CSC304 Lecture 8

Mechanism Design w/ Money: Vickrey auction (single-item, general case)

Mechanism Design Recap

- Goal: Maximize social welfare $\max_a \sum_i v_i(a)$

- Method: Truthful Direct Revelation Mechanism
  1. Declare (f, p)
  2. Elicit valuations $v = (v_i)_{i=1}^n$
  3. $f(v)$ chooses the social welfare maximizing outcome
  4. $p(v)$ sets the payments charged to agents in a way that every agent wants to reveal their v_i truthfully

Mechanism Design Recap

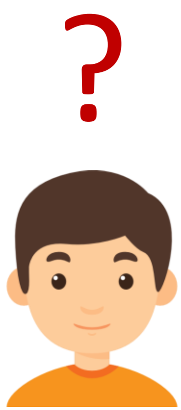
- Revelation Principle $\rightarrow$ without loss of generality
  - If a mechanism selects outcome a, payments p in eq., so does some truthful direct revelation mechanism.

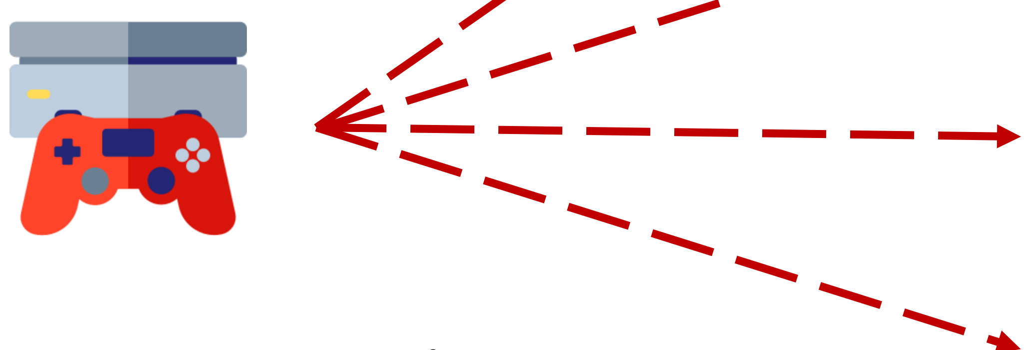
Note:

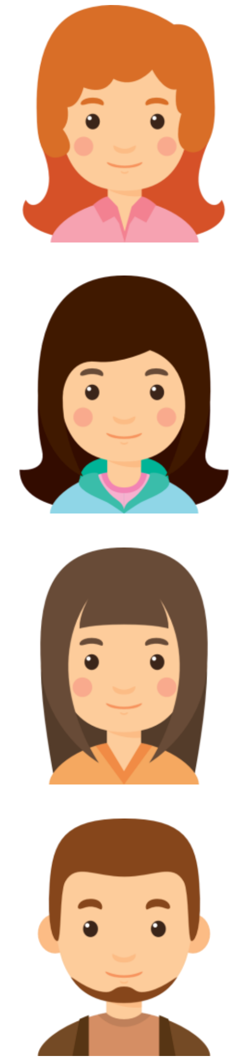
- Principal wants to maximize social welfare $\sum_i v_i(a)$
- Each agent i wants to maximize his net utility
  - If outcome a is chosen and agent i pays p_i, his net utility is $v_i(a) - p_i$

Single-Item Auction

Objective: The one who really needs it more should have it.





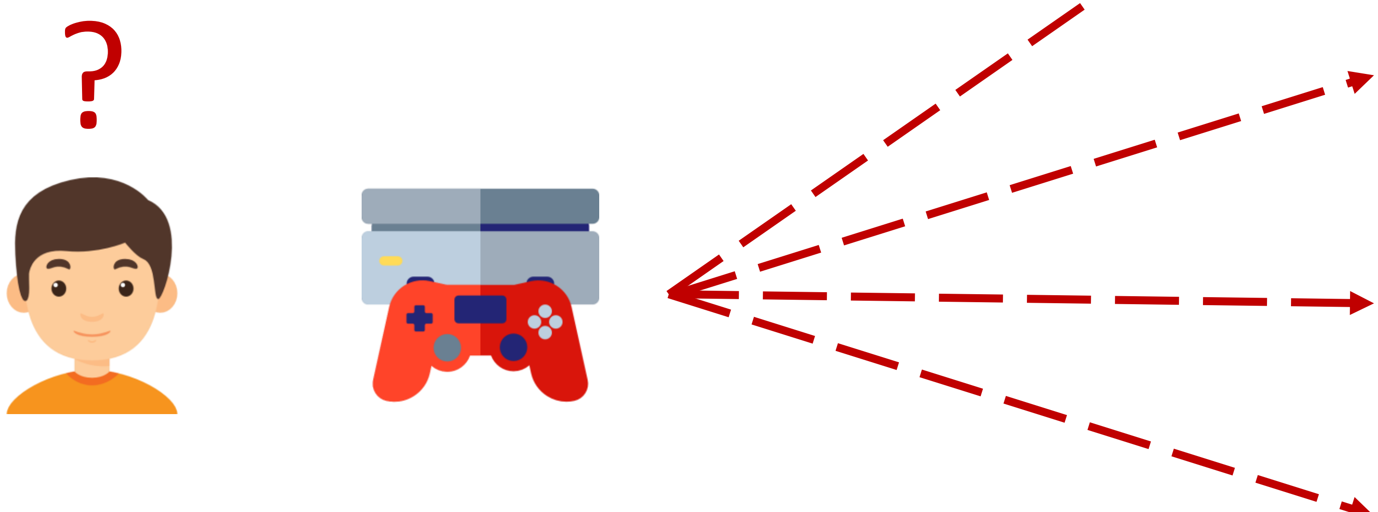


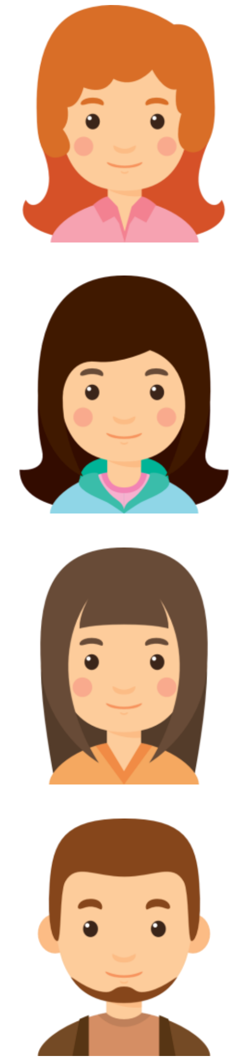
Rule 1: Each would tell me his/her value.
I'll give it to the one with the higher value.

Image Courtesy: Freepik

Single-Item Auction

Objective: The one who really needs it more should have it.



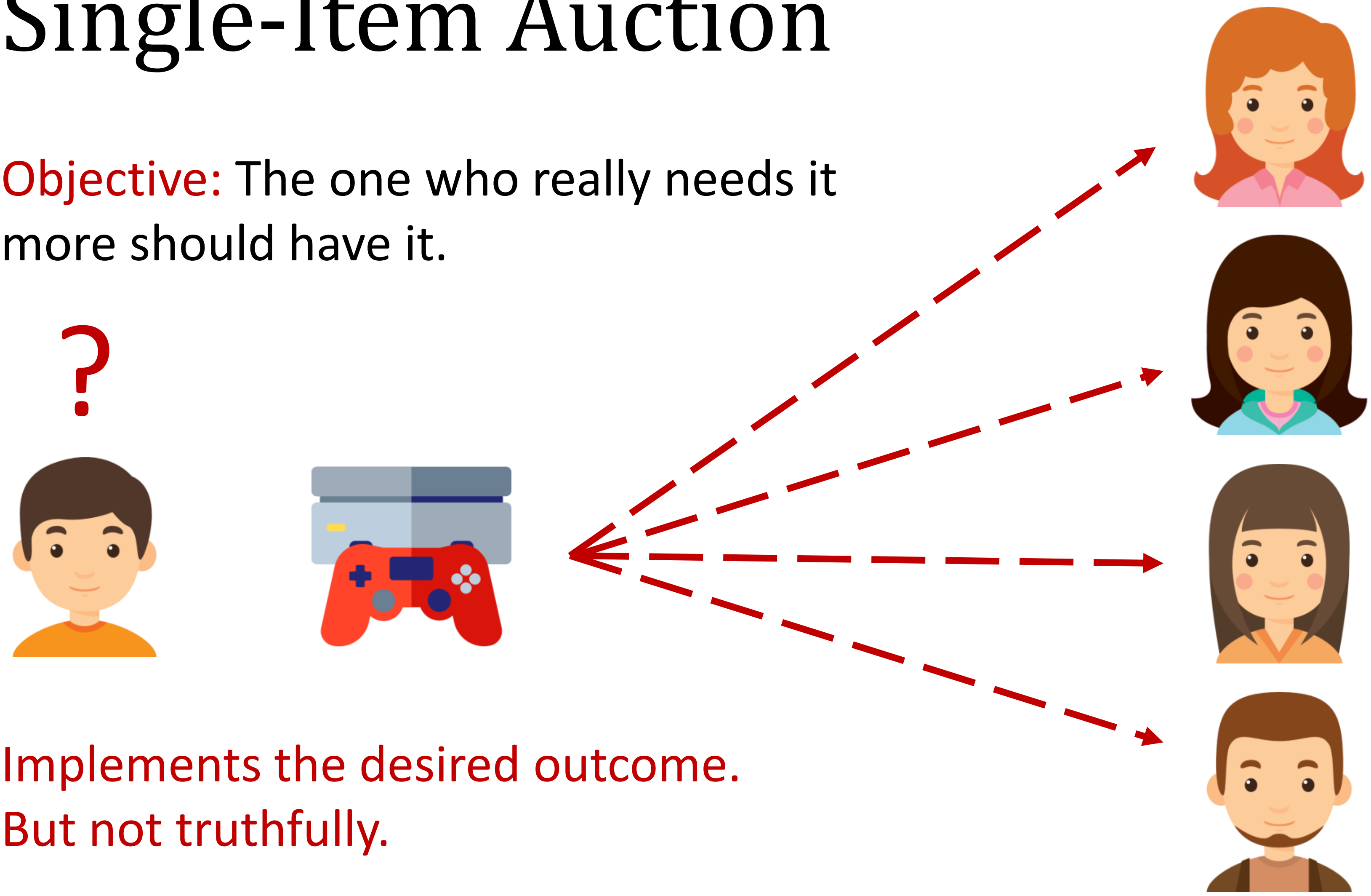


Rule 2: Each would tell me his/her value. I'll give it to the one with the higher value, but they have to pay me that value.

Image Courtesy: Freepik

Single-Item Auction

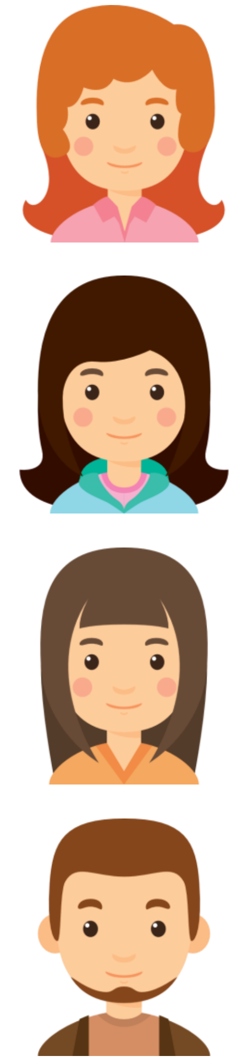
Objective: The one who really needs it more should have it.



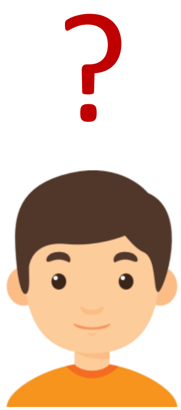
Implements the desired outcome.
But not truthfully.

Image Courtesy: Freepik

Single-Item Auction



Objective: The one who really needs it more should have it.



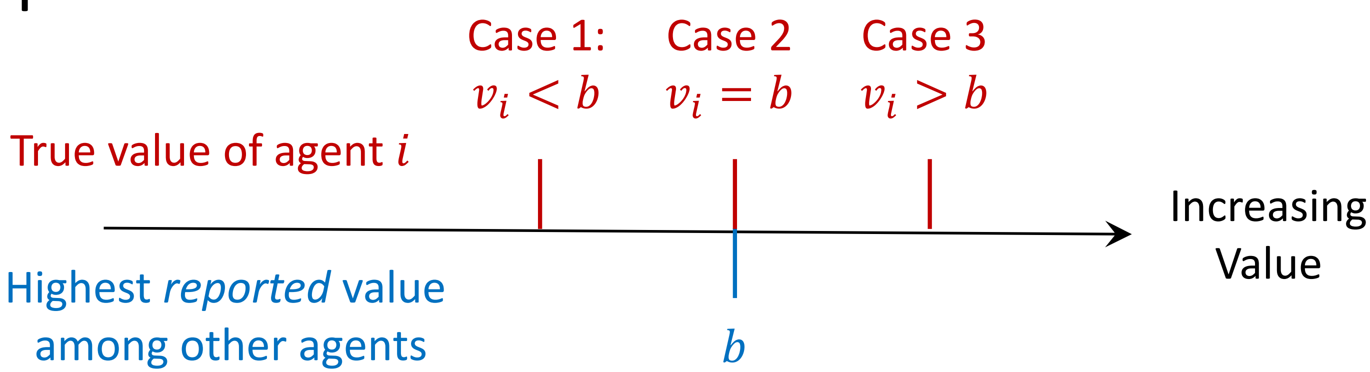
Rule 3: Each would tell me his/her value. I'll give it to the one with the highest value, and charge them the second highest value.

Image Courtesy: Freepik

Vickrey Auction: Single-Item

- f : give the item to agent $i^* \in \text{argmax}_i\, v_i$
- p : $p_{i^*} = \max_{j \neq i^*} v_j$, nothing to other agents

Theorem:
Vickrey auction is dominant strategy incentive compatible.



Vickrey Auction: Identical Items

- Two identical xboxes
  - Each agent i only wants one, has value v_i
  - Goal: give to the agents with the two highest values
- Attempt 1
  - To agent with highest value, charge 2nd highest value.
  - To agent with 2nd highest value, charge 3rd highest value.
- Attempt 2
  - To agents with highest and 2nd highest values, charge the 3rd highest value.
- Piazza Question: Which attempt(s) would be DSIC?
  - Both, 1, 2, None.

Vickrey Auction

- What if I want to give away an xbox AND a ps4?

- Each agent still wants only one of them
  - But has different values for the two
  - $v_i(xbox),\ v_i(ps4)$
  - $v_i(\{xbox, ps4\}) = \max\left(v_i(xbox), v_i(ps4)\right)$

- Who gets the xbox? Who gets the ps4? How much should I charge them?

Vickrey Auction

- Recall:
  - Set of alternatives A
  - Valuations $v = (v_i)_{i=1}^n$
  - Social choice function $f(v)$
  - Payment rule $p(v)$

- Vickrey Auction
  - $f(v) = \text{argmax}_{a \in A} \sum_i v_i(a)$
  - $p_i(v) = -\sum_{j \neq i} v_j\big(f(v)\big)$

As always, do what maximizes social welfare.

Pay (not charge!) to each agent the total value to others

Vickrey Auction

- Why is this truthful (DSIC)?
  - Suppose agent $j \neq i$ reports $\hat{v}_j$

- Utility to agent i when reporting v_i'
  - Let $f(v_i', \hat{v}_{-i}) = a$
  - $u_i = v_i(a) - \left(-\sum_{j \neq i} \hat{v}_j(a)\right)$
  - Agent i wants a to maximize $v_i(a) + \sum_{j \neq i} \hat{v}_j(a)$
  - f chooses a that maximizes $v_i'(a) + \sum_{j \neq i} \hat{v}_j(a)$
  - Simple! Report $v_i' = v_i$

Vickrey Auction

- Problem: Even to give away my single xbox, I need to pay each friend who doesn't get it the value of the friend who gets it
  - OK, I'm not that rich!

- Want two properties in addition to DSIC
  - Agents should pay the principal: $p_i(v) \geq 0$
  - Agents shouldn't pay too much: $p_i(v) \leq v_i\big(f(v)\big)$
    - Individual Rationality (IR)

Idea

- Vickrey auction
  - $f(v) = \text{argmax}_{a \in A} \sum_i v_i(a)$
  - $p_i(v) = -\sum_{j \neq i} v_j\big(f(v)\big)$

- A slight modification
  - $f(v) = \text{argmax}_{a \in A} \sum_i v_i(a)$
  - $p_i(v) = h_i(v_{-i}) - \sum_{j \neq i} v_j\big(f(v)\big)$

- Still truthful. Agent i has no control over his additional payment $h_i(v_{-i})$

VCG

- Clarke's pivot rule
  - $h_i(v_{-i}) = \max_a \sum_{j \neq i} v_j(a)$
  - Maximum welfare to others if agent i wasn't there

- VCG (Vickrey-Clarke-Groves Auction)
  - $f(v) = a^* = \operatorname{argmax}_{a \in A} \sum_i v_i(a)$
  - $p_i(v) = \left[\max_a \sum_{j \neq i} v_j(a)\right] - \left[\sum_{j \neq i} v_j(a^*)\right]$

- Payment charged to agent i = loss in welfare caused to others due to presence of agent i

VCG

- $f(v) = a^* = \text{argmax}_{a \in A} \sum_i v_i(a)$
- $p_i(v) = \left[\max_a \sum_{j \neq i} v_j(a)\right] - \left[\sum_{j \neq i} v_j(a^*)\right]$

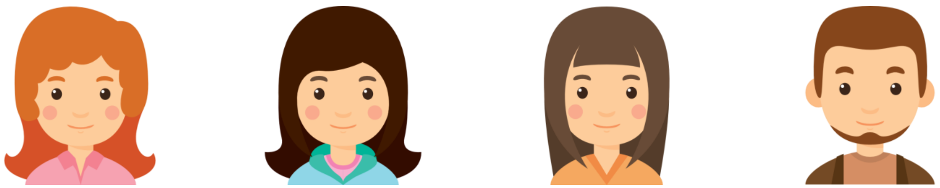
- We already saw that this is DSIC.

- Why is $p_i(v) \geq 0$?

- Why is $p_i(v) \leq v_i\big(f(v)\big)$?

VCG: Simple Example

- Let’s go back to giving away an xbox and a ps4.

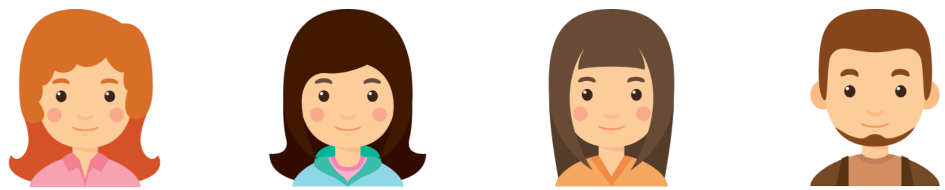


	A1	A2	A3	A4
XBox	3	4	8	7
PS4	4	2	6	1

Q: Who gets the xbox and who gets the PS4?

Q: How much do they pay?

VCG: Simple Example

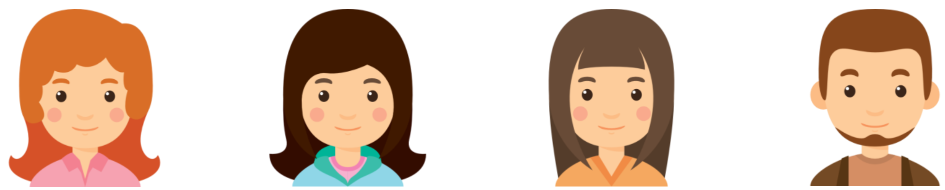


	A1	A2	A3	A4
XBox	3	4	8	7
PS4	4	2	6	1

Allocation:

- A4 gets XBox, A3 gets PS4
- Achieves maximum welfare of $7 + 6 = 13$

VCG: Simple Example

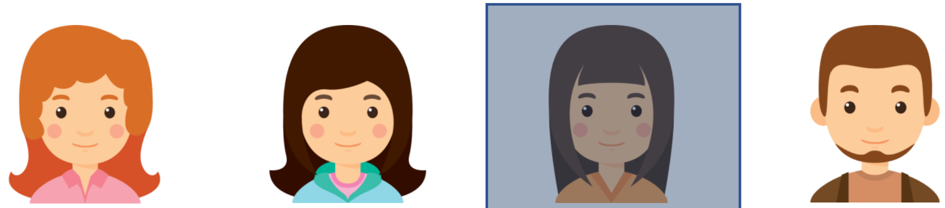


	A1	A2	A3	A4
XBox	3	4	8	7
PS4	4	2	6	1

Payments:

- Zero payments charged to A1 and A2
- “Deleting” either of them does not change the outcome or payments for others
- Can also be seen by individual rationality

VCG: Simple Example

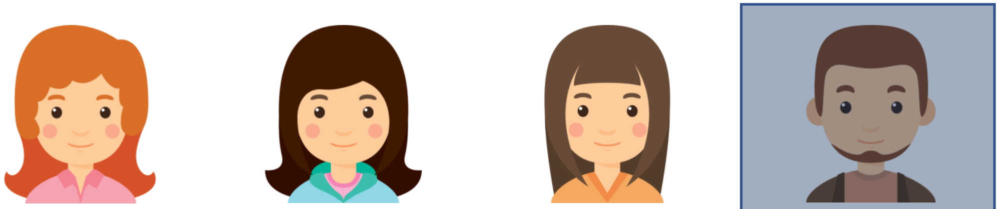


	A1	A2	A3	A4
XBox	3	4	8	7
PS4	4	2	6	1

Payments:

- Payment charged to A3 $= 11 - 7 = 4$
- Max welfare to others if A3 absent: $7 + 4 = 11$
  - Give XBox to A4 and PS4 to A1
- Welfare to others if A3 present: 7

VCG: Simple Example

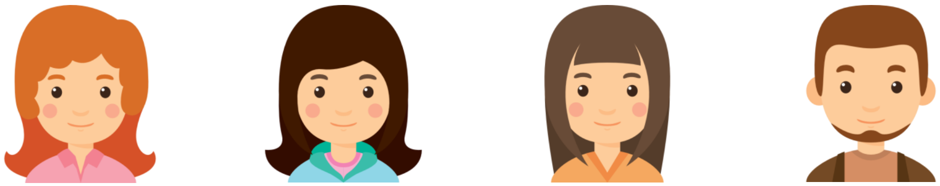


	A1	A2	A3	A4
XBox	3	4	8	7
PS4	4	2	6	1

Payments:

- Payment charged to A4 $= 12 - 6 = 6$
- Max welfare to others if A4 absent: $8 + 4 = 12$
  - Give XBox to A3 and PS4 to A1
- Welfare to others if A4 present: 6

VCG: Simple Example



	A1	A2	A3	A4
XBox	3	4	8	7
PS4	4	2	6	1

Final Outcome:

- **Allocation:** A3 gets PS4, A4 gets XBox
- **Payments:** A3 pays 4, A4 pays 6
- **Net utilities:** A3 gets $6 - 4 = 2$, A4 gets $7 - 6 = 1$

Recap

- Four properties
  - Maximize social welfare
  - Dominant strategy incentive compatibility (DSIC)
  - No payments to agents
  - Individual rationality (IR)

- Vickrey auction satisfies the first two
- VCG adds Clarke’s pivot rule to satisfy all four