CSC304 Lecture 5

Game Theory : Zero-Sum Games, The Minimax Theorem

Recap

- Last lecture
  - Cost-sharing games
    - Price of anarchy (PoA) can be n
    - Price of stability (PoS) is $O(\log n)$
  - Potential functions and pure Nash equilibria
  - Congestion games
  - Braess' paradox
  - Updated (slightly more detailed) slides
- Assignment 1 to be posted
- Volunteer note-taker

Zero-Sum Games

- Total reward constant in all outcomes (w.l.o.g. 0)
  - Common term: “zero-sum situation”
  - Psychology literature: “zero-sum thinking”
  - “Strictly competitive games”

- Focus on two-player zero-sum games (2p-zs)
  - “The more I win, the more you lose”

Zero-Sum Games

Zero-sum game: Rock-Paper-Scissor

P2 / P1	Rock	Paper	Scissor
Rock	(0 , 0)	(-1 , 1)	(1 , -1)
Paper	(1 , -1)	(0 , 0)	(-1 , 1)
Scissor	(-1 , 1)	(1 , -1)	(0 , 0)

Non-zero-sum game: Prisoner's dilemma

John / Sam	Stay Silent	Betray
Stay Silent	(-1 , -1)	(-3 , 0)
Betray	(0 , -3)	(-2 , -2)

Zero-Sum Games

- Why are they interesting?
  - Most games we play are zero-sum: chess, tic-tac-toe, rock-paper-scissor, …
  - (win, lose), (lose, win), (draw, draw)
  - (1, -1), (-1, 1), (0, 0)

- Why are they technically interesting?
  - Relation between the rewards of P1 and P2
  - P1 maximizes his reward
  - P2 maximizes his reward = minimizes reward of P1

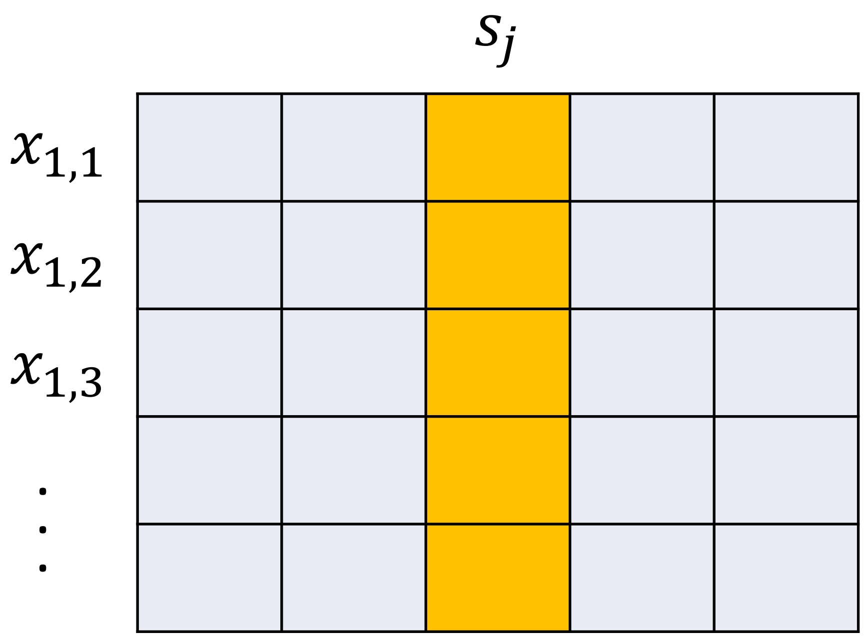
Zero-Sum Games

- Reward for P2 = - Reward for P1
  - Only need a single matrix A : reward for P1
  - P1 wants to maximize, P2 wants to minimize

P1 \ P2	Rock	Paper	Scissor
Rock	0	-1	1
Paper	1	0	-1
Scissor	-1	1	0

Rewards in Matrix Form

- Say P1 uses mixed strategy $x_1 = (x_{1,1}, x_{1,2}, \ldots)$
  - What are the rewards for P1 corresponding to different possible actions of P2?

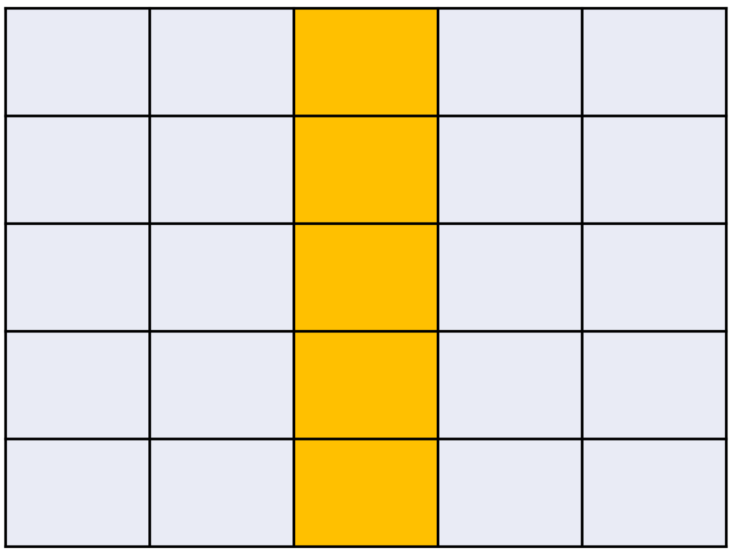


Rewards in Matrix Form

- Say P1 uses mixed strategy $x_1 = (x_{1,1}, x_{1,2}, \dots)$
  - ➢ What are the rewards for P1 corresponding to different possible actions of P2?

$$[x_{1,1}, x_{1,2}, x_{1,3}, \dots] \quad *$$

s_j



❖ Reward for P1 when P2 chooses s_j = $\left(x_1^T * A\right)_j$

Rewards in Matrix Form

- Reward for P1 when…
  - P1 uses mixed strategy x_1
  - P2 uses mixed strategy x_2

$$\left[\left(x_1^T * A\right)_1, \left(x_1^T * A\right)_2, \left(x_1^T * A\right)_3 \dots\right] \quad * \quad \begin{bmatrix} x_{2,1} \\ x_{2,2} \\ x_{2,3} \\ \vdots \end{bmatrix}$$

$$= \quad x_1^T * A * x_2$$

How would the two players act do in this zero-sum game?

John von Neumann, 1928

Maximin Strategy

- Worst-case thinking by P1…
  - If I choose mixed strategy x_1…
  - P2 would choose x_2 to minimize my reward (i.e., maximize his reward)
  - Let me choose x_1 to maximize this "worst-case reward"

$$V_1^* = \max_{x_1} \min_{x_2} x_1^T * A * x_2$$

Maximin Strategy

$$V_1^* = \max_{x_1} \min_{x_2} x_1^T * A * x_2$$

- V_1^* : maximin value of P1
- x_1^* (maximizer) : maximin strategy of P1
- "By playing x_1^*, I guarantee myself at least V_1^*"

- But if P1 $\rightarrow x_1^*$, P2's best response $\rightarrow \hat{x}_2$
  - Will x_1^* be the best response to $\hat{x}_2$?

Maximin vs Minimax

Player 1

Choose my strategy to maximize my reward, worst-case over P2's response

$$V_1^* = \max_{x_1} \min_{x_2} x_1^T * A * x_2$$

x_1^*

Player 2

Choose my strategy to minimize P1's reward, worst-case over P1's response

$$V_2^* = \min_{x_2} \max_{x_1} x_1^T * A * x_2$$

x_2^*

Question: Relation between V_1^* and V_2^*?

Maximin vs Minimax

$$V_1^* = \max_{x_1} \min_{x_2} x_1^T * A * x_2 \qquad V_2^* = \min_{x_2} \max_{x_1} x_1^T * A * x_2$$

x_1^* x_2^*

- What if (P1,P2) play (x_1^*, x_2^*)?
  - P1 must get at least V_1^* (ensured by P1)
  - P1 must get at most V_2^* (ensured by P2)
  - $V_1^* \leq V_2^*$

The Minimax Theorem

- Jon von Neumann [1928]

- Theorem: For any 2p-zs game,
  - $V_1^* = V_2^* = V^*$ (called the minimax value of the game)
  - Set of Nash equilibria =

    $\{(\mathrm{x}_1^*, \mathrm{x}_2^*) : \mathrm{x}_1^* = \text{maximin for P1}, \mathrm{x}_2^* = \text{minimax for P2}\}$

- Corollary: x_1^* is best response to x_2^* and vice-versa.

The Minimax Theorem

- Jon von Neumann [1928]

  "As far as I can see, there could be no theory of games … without that theorem …

  I thought there was nothing worth publishing until the Minimax Theorem was proved"

- An unequivocal way to "solve" zero-sum games
  - Optimal strategies for P1 and P2 (up to ties)
  - Optimal rewards for P1 and P2 under a rational play

Proof of the Minimax Theorem

- Simpler proof using Nash's theorem
  - But predates Nash's theorem

- Suppose $(\tilde{x}_1, \tilde{x}_2)$ is a NE

- P1 gets value $\tilde{v} = (\tilde{x}_1)^T A\, \tilde{x}_2$

- $\tilde{x}_1$ is best response for P1 : $\tilde{v} = \max_{x_1} (x_1)^T A\, \tilde{x}_2$

- $\tilde{x}_2$ is best response for P2 : $\tilde{v} = \min_{x_2} (\tilde{x}_1)^T A\, x_2$

Proof of the Minimax Theorem

$$V_2^* = \min_{x_2} \max_{x_1} x_1^T * A * x_2 \leq$$

$$\max_{x_1} (x_1)^T A\, \tilde{x}_2 = \tilde{v} = \min_{x_2} (\tilde{x}_1)^T A\, x_2$$

$$\leq \max_{x_1} \min_{x_2} x_1^T * A * x_2 = V_1^*$$

- But we already saw $V_1^* \leq V_2^*$
  - $V_1^* = V_2^*$

Proof of the Minimax Theorem

$$V_2^* = \min_{x_2} \max_{x_1} x_1^T * A * x_2 =$$

$$\max_{x_1} (x_1)^T A\, \tilde{x}_2 = \tilde{v} = \max_{x_2} (\tilde{x}_1)^T A\, x_2$$

$$= \max_{x_1} \min_{x_2} x_1^T * A * x_2 = V_1^*$$

- When $(\tilde{x}_1, \tilde{x}_2)$ is a NE, $\tilde{x}_1$ and $\tilde{x}_2$ must be maximin and minimax strategies for P1 and P2, respectively.
- The reverse direction is also easy to prove.

Computing Nash Equilibria

- Can I practically compute a maximin strategy (and thus a Nash equilibrium of the game)?
- Wasn't it computationally hard even for 2-player games?
- For 2p-zs games, a Nash equilibrium can be computed in polynomial time using linear programming.
  - Polynomial in #actions of the two players: m_1 and m_2

Computing Nash Equilibria

Maximize v

Subject to

$$\left(x_1^T A\right)_j \geq v,\ j \in \{1, \dots, m_2\}$$

$$x_1(1) + \cdots + x_1(m_1) = 1$$

$$x_1(i) \geq 0, i \in \{1, \dots, m_1\}$$

Minimax Theorem in Real Life?

- If you were to play a 2-player zero-sum game (say, as player 1), would you always play a maximin strategy?

- What if you were convinced your opponent is an idiot?

- What if you start playing the maximin strategy, but observe that your opponent is not best responding?

Minimax Theorem in Real Life?

Kicker \ Goalie	L	R
L	0.58	0.95
R	0.93	0.70

Kicker

Maximize v

Subject to

$0.58p_L + 0.93p_R \geq v$

$0.95p_L + 0.70p_R \geq v$

$p_L + p_R = 1$

$p_L \geq 0, p_R \geq 0$

Goalie

Minimize v

Subject to

$0.58q_L + 0.95q_R \leq v$

$0.93q_L + 0.70q_R \leq v$

$q_L + q_R = 1$

$q_L \geq 0, q_R \geq 0$

Minimax Theorem in Real Life?

Kicker \ Goalie	L	R
L	0.58	0.95
R	0.93	0.70

Kicker
Maximin:
$p_L = 0.38, p_R = 0.62$

Reality:
$p_L = 0.40, p_R = 0.60$

Goalie
Maximin:
$q_L = 0.42, q_R = 0.58$

Reality:
$p_L = 0.423, q_R = 0.577$

Some evidence that people may play minimax strategies.

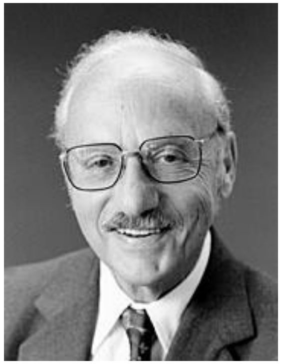
Minimax Theorem

- We proved it using Nash's theorem
  - Cheating. Typically, Nash's theorem (for the special case of 2p-zs games) is proved using the minimax theorem.

- Useful for proving Yao's principle, which provides lower bound for randomized algorithms

- Equivalent to linear programming duality

John von Neumann



George Dantzig

von Neumann and Dantzig

George Dantzig loves to tell the story of his meeting with John von Neumann on October 3, 1947 at the Institute for Advanced Study at Princeton. Dantzig went to that meeting with the express purpose of describing the linear programming problem to von Neumann and asking him to suggest a computational procedure. He was actually looking for methods to benchmark the simplex method. Instead, he got a 90-minute lecture on Farkas Lemma and Duality (Dantzig's notes of this session formed the source of the modern perspective on linear programming duality). Not wanting Dantzig to be completely amazed, von Neumann admitted:

"I don't want you to think that I am pulling all this out of my sleeve like a magician. I have recently completed a book with Morgenstern on the theory of games. What I am doing is conjecturing that the two problems are equivalent. The theory that I am outlining is an analogue to the one we have developed for games.“

- (Chandru & Rao, 1999)