# CSC304 Lecture 12

## Mechanism Design w/ Money: Myerson's Auction

# Recap: Revenue Maximization

- Single item auctions

- One bidder
  - Value $v$ is drawn from distribution with CDF $F$
  - Strategyproof = post a price $r$
  - Optimal $r^* = \text{argmax}_r\, r \cdot \big(1 - F(r)\big)$

- Two bidders
  - Values $v_1$ drawn from $F_1$, $v_2$ drawn from $F_2$
  - ??

# Single-Parameter Environments

- Roger B. Myerson solved revenue optimal auctions in “single-parameter environments”
- Proposed a simple auction that maximizes expected revenue

# Single-Parameter Environments

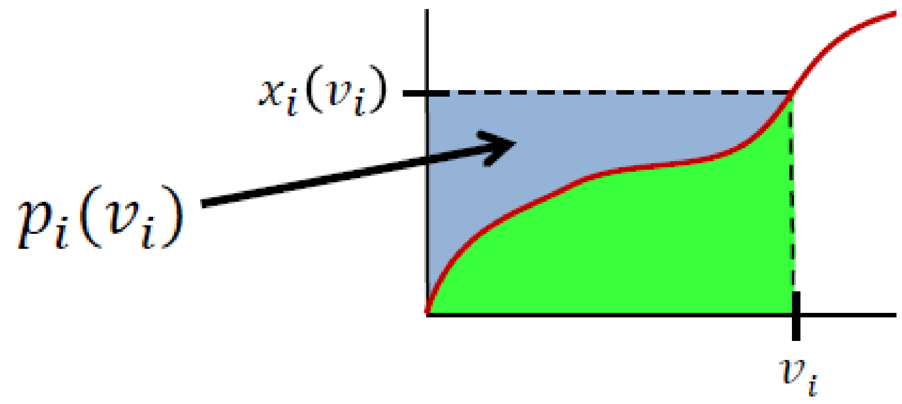
- Each agent $i$...
  - has a private value $v_i$ drawn from a distribution with CDF $F_i$ and PDF $f_i$
  - is "satisfied" at some level $x_i \in [0,1]$, which gives the agent value $x_i \cdot v_i$
  - is asked to pay $p_i$
- Examples
  - Single divisible item
  - Single indivisible item ($x_i \in \{0,1\}$ – this is okay too!)
  - Many items, single-minded bidders (again $x_i \in \{0,1\}$)

# Myerson's Lemma

- Myerson's Lemma:
  For a single-parameter environment, a mechanism is strategyproof if and only if for all $i$
  1. $x_i$ is monotone non-decreasing in $v_i$
  2. $p_i = v_i \cdot x_i(v_i) - \int_0^{v_i} x_i(z) dz + p_i(0)$

  (typically, $p_i(0) = 0$)

- Generalizes critical payments
  - For every "$\delta$" allocation, pay the lowest value that would have won it



# Myerson's Lemma

- Note: allocation determines unique payments

$$p_i = v_i \cdot x_i(v_i) - \int_0^{v_i} x_i(z)dz + p_i(0)$$

- A corollary: revenue equivalence
  - If two mechanisms use the same allocation $x_i$, they "essentially" have the same expected revenue
- Another corollary: optimal revenue auctions
  - Optimizing revenue = optimizing some function of allocation (easier to analyze)

# Myerson's Theorem

- "Expected Revenue = Expected Virtual Welfare"
  - Recall: $p_i = v_i \cdot x_i(v_i) - \int_0^{v_i} x_i(z)dz + p_i(0)$
  - Take expectation over draw of valuations + lots of calculus

$$E_{\{v_i \sim F_i\}}[\Sigma_i \, p_i] = E_{\{v_i \sim F_i\}}[\Sigma_i \, \varphi_i \cdot x_i \,]$$

- $\varphi_i = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ = virtual value of bidder $i$

- $\sum_i \varphi_i \cdot x_i$ = virtual welfare

# Myerson's Theorem

- Myerson's auction:
  - A strategyproof auction maximizes the (expected) revenue if its allocation rule maximizes the virtual welfare subject to monotonicity and it charges critical payments.

- Charging critical payments is easy.
- But maximizing virtual welfare *subject to monotonicity* is tricky.
  - Let's get rid of the monotonicity requirement!

# Myerson's Theorem Simplified

- Regular Distributions
  - A distribution $F$ is regular if its virtual value function $\varphi(v) = v - (1 - F(v))/f(v)$ is non-decreasing in $v$.
  - Many important distributions are regular, e.g., uniform, exponential, Gaussian, power-law, …

- Lemma
  - If all $F_i$'s are regular, the allocation rule maximizing virtual welfare is already monotone.

- Myerson's Corollary:
  - When all $F_i$'s are regular, the strategyproof auction maximizes virtual welfare and charges critical payments.

# Single Item + Single Bidder

- Setup:
  - Single indivisible item, single bidder, value $v$ drawn from a regular distribution with CDF $F$ and PDF $f$
- Goal:
  - Maximize $\varphi \cdot x$, where $\varphi = v - \frac{1-F(v)}{f(v)}$ and $x \in \{0,1\}$
- Optimal auction:
  - $x = 1$ iff $\varphi \geq 0 \Leftrightarrow v \geq \frac{1-F(v)}{f(v)} \Leftrightarrow v \geq v^*$ where $v^* = \frac{1-F(v^*)}{f(v^*)}$
  - Critical payment: $v^*$
  - This is VCG with a reserve price of $\varphi^{-1}(0)$!

# Example

- Optimal auction:
  - $x = 1$ iff $\varphi \geq 0 \Leftrightarrow v \geq \frac{1-F(v)}{f(v)}$
  - Critical payment: $v^*$ such that $v^* = \frac{1-F(v^*)}{f(v^*)}$

- Distribution is $U[0,1]$:
  - $x = 1$ iff $v \geq \frac{1-v}{1} \Leftrightarrow v \geq \frac{1}{2}$
  - Critical payment = $\frac{1}{2}$
  - That is, we post the optimal price of 0.5

# Single Item + $n$ Bidders

- Setup:
  - Single indivisible item, each bidder $i$ has value $v_i$ drawn from a regular distribution with CDF $F_i$ and PDF $f_i$

- Goal:
  - Maximize $\sum_i \varphi_i \cdot x_i$ where $\varphi_i = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ and $x_i \in \{0,1\}$ such that $\sum_i x_i \leq 1$

# Single Item + $n$ Bidders

- Optimal auction:
  - If all $\varphi_i < 0$:
    - Nobody gets the item, nobody pays anything
    - For all $i$, $x_i = p_i = 0$
  - If some $\varphi_i \geq 0$:
    - Agent with highest $\varphi_i$ wins the item, pays critical payment
    - $i^* \in argmax_i\ \varphi_i(v_i), x_{i^*} = 1,\ x_i = 0\ \forall i \neq i^*$
    - $p_{i^*} = \varphi_{i^*}^{-1}\left(\max\left(0, \max_{j \neq i^*} \varphi_j(v_j)\right)\right)$

- Note: The item doesn't necessarily go to the highest value agent!

# Special Case: iid Values

- Suppose all distributions are identical (say CDF $F$ and PDF $f$)

- Check that the auction simplifies to the following
  - Allocation: item goes to bidder $i^*$ with highest value if her value $v_{i^*} \geq \varphi^{-1}(0)$
  - Payment charged = $\max\left(\varphi^{-1}(0), \max_{j \neq i^*} v_j\right)$

- This is again VCG with a reserve price of $\varphi^{-1}(0)$

# Example

- Two bidders, both drawing iid values from $U[0,1]$
  - $\varphi(v) = v - \frac{1-v}{1} = 2v - 1$
  - $\varphi^{-1}(0) = 1/2$

- Auction:
  - Give the item to the highest bidder if their value is at least ½
  - Charge them $\max(½, 2^{\text{nd}} \text{ highest bid})$

# Example

- Two bidders, one with value from $U[0,1]$, one with value from $U[3,5]$
  - $\varphi_1(v_1) = 2v_1 - 1$
  - $\varphi_2(v_2) = v_2 - \frac{1-F_2(v_2)}{f_2(v_2)} = v_2 - \frac{1-\frac{v_2-3}{2}}{1/2} = 2v_2 - 5$

- Auction:
  - If $v_1 < ½$ and $v_2 < 5/2$, the item remains unallocated.
  - Otherwise…
    - If $2v_1 - 1 > 2v_2 - 5$, agent 1 gets it and pays $\max(½, v_2 - 2)$
    - If $2v_1 - 1 < 2v_2 - 5$, agent 2 gets it and pays $\max(5/2\,, v_1 + 2)$

# Extensions

- Irregular distributions:
  - E.g., multi-modal or extremely heavy tail distributions
  - Need to add the monotonicity constraint
  - Turns out, we can "iron" irregular distributions to make them regular and then use Myerson's framework
- Relaxing DSIC to BNIC
  - Myerson's mechanism has optimal revenue among all DSIC mechanisms
  - Turns out, it also has optimal revenue among the much larger class of BNIC mechanisms!

# Approx. Optimal Auctions

- Optimal auctions become unintuitive and difficult to understand with unequal distributions, even if they are regular
  - Simpler auctions preferred in practice
  - We still want approximately optimal revenue

- Theorem [Hartline & Roughgarden, 2009]:
  - For iid values from regular distributions, VCG with bidder-specific reserve prices gives a 2-approximation of the optimal revenue.

# Approximately Optimal

- Still relies on knowing bidders' distributions
- Theorem [Bulow and Klemperer, 1996]:
  - For i.i.d. values, $E$[Revenue of VCG with $n+1$ bidders] $\geq E$[Optimal revenue with $n$ bidders]
- "Spend that effort in getting one more bidder than in figuring out the optimal auction"

# Simple proof

- One can show that VCG with $n+1$ bidders has the max revenue among all $n+1$ bidder strategyproof auctions *that always allocate the item*
  - Via revenue equivalence

- Consider the auction: “Run $n$-bidder Myerson on the first $n$ bidders. If the item is unallocated, give it to agent $n+1$ for free.”
  - $n+1$ bidder DSIC auction
  - As much revenue as $n$-bidder Myerson auction

# Optimizing Revenue is Hard

- Slow progress beyond single-parameter setting
  - Even with just two items and one bidder with i.i.d. values for both items, the optimal auction DOES NOT run Myerson's auction on individual items!
  - "Take-it-or-leave-it" offers for the two items bundled might increase revenue

- But nowadays, the focus is on simple, approximately optimal auctions instead of complicated, optimal auctions.