CSC304 Lecture 4
Guest Lecture: Prof. Allan Borodin

Game Theory
(Cost sharing & congestion games, Potential function, Braess' paradox)

Recap

- Finding pure and mixed Nash equilibria
  - Best response diagrams
  - Indifference principle

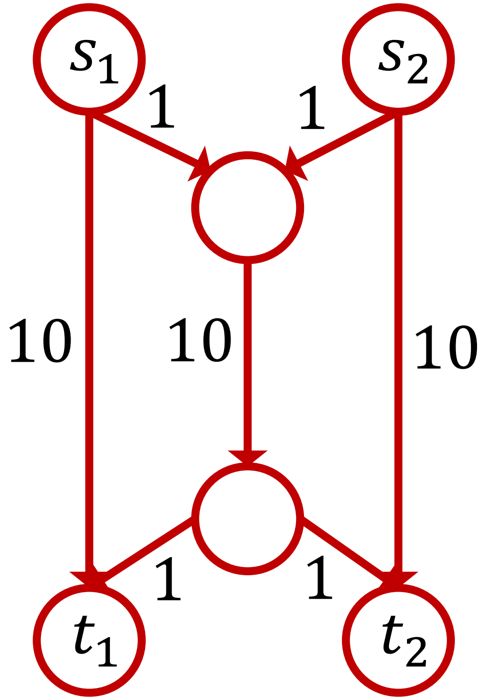
- Price of Anarchy (PoA) and Price of Stability (PoS)
  - How does the Nash equilibrium compare to the social optimum, in the worst case and in the best case?

Cost Sharing Game

- n players on directed weighted graph G

- Player i
  - Wants to go from s_i to t_i
  - Strategy set S_i = {directed $s_i \rightarrow t_i$ paths}
  - Denote his chosen path by $P_i \in S_i$

- Each edge e has cost c_e (weight)
  - Cost is split among all players taking edge e
  - That is, among all players i with $e \in P_i$

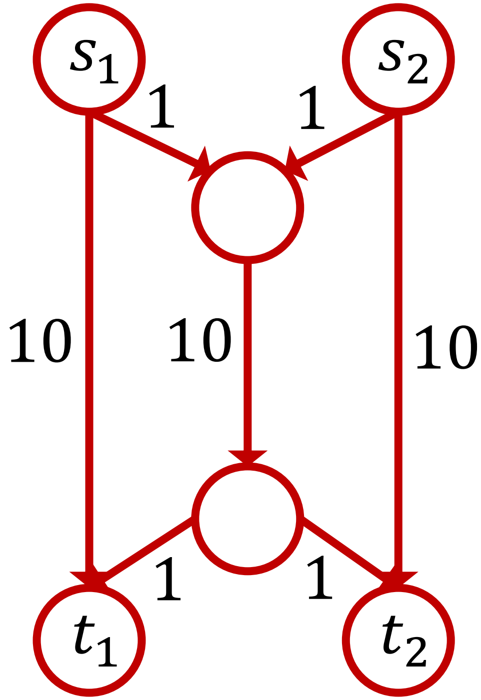


Cost Sharing Game

- Given strategy profile $\vec{P}$, cost $c_i\left(\vec{P}\right)$ to player i is sum of his costs for edges $e \in P_i$

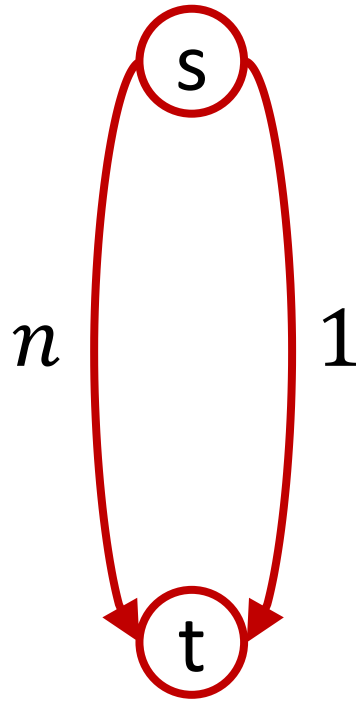
- Social cost $C\left(\vec{P}\right) = \sum_i c_i\left(\vec{P}\right)$
  - Note that $C\left(\vec{P}\right) = \sum_{e \in E(\vec{P})} c_e$, where $E(\vec{P})$={edges taken in $\vec{P}$ by at least one player}

- In the example on the right:
  - What if both players take the direct paths?
  - What if both take the middle paths?
  - What if only one player takes the middle path while the other takes the direct path?



Cost Sharing: Simple Example

- Example on the right: n players
- Two pure NE
  - All taking the n-edge: social cost = n
  - All taking the 1-edge: social cost = 1
    - Also the social optimum



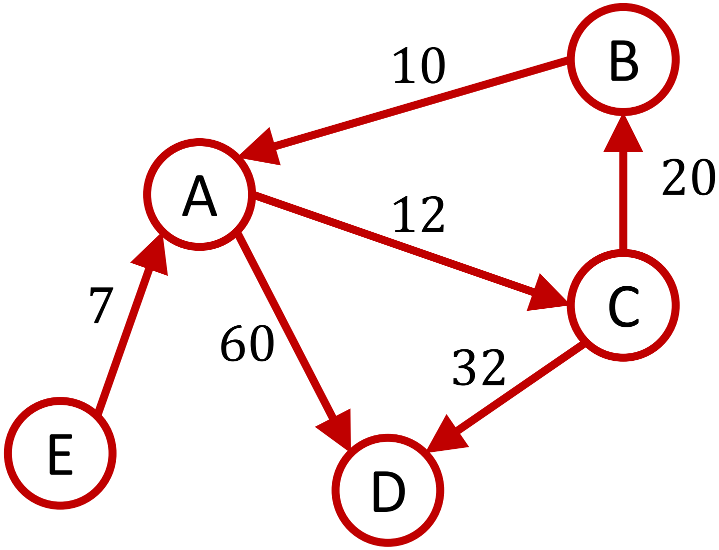
- In this game, price of anarchy $\geq n$
- We can show that for all cost sharing games, price of anarchy $\leq n$

Cost Sharing: PoA

- **Theorem:** The price of anarchy of a cost sharing game is at most n.
- **Proof:**
  - Suppose the social optimum is $(P_1^*, P_2^*, \dots, P_n^*)$, in which the cost to player i is c_i^*.
  - Take any NE with cost c_i to player i.
  - Let c_i' be his cost if he switches to P_i^*.
  - NE $\Rightarrow c_i' \geq c_i$ (Why?)
  - But : $c_i' \leq n \cdot c_i^*$ (Why?)
  - $c_i \leq n \cdot c_i^*$ for each i $\Rightarrow$ no worse than $n \times$ optimum ■

Cost Sharing

- Price of anarchy
  - All cost-sharing games: PoA $\leq n$
  - Example game where PoA $= n$
- Price of stability? Later…
- Both examples we saw had pure Nash equilibria
  - What about more complex games, like the one on the right?



10 players: $E \rightarrow C$
27 players: $B \rightarrow D$
19 players: $C \rightarrow D$

Good News

- Theorem: All cost sharing games have a pure Nash eq.

- Proof:
  - Via "potential function" argument

Step 1: Define Potential Fn

- Potential function: $\Phi : \prod_i S_i \rightarrow \mathbb{R}_+$
  - For all pure strategy profiles $\vec{P} = (P_1, \dots, P_n) \in \prod_i S_i$, …
  - all players i, and …
  - all alternative strategies $P_i' \in S_i$ for player i…

$$c_i\left(P_i', \vec{P}_{-i}\right) - c_i\left(\vec{P}\right) = \Phi\left(P_i', \vec{P}_{-i}\right) - \Phi\left(\vec{P}\right)$$

- When a single player changes his strategy, the change in *his* cost is equal to the change in the potential function
  - Do not care about the changes in the costs to others

Step 2: Potential $F^n \rightarrow$ pure Nash Eq

- All games that admit a potential function have a pure Nash equilibrium. Why?
  - Think about $\vec{P}$ that minimizes the potential function.
  - What happens when a player deviates?
    - If his cost decreases, the potential function value must also decrease.
    - $\vec{P}$ already minimizes the potential function value.

- Pure strategy profile minimizing potential function is a pure Nash equilibrium.

Step 3: Potential F^n for Cost-Sharing

- Recall: $E(\vec{P}) = \{\text{edges taken in } \vec{P} \text{ by at least one player}\}$

- Let $n_e(\vec{P})$ be the number of players taking e in $\vec{P}$

$$\Phi\left(\vec{P}\right) = \sum_{e \in E(\vec{P})} \sum_{k=1}^{n_e(\vec{P})} \frac{c_e}{k}$$

- Note: The cost of edge e to each player taking e is $c_e/n_e(\vec{P})$. But the potential function includes all fractions: $c_e/1, c_e/2, \ldots, c_e/n_e\left(\vec{P}\right)$.

Step 3: Potential F^{n} for Cost-Sharing

$$\Phi\left(\vec{P}\right) = \sum_{e \in E(\vec{P})} \sum_{k=1}^{n_e(\vec{P})} \frac{c_e}{k}$$

- Why is this a potential function?
  - If a player changes path, he pays $\frac{c_e}{n_e(\vec{P})+1}$ for each new edge e, gets back $\frac{c_f}{n_f(\vec{P})}$ for each old edge f.
  - This is precisely the change in the potential function too.
  - So $\Delta c_i = \Delta\Phi$.

■

Potential Minimizing Eq.

- There could be multiple pure and multiple mixed Nash equilibria
  - Pure Nash equilibria are "local minima" of the potential function.
  - *A single player* deviating should not decrease the function value.

- Minimizing the potential function just gives one of the pure Nash equilibria
  - Is this equilibrium special? Yes!

Potential Minimizing Eq.

- $$\sum_{e \in E(\vec{P})} c_e \leq \Phi\left(\vec{P}\right) = \sum_{e \in E(\vec{P})} \sum_{k=1}^{n_e(\vec{P})} \frac{c_e}{k} \leq \sum_{e \in E(\vec{P})} c_e * \sum_{k=1}^{n} \frac{1}{k}$$

Social cost

Harmonic function $H(n)$
$= \sum_{k=1}^{n} 1/n = O(\log n)$

- $$\forall \vec{P},\ C\left(\vec{P}\right) \leq \Phi\left(\vec{P}\right) \leq C\left(\vec{P}\right) * H(n)$$

- $$C\left(\vec{P}^*\right) \leq \Phi\left(\vec{P}^*\right) \leq \Phi(OPT) \leq C(OPT) * H(n)$$

Potential minimizing eq.

Social optimum

Potential Minimizing Eq.

- Potential minimizing equilibrium gives $O(\log n)$ approximation to the social optimum
  - Price of stability is $O(\log n)$
  - Compare to the price of anarchy, which can be n

Congestion Games

- Generalize cost sharing games
- n players, m resources (e.g., edges)
- Each player i chooses a set of resources P_i (e.g., $s_i \rightarrow t_i$ paths)
- When n_j player use resource j, each of them get a cost $f_j(n_j)$
- Cost to player is the sum of costs of resources used

Congestion Games

- Theorem [Rosenthal 1973]: Every congestion game is a potential game.

- Potential function:

$$\Phi\left(\vec{P}\right) = \sum_{j \in E(\vec{P})} \sum_{k=1}^{n_j(\vec{P})} f_j(k)$$

- Theorem [Monderer and Shapley 1996]: Every potential game is equivalent to a congestion game.

Potential Functions

- Potential functions are useful for deriving various results
  - E.g., used for analyzing amortized complexity of algorithms

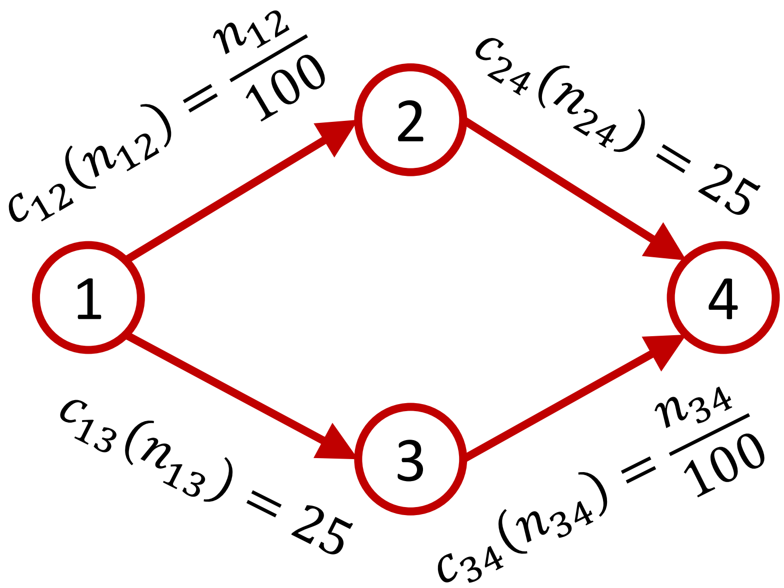
- Bad news: Finding a potential function that works may be hard.

The Braess' Paradox

- In cost sharing, f_j is decreasing
  - The more people use a resource, the less the cost to each.
- f_j can also be increasing
  - Road network, each player going from home to work
  - Uses a sequence of roads
  - The more people on a road, the greater the congestion, the greater the delay (cost)
- Can lead to unintuitive phenomena

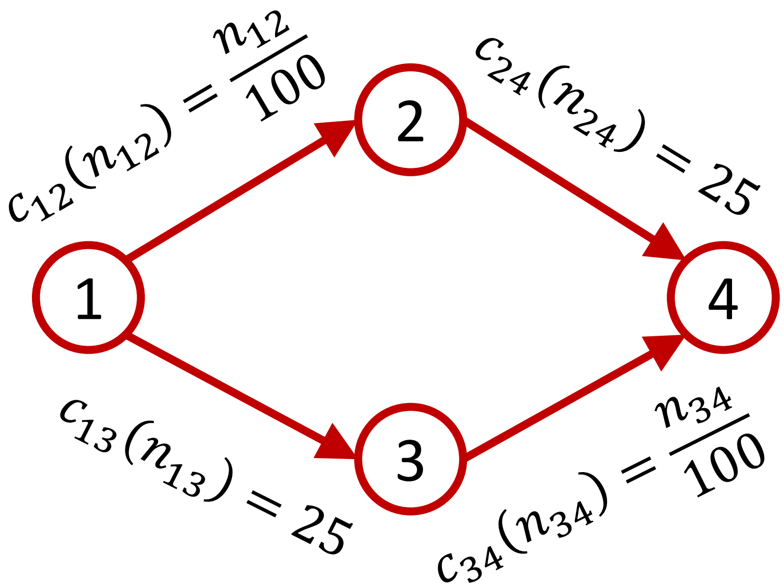
The Braess' Paradox

- Parkes-Seuken Example:
  - 2000 players want to go from 1 to 4
  - $1 \rightarrow 2$ and $3 \rightarrow 4$ are "congestible" roads
  - $1 \rightarrow 3$ and $2 \rightarrow 4$ are "constant delay" roads



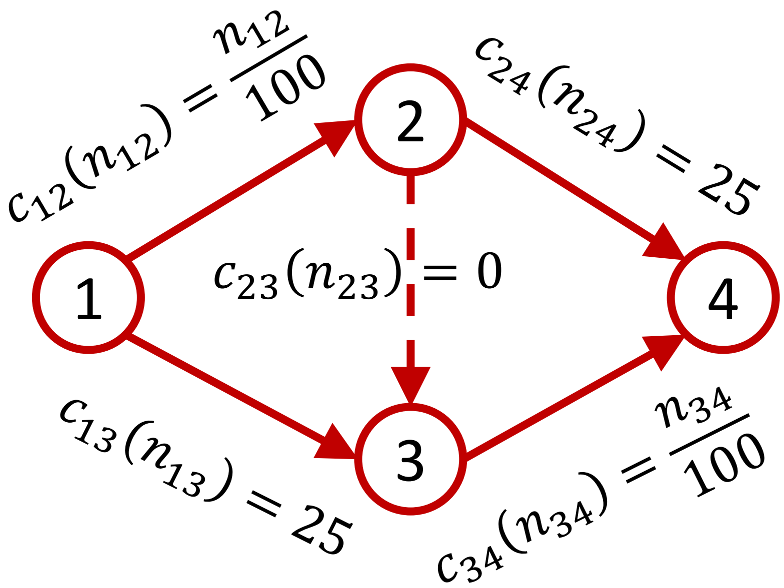
The Braess' Paradox

- Pure Nash equilibrium?
  - 1000 take $1 \to 2 \to 4$, 1000 take $1 \to 3 \to 4$
  - Each player has cost $10 + 25 = 35$
  - Anyone switching to the other creates a greater congestion on it, and faces a higher cost



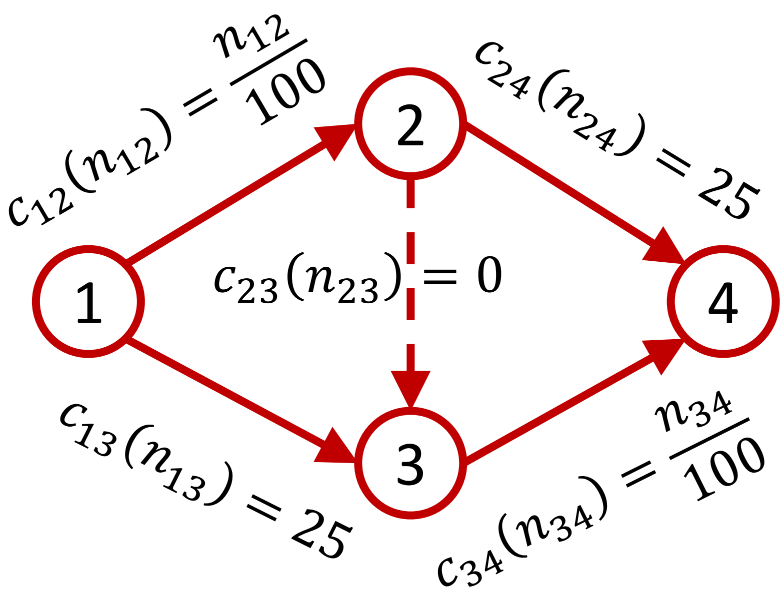
The Braess' Paradox

- What if we add a zero-cost connection $2 \rightarrow 3$?
  - Intuitively, adding more roads should only be helpful
  - In reality, it leads to a greater delay for everyone in the unique equilibrium!



The Braess' Paradox

- Nobody chooses $1 \rightarrow 3$ as $1 \rightarrow 2 \rightarrow 3$ is better irrespective of how many other players take it
- Similarly, nobody chooses $2 \rightarrow 4$
- Everyone takes $1 \rightarrow 2 \rightarrow 3 \rightarrow 4$, faces delay = 40!



The Braess' Paradox

- In fact, what we showed is:
  - In the new game, $1 \rightarrow 2 \rightarrow 3 \rightarrow 4$ is a strictly dominant strategy for each firm!