CSC304 Lecture 2

Game Theory
(Basic Concepts)

Game Theory

- How do rational, self-interested agents act?
- Each agent has a set of possible actions
- Rules of the game:
  - Rewards for the agents as a function of the actions taken by different agents

- We focus on noncooperative games
  - No external force or agencies enforcing coalitions

Normal Form Games

- A set of players $\mathrm{N} = \{1, \dots, n\}$
- A set of actions S
  - Action of player $i \rightarrow s_i$
  - Action profile $\vec{s} = (s_1, \dots, s_n)$

- For each player i, utility function $u_i : S^n \rightarrow \mathbb{R}$
  - Given action profile $\vec{s} = (s_1, \dots, s_n)$, each player i gets reward $u_i(s_1, \dots, s_n)$

Normal Form Games

Recall: Prisoner's dilemma $\quad S = \{\text{Silent}, \text{Betray}\}$

John's Actions / Sam's Actions	Stay Silent	Betray
Stay Silent	**(-1 , -1)**	**(-3 , 0)**
Betray	**(0 , -3)**	**(-2 , -2)**

$$\boxed{u_{Sam}(\underbrace{Betray}_{s_{Sam}}, \underbrace{Silent}_{s_{John}})} \qquad \boxed{u_{John}(Betray, Silent)}$$

Player Strategies

- Pure strategy
  - Choose an action to play
  - E.g., "Betray"
  - For our purposes, simply an action.
    - In repeated or multi-move games (like Chess), need to choose an action to play at every step of the game based on history.

- Mixed strategy
  - Choose a probability distribution over actions
  - Randomize over pure strategies
  - E.g., "Betray with probability 0.3, and stay silent with probability 0.7"

Dominant Strategies

- For player i, s_i dominates s_i' if playing s_i "is better than" playing s_i' irrespective of the strategies of the other players.

- Two variants: Weakly dominate / Strictly dominate
  - $u_i(s_i, \vec{s}_{-i}) \geq u_i(s_i', \vec{s}_{-i}), \forall \vec{s}_{-i}$
  - Strict inequality for some $\vec{s}_{-i}$ ← Weak
  - Strict inequality for all $\vec{s}_{-i}$ ← Strict

Dominant Strategies

- s_i is a strictly (or weakly) dominant strategy for player i if
  - it strictly (or weakly) dominates every other strategy

- If there exists a strictly dominant strategy
  - Only makes sense to play it

- If every player has a strictly dominant strategy
  - Determines the rational outcome of the game

Example: Prisoner's Dilemma

- Recap:

John's Actions / Sam's Actions	Stay Silent	Betray
Stay Silent	**(-1 , -1)**	**(-3 , 0)**
Betray	**(0 , -3)**	**(-2 , -2)**

- Each player strictly wants to
  - Betray if the other player will stay silent
  - Betray if the other player will betray

- Betray = strictly dominant strategy for each player

Iterated Elimination

- What if there are no dominant strategies?
  - No single strategy dominates every other strategy
  - But some strategies might still be dominated

- Assuming everyone knows everyone is rational…
  - Can remove their dominated strategies
  - Might reveal a newly dominant strategy

- Eliminating only strictly dominated vs eliminating weakly dominated

Iterated Elimination

- Toy example:
  - Microsoft vs Startup
  - Enter the market or stay out?

Microsoft \ Startup	Enter	Stay Out
Enter	(2 , -2)	(4 , 0)
Stay Out	(0 , 4)	(0 , 0)

- Q: Is there a dominant strategy for startup?
- Q: Do you see a rational outcome of the game?

Iterated Elimination

- More serious: “Guess 2/3 of average”
  - Each student guesses a real number between 0 and 100 (inclusive)
  - The student whose number is the closest to 2/3 of the average of all numbers wins!

- Q: What would you do?

Nash Equilibrium

- If you can find strictly dominant strategies…
  - Either directly, or by iteratively eliminating dominated strategies
  - Rational outcome of the game

- What if this doesn't help?

Students \ Professor	Attend	Be Absent
Attend	(3 , 1)	(-1 , -3)
Be Absent	(-1 , -1)	(0 , 0)

Nash Equilibrium

- Domination
  - X dominates Y = "Play X instead of Y irrespective of what others are doing"
  - Too strong
  - Replace by "given what others are doing"

- Nash Equilibrium
  - A strategy profile $\vec{s}$ is in Nash equilibrium if s_i is the best action for player i given that other players are playing $\vec{s}_{-i}$

$$u_i(s_i, \vec{s}_{-i}) \geq u_i(s_i', \vec{s}_{-i}), \forall s_i'$$

No quantifier on $\vec{s}_{-i}$

Recap: Prisoner's Dilemma

John's Actions / Sam's Actions	Stay Silent	Betray
Stay Silent	(-1 , -1)	(-3 , 0)
Betray	(0 , -3)	(-2 , -2)

- Nash equilibrium?

- Q: If player i has a strictly dominant strategy…
  a) It has nothing to do with Nash equilibria.
  b) It must be part of some Nash equilibrium.
  c) It must be part of all Nash equilibria.

Recap: Prisoner's Dilemma

John's Actions / Sam's Actions	Stay Silent	Betray
Stay Silent	(-1 , -1)	(-3 , 0)
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- Nash equilibrium?

- Q: If player i has a weakly dominant strategy…
  a) It has nothing to do with Nash equilibria.
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Recap: Microsoft vs Startup

Microsoft \ Startup	Enter	Stay Out
Enter	(2 , -2)	(4 , 0)
Stay Out	(0 , 4)	(0 , 0)

- Nash equilibrium?

- Q: Removal of strictly dominated strategies…
  a) Might remove existing Nash equilibria.
  b) Might add new Nash equilibria.
  c) Both of the above.
  d) None of the above.

Recap: Microsoft vs Startup

Microsoft \ Startup	Enter	Stay Out
Enter	(2 , -2)	(4 , 0)
Stay Out	(0 , 4)	(0 , 0)

- Nash equilibrium?

- Q: Removal of weakly dominated strategies…
  a) Might remove existing Nash equilibria.
  b) Might add new Nash equilibria.
  c) Both of the above.
  d) None of the above.

Recap: Attend or Not

Students \ Professor	Attend	Be Absent
Attend	(3 , 1)	(-1 , -3)
Be Absent	(-1 , -1)	(0 , 0)

- Nash equilibrium?

Example: Stag Hunt

Hunter 2 \ Hunter 1	Stag	Hare
Stag	(4 , 4)	(0 , 2)
Hare	(2 , 0)	(1 , 1)

- Game:
  - Each hunter decides to hunt stag or hare.
  - Stag = 8 days of food, hare = 2 days of food
  - Catching stag requires both hunters, catching hare requires only one.
  - If they catch only one animal, they share.
- Nash equilibrium?

Example: Rock-Paper-Scissor

P1 / P2	Rock	Paper	Scissor
Rock	(0 , 0)	(-1 , 1)	(1 , -1)
Paper	(1 , -1)	(0 , 0)	(-1 , 1)
Scissor	(-1 , 1)	(1 , -1)	(0 , 0)

- Nash equilibrium?

Example: Inspect Or Not

Inspector / Driver	Inspect	Don't Inspect
Pay Fare	**(-10 , -1)**	**(-10 , 0)**
Don't Pay Fare	**(-90 , 29)**	**(0 , -30)**

- Game:
  - Fare = 10
  - Cost of inspection = 1
  - Fine if fare not paid = 30
  - Total cost to driver if caught = 90

- Nash equilibrium?

Nash's Beautiful Result

- Theorem: Every normal form game admits a mixed-strategy Nash equilibrium.

- What about Rock-Paper-Scissor?

P2 \ P1	Rock	Paper	Scissor
Rock	(0 , 0)	(-1 , 1)	(1 , -1)
Paper	(1 , -1)	(0 , 0)	(-1 , 1)
Scissor	(-1 , 1)	(1 , -1)	(0 , 0)

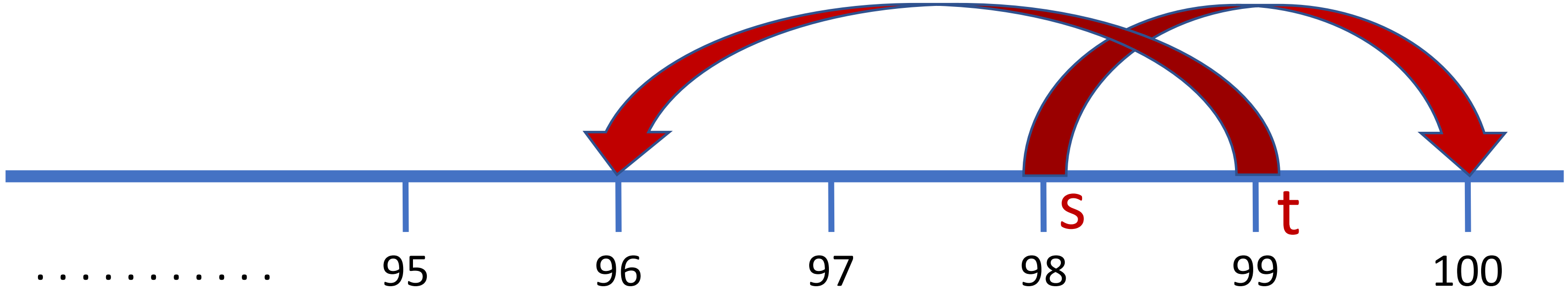
Indifference Principle

- *If the mixed strategy of player i in a Nash equilibrium randomizes over a set of pure strategies T_i, then the expected payoff to player i from each pure strategy in T_i must be identical.*

- Derivation of rock-paper-scissor on the blackboard.

Extra Fun 1: Cunning Airlines

- Two travelers lose their luggage.
- Airline agrees to refund up to $100 to each.
- Policy: Both travelers would submit a number between 2 and 99 (inclusive).
  - If both report the same number, each gets this value.
  - If one reports a lower number (s) than the other (t), the former gets s+2, the latter gets s-2.



Extra Fun 2: Ice Cream Shop

- Two brothers, each wants to set up an ice cream shop on the beach ([0,1]).
- If the shops are at s, t (with $s \leq t$)
  - ➢ The brother at s gets $\left[0, \frac{s+t}{2}\right]$, the other gets $\left[\frac{s+t}{2}, 1\right]$