CSC304 Lecture 19

Fair Division 2: Cake-cutting, Indivisible goods

Recall: Cake-Cutting

- A heterogeneous, divisible good
  - Represented as $[0,1]$
- Set of players $N = \{1, \dots, n\}$
  - Each player i has valuation V_i
- Allocation $A = (A_1, \dots, A_n)$
  - Disjoint partition of the cake

Recall: Cake-Cutting

- We looked at two measures of fairness:

- Proportionality: $\forall i \in N: V_i(A_i) \geq {}^{1}/_{n}$
  - "Every agent should get her fair share."

- Envy-freeness: $\forall i, j \in N: V_i(A_i) \geq V_i(A_j)$
  - "No agent should prefer someone else's allocation."

Other Desiderata

- There are two more properties that we often desire from an allocation.

- Pareto optimality (PO)
  - Notion of efficiency
  - Informally, it says that there should be no “obviously better” allocation

- Strategyproofness (SP)
  - No player should be able to gain by misreporting her valuation

Strategyproofness (SP)

- For deterministic mechanisms
  - "Strategyproof": No player should be able to increase her *utility* by misreporting her valuation, irrespective of what other players report.

- For randomized mechanisms
  - "Strategyproof-in-expectation": No player should be able to increase her *expected utility* by misreporting.
  - For simplicity, we'll call this strategyproofness, and assume we mean "in expectation" if the mechanism is randomized.

Strategyproofness (SP)

- Deterministic
  - Bad news!
  - Theorem [Menon & Larson '17] : No deterministic SP mechanism is (even approximately) proportional.
- Randomized
  - Good news!
  - Theorem [Chen et al. '13, Mossel & Tamuz '10]: There is a randomized SP mechanism that always returns an envy-free allocation.

Perfect Partition

- Theorem [Lyapunov '40]:
  - There always exists a "perfect partition" $(B_1, \ldots, B_n)$ of the cake such that $V_i(B_j) = {}^1/_n$ for every $i, j \in [n]$.
  - Every agent values every bundle equally.

- Theorem [Alon '87]:
  - There exists a perfect partition that only cuts the cake at $poly(n)$ points.
  - In contrast, Lyapunov's proof is non-constructive, and might need an unbounded number of cuts.

Perfect Partition

- Q: Can you use an algorithm for computing a perfect partition as a black-box to design a randomized SP+EF mechanism?
  - Yes! Compute a perfect partition, and assign the n bundles to the n players uniformly at random.
  - Why is this EF?
    - Every agent values every bundle at $^1/_n$.
  - Why is this SP-in-expectation?
    - Because an agent is assigned a random bundle, her expected utility is $^1/_n$, irrespective of what she reports.

Pareto Optimality (PO)

- Definition
  - We say that an allocation $A = (A_1, \ldots, A_n)$ is PO if there is no alternative allocation $B = (B_1, \ldots, B_n)$ such that
  1. Every agent is at least as happy: $V_i(B_i) \geq V_i(A_i), \forall i \in N$
  2. Some agent is strictly happier: $V_i(B_i) > V_i(A_i), \exists i \in N$

  - I.e., an allocation is PO if there is no "better" allocation.

- Q: Is it PO to give the entire cake to player 1?
- A: Not necessarily. But yes if player 1 values "every part of the cake positively".

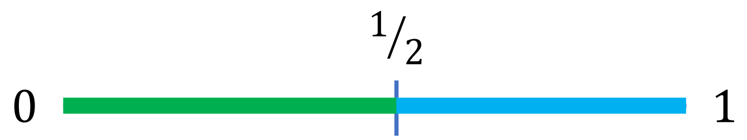
PO + EF

- Theorem [Weller '85]:
  - There always exists an allocation of the cake that is both envy-free and Pareto optimal.

- One way to achieve PO+EF:
  - Nash-optimal allocation: $\text{argmax}_A \prod_{i \in N} V_i(A_i)$
  - Obviously, this is PO. The fact that it is EF is non-trivial.
  - This is named after John Nash.
    - Nash social welfare = product of utilities
    - Different from utilitarian social welfare = sum of utilities

Nash-Optimal Allocation

- Example:
  - Green player has value 1 distributed over $[0, {}^2/_3]$
  - Blue player has value 1 distributed over $[0,1]$
  - Without loss of generality (why?) suppose:
    - Green player gets x fraction of $[0, {}^2/_3]$
    - Blue player gets the remaining $1 - x$ fraction of $[0, {}^2/_3]$ AND all of $[{}^2/_3, 1]$.
  - Green's utility = x, blue's utility = $(1 - \mathrm{x}) \cdot \frac{2}{3} + \frac{1}{3} = \frac{3-2x}{3}$
  - Maximize: $x \cdot \frac{3-2x}{3} \Rightarrow x = {}^3/_4$ (${}^3/_4$ fraction of ${}^2/_3$ is ${}^1/_2$).

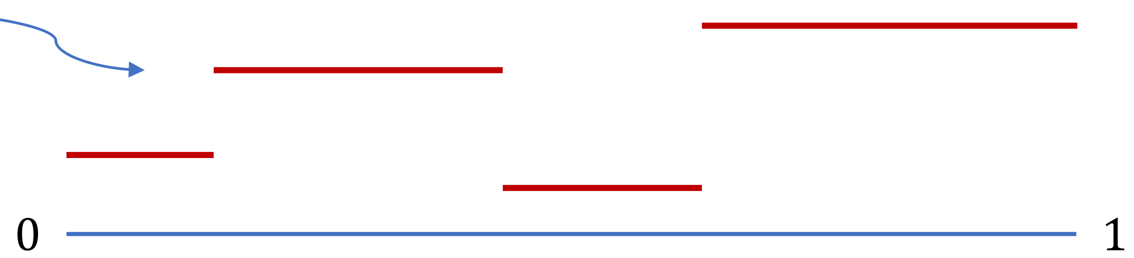
Allocation  Each player's utility = ${}^3/_4$

Problem with Nash Solution

- Difficult to compute in general
  - I believe it should require an unbounded number of queries in the Robertson-Webb model. But I can't find such a result in the literature.

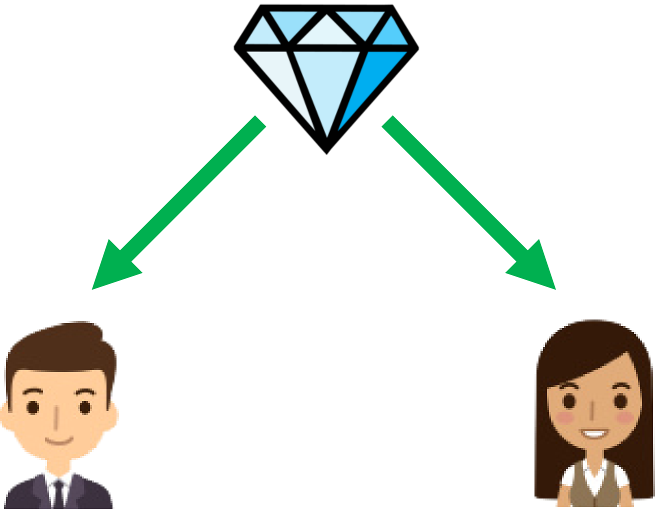
- Theorem [Aziz & Ye '14]:
  - For *piecewise constant* valuations, the Nash-optimal solution can be computed in polynomial time.

The density function of a piecewise constant valuation looks like this



Indivisible Goods

- Goods cannot be shared / divided among players
  - E.g., house, painting, car, jewelry, …
- Problem: Envy-free allocations may not exist!



Indivisible Goods: Setting

				
	8	7	20	5
	9	11	12	8
	9	10	18	3

Given such a matrix of numbers, assign each good to a player.

We assume additive values. So, e.g., V$(\{$, $\}) = 8 + 7 = 15$

Indivisible Goods

- Envy-freeness up to one good (EF1):

$$\forall i, j \in N, \exists g \in A_j : V_i(A_i) \geq V_i\left(A_j \setminus \{g\}\right)$$

  - Technically, we need either this or $A_j = \emptyset$.
  - "If i envies j, there must be some good in j's bundle such that removing it would make i envy-free of j."

- Does there always exist an EF1 allocation?

EF1

- Yes! We can use Round Robin.
  - Agents take turns in cyclic order: $1,2,\ldots,n,1,2,\ldots,n,\ldots$
  - In her turn, an agent picks the good she likes the most among the goods still not picked by anyone.

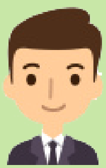
- Observation: This always yields an EF1 allocation.
  - Informal proof on the board.

- Sadly, on some instances, this returns an allocation that is not Pareto optimal.

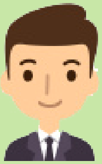
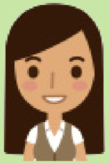
EF1+PO?

- Nash welfare to rescue!
- Theorem [Caragiannis et al. '16]:
  - The allocation $\text{argmax}_A \prod_{i \in N} V_i(A_i)$ is EF1 + PO.
  - Note: This maximization is over only "integral" allocations that assign each good to some player in whole.
  - Note: Subtle tie-breaking if all allocations have zero Nash welfare.
    - Step 1: Choose a subset of players $S \subseteq N$ with largest $|S|$ such that it is possible to give a positive utility to every player in S simultaneously.
    - Step 2: Choose $\text{argmax}_A \prod_{i \in S} V_i(A_i)$

Integral Nash Allocation?

				
	8	7	20	5
	9	11	12	8
	9	10	18	3

20 * (11+8) * 9 = 3420
is the maximum possible product

				
	8	7	20	5
	9	11	12	8
	9	10	18	3

Computation

- For indivisible goods, Nash-optimal solution is strongly NP-hard to compute
  - That is, remains NP-hard even if all values in the matrix are bounded

- Open Question: If our goal is EF1+PO, is there a different polynomial time algorithm?
  - Not sure. But a recent paper gives a pseudo-polynomial time algorithm for EF1+PO
    - Time is polynomial in n, m, and $\max_{i,g} V_i(\{g\})$.

Stronger Fairness

- Open Question: Does there always exist an EFx allocation?

- EF1: $\forall i, j \in N, \exists g \in A_j : V_i(A_i) \geq V_i\left(A_j \backslash \{g\}\right)$
  - Note: Or $A_j = \emptyset$ also allowed.
  - Intuitively, i doesn't envy j if she gets to remove her most valued item from j's bundle.

- EFx: $\forall i, j \in N, \forall g \in A_j : V_i(A_i) \geq V_i\left(A_j \backslash \{g\}\right)$
  - Note: $\forall g \in A_j$ such that $V_i(\{g\}) > 0$.
  - Intuitively, i doesn't envy j even if she removes her least positively valued item from j's bundle.

Stronger Fairness

- To clarify the difference between EF1 and EFx:
  - Suppose there are two players and three goods with values as follows.

	A	B	C
P1	5	1	10
P2	0	1	10

  - If you give {A} → P1 and {B,C} → P2, it's EF1 but not EFx.
    - EF1 because if P1 removes C from P2's bundle, all is fine.
    - Not EFx because removing B doesn't eliminate envy.
  - Instead, {A,B} → P1 and {C} → P2 would be EFx.