theta(n^2)$
4. $\Theta(n^2 \log n)$

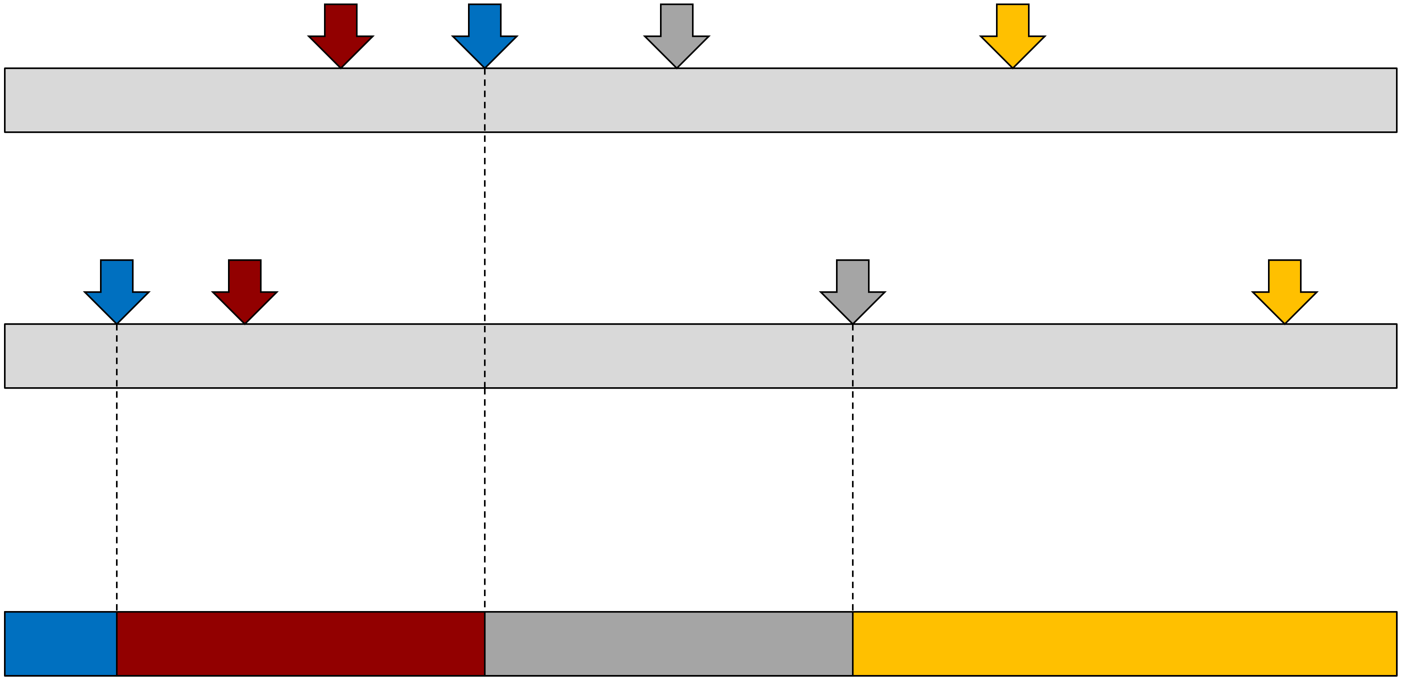
EVEN-PAZ (RECURSIVE)

- Input: Interval $[x, y]$, number of players n
  - For simplicity, assume $n = 2^k$ for some k
- If $n = 1$, give $[x, y]$ to the single player.
- Otherwise, let each player i mark z_i s.t.

$$V_i([x, z_i]) = \frac{1}{2} V_i([x, y])$$

- Let z^* be mark $n/2$ from the left.
- Recurse on $[x, z^*]$ with the left $n/2$ players, and on $[z^*, y]$ with the right $n/2$ players.

EVEN-PAZ



EVEN-PAZ

- Theorem: EVEN-PAZ returns a Prop allocation.
- Inductive Proof:
  - Hypothesis: With n players, EVEN-PAZ ensures that for each player i, $V_i(A_i) \geq (1/n) \cdot V_i([x, y])$
    - Prop follows because initially $V_i([x, y]) = V_i([0,1]) = 1$
  - Base case: $n = 1$ is trivial.
  - Suppose it holds for $n = 2^{k-1}$. We prove for $n = 2^k$.
  - Take the 2^{k-1} left players.
    - Every left player i has $V_i([x, z^*]) \geq (1/2)\, V_i([x, y])$
    - If it gets A_i, by induction, $V_i(A_i) \geq \frac{1}{2^{k-1}}\, V_i([x, z^*]) \geq \frac{1}{2^k}\, V_i([x, y])$

Even-Paz

- Theorem: Even-Paz uses $O(n \log n)$ queries.
- Simple Proof:
  - Protocol runs for $\log n$ rounds.
  - In each round, each player is asked one cut query.
  - QED!

Complexity of Proportionality

- Theorem [Edmonds and Pruhs, 2006]: Any proportional protocol needs $\Omega(n \log n)$ operations in the Robertson-Webb model.

- Thus, the EVEN-PAZ protocol is (asymptotically) provably optimal!

Envy-Freeness?

- "I suppose you are also going to give such cute algorithms for finding envy-free allocations?"

- Bad luck. For n-player EF cake-cutting:
  - [Brams and Taylor, 1995] give an unbounded EF protocol.
  - [Procaccia 2009] shows $\Omega(n^2)$ lower bound for EF.
  - Last year, the long-standing major open question of "bounded EF protocol" was resolved!
  - [Aziz and Mackenzie, 2016]: $O\left(n^{n^{n^{n^{n^{n}}}}}\right)$ protocol!
    - Yes, it's not a typo!

Next Lecture

- More desiderata
- Allocation of indivisible goods