

CSC304 Lecture 17

Voting 4: Impartial selection

Recap

- The Gibbard-Satterthwaite theorem says that we cannot design strategyproof voting rules that are also nondictatorial and onto.
- Restricted settings (e.g., facility location on a line)
 - There exist strategyproof, nondictatorial, and onto rules.
 - They can be used to (perfectly or approximately) optimize the societal goal
- Today, we will study another interesting setting called *impartial selection*

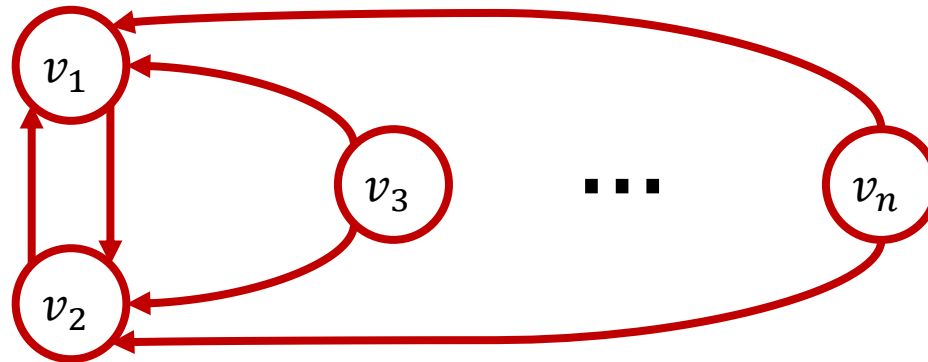
Impartial Selection

- “How can we select k people out of n people?”
 - Applications: electing a student representation committee, selecting k out of n grant applications to fund using peer review, ...
- Model
 - Input: a *directed* graph $G = (V, E)$
 - Nodes $V = \{v_1, \dots, v_n\}$ are the n people
 - Edge $e = (v_i, v_j) \in E$: v_i supports/approves of v_j
 - We do not allow or ignore self-edges (v_i, v_i)
 - Output: a subset $V' \subseteq V$ with $|V'| = k$
 - $k \in \{1, \dots, n - 1\}$ is given

Impartial Selection

- **Impartiality:** A k -selection rule f is *impartial* if $v_i \in f(G)$ does not depend on the outgoing edges of v_i
 - v_i cannot manipulate his outgoing edges to get selected
 - **Q:** But the definition says v_i can neither go from $v_i \notin f(G)$ to $v_i \in f(G)$, nor from $v_i \in f(G)$ to $v_i \notin f(G)$. Why?
- **Societal goal:** maximize the sum of in-degrees of selected agents $\sum_{v \in f(G)} |in(v)|$
 - $in(v)$ = set of nodes that have an edge to v
 - $out(v)$ = set of nodes that v has an edge to
 - **Note:** OPT will pick the k nodes with the highest indegrees

Optimal \neq Impartial



- An optimal 1-selecton rule must select v_1 or v_2
- The other node can remove his edge to the winner, and make sure the optimal rule selects him instead
- This violates impartiality

Goal: Approximately Optimal

- **α -approximation:** We want a k -selection system that always returns a set with total indegree at least α times the total indegree of the optimal set
- **Q:** For $k = 1$, what about the following rule?
Rule: “Select the lowest index vertex in $out(v_1)$.
If $out(v_1) = \emptyset$, select v_2 .”
 - A. Impartial + constant approximation
 - **B.** Impartial + bad approximation
 - C. Not impartial + constant approximation
 - D. Not impartial + bad approximation

No Finite Approximation ☹️

- **Theorem** [Alon et al. 2011]
For every $k \in \{1, \dots, n - 1\}$, there is no impartial k -selection rule with a finite approximation ratio.
- **Proof:**
 - For small k , this is trivial. E.g., consider $k = 1$.
 - What if G has two nodes v_1 and v_2 that point to each other, and there are no other edges?
 - For finite approximation, the rule must choose either v_1 or v_2
 - Say it chooses v_1 . If v_2 now removes his edge to v_1 , the rule must choose v_2 for any finite approximation.
 - Same argument as before. But applies to any “finite approximation rule”, and not just the optimal rule.

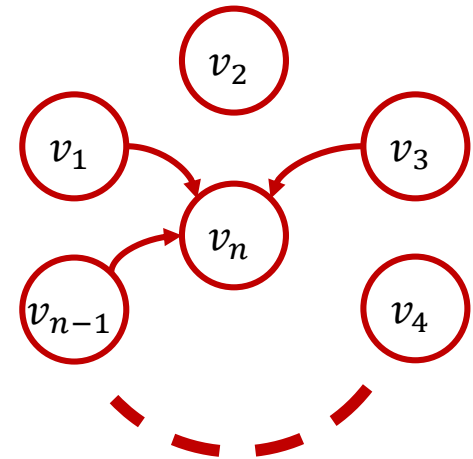
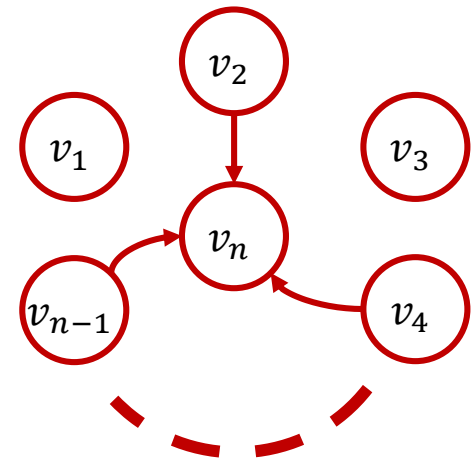
No Finite Approximation ☹️

- **Theorem** [Alon et al. 2011]
For every $k \in \{1, \dots, n - 1\}$, there is no impartial k -selection rule with a finite approximation ratio.
- **Proof:**
 - Proof is more intricate for larger k . Let's do $k = n - 1$.
 - $k = n - 1$: given a graph, “eliminate” a node.
 - Suppose for contradiction that there is such a rule f .
 - W.l.o.g., say v_n is eliminated in the empty graph.
 - Consider a family of graphs in which a subset of $\{v_1, \dots, v_{n-1}\}$ have edges to v_n .

No Finite Approximation ☹️

- **Proof ($k = n - 1$ continued):**

- Consider *star graphs* in which a non-empty subset of $\{v_1, \dots, v_{n-1}\}$ have edge to v_n , and there are no other edges
 - Represented by bit strings $\{0,1\}^{n-1} \setminus \{\vec{0}\}$
- v_n cannot be eliminated in any star graph
 - Otherwise we have infinite approximation
- f maps $\{0,1\}^{n-1} \setminus \{\vec{0}\}$ to $\{1, \dots, n - 1\}$
 - “Who will be eliminated?”
- Impartiality: $f(\vec{x}) = i \Leftrightarrow f(\vec{x} + \vec{e}_i) = i$
 - \vec{e}_i has 1 at i^{th} coordinate, 0 elsewhere
 - In words, i cannot prevent elimination by adding or removing his edge to v_n



No Finite Approximation ☹️

- **Proof ($k = n - 1$ continued):**

- $f: \{0,1\}^{n-1} \setminus \{\vec{0}\} \rightarrow \{1, \dots, n-1\}$

- $f(\vec{x}) = i \Leftrightarrow f(\vec{x} + \vec{e}_i) = i$
 - \vec{e}_i has 1 only in i^{th} coordinate

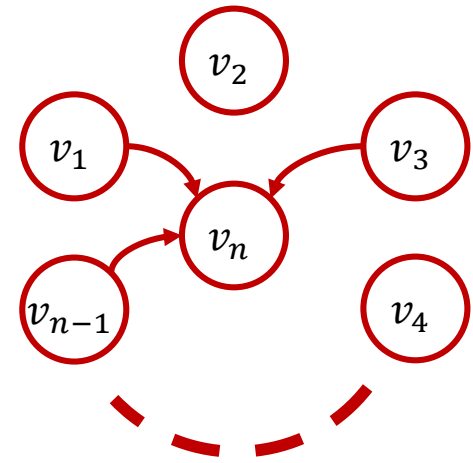
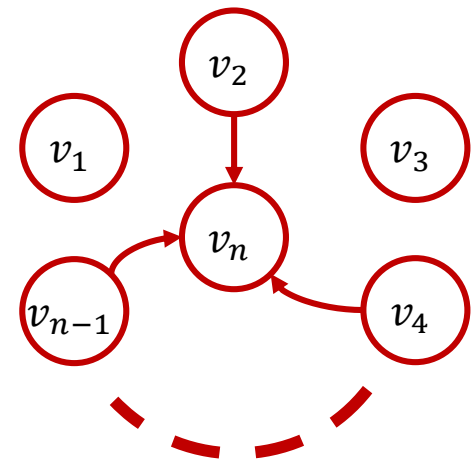
- **Pairing implies...**

- The number of strings on which f outputs i is even, for every i .

- Thus, total number of strings in the domain must be even too.

- But total number of strings is $2^{n-1} - 1$ (odd)

- So impartiality must be violated for some pair of \vec{x} and $\vec{x} + \vec{e}_i$



Back to Impartial Selection

- **Question:** So what *can* we do to select impartially?
- **Answer:** Randomization!
 - Impartiality now requires that the probability of an agent being selected be independent of his outgoing edges.
- **Examples:** Randomized Impartial Mechanisms
 - Choose k nodes uniformly at random
 - Sadly, this still has arbitrarily bad approximation.
 - Imagine having k special nodes with indegree $n - 1$, and all other nodes having indegree 0.
 - Mechanism achieves $(k/n) * OPT \Rightarrow$ approximation = n/k
 - Good when k is comparable to n , but bad when k is small.

Random Partition

- **Idea:**

- What if we partition V into V_1 and V_2 , and select k nodes from V_1 based only on edges coming to them from V_2 ?

- **Mechanism:**

- Assign each node to V_1 or V_2 i.i.d. with probability $\frac{1}{2}$
- Choose $V_i \in \{V_1, V_2\}$ at random
- Choose k nodes from V_i that have most incoming edges from nodes in V_{3-i}

Random Partition

- **Analysis:**

- We want to approximate $I = \#$ edges incoming to nodes in OPT .
 - Let $OPT_1 = OPT \cap V_1$, and $OPT_2 = OPT \cap V_2$.
 - Let $I_1 = \#$ edges incoming to OPT_1 from V_2 .
 - Let $I_2 = \#$ edges incoming to OPT_2 from V_1 .
- Note that $E[I_1 + I_2] = I/2$. (WHY?)
- With probability $1/2$, mechanism picks k nodes from V_1 that have most incoming edges from V_2 (thus at least I_1 incoming edges).
 - Because they're at least as good as OPT_1 .
- With probability $1/2$, mechanism picks k nodes from V_2 that have most incoming edges from V_1 (thus at least I_2 incoming edges).
- The expected total incoming edges is at least
 - $E\left[\left(\frac{1}{2}\right) \cdot I_1 + \left(\frac{1}{2}\right) \cdot I_2\right] = \left(\frac{1}{2}\right) \cdot E[I_1 + I_2] = \left(\frac{1}{2}\right) \cdot \frac{I}{2} = \frac{I}{4}$

Random Partition

- **Improvement**

- The approximation ratio improves if we pick $k/2$ nodes from each part instead of picking all k from one part.
- More generally, we can divide into ℓ parts, and pick k/ℓ nodes from each part based on incoming edges from all other parts.

- **Theorem [Alon et al. 2011]:**

- $\ell = 2$ gives a 4-approximation.
- For $k \geq 2$, $\ell \sim k^{1/3}$ gives $1 + O\left(\frac{1}{k^{1/3}}\right)$ approximation.

Better Approximations

- [Alon et al. 2011] conjectured that for randomized impartial 1-selection...
 - (For which their mechanism is a 4-approximation)
 - It should be possible to achieve a 2-approximation.
 - Recently proved by [Fischer & Klimm, 2014]
 - **Permutation mechanism:**
 - Select a random permutation $(\pi_1, \pi_2, \dots, \pi_n)$ of the vertices.
 - Start by selecting $y = \pi_1$ as the “current answer”.
 - At any iteration t , let $y \in \{\pi_1, \dots, \pi_t\}$ be the current answer.
 - From $\{\pi_1, \dots, \pi_t\} \setminus \{y\}$, if there are more edges to π_{t+1} than to y , change the current answer to $y = \pi_{t+1}$.

Better Approximations

- 2-approximation is tight.
 - In an n -node graph, fix u and v , and suppose no other nodes have any incoming/outgoing edges.
 - Three cases: only $u \rightarrow v$ edge, only $v \rightarrow u$, or both.
 - The best impartial mechanism selects u and v with probability $\frac{1}{2}$ in every case, and achieves 2-approximation.
- But this is because $n - 2$ nodes are not voting!
 - What if every node must have an outgoing edge?
 - [Fischer & Klimm]:
 - Permutation mechanism gives $\frac{12}{7} = 1.714$ approximation.
 - No mechanism gives better than $\frac{2}{3}$ approximation.
 - Open question to achieve better than $\frac{12}{7}$.

The rest of this lecture is not part of the syllabus.

PageRank

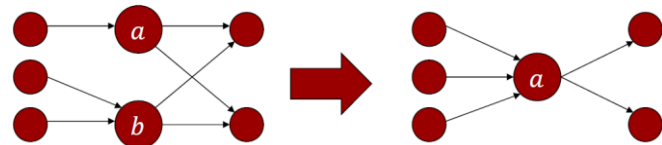
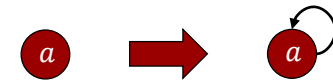
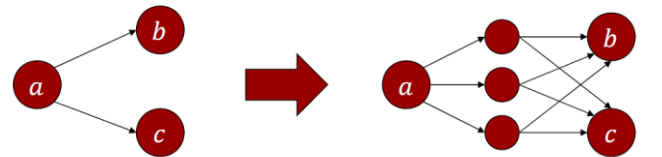
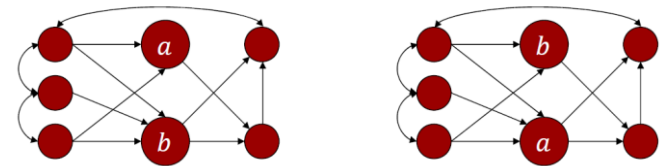
- An extension of the impartial selection problem
 - Instead of selecting k nodes, we want to *rank* all nodes
- The PageRank Problem: Given a directed graph, rank all nodes by their “importance”.
 - Think of the web graph, where nodes are webpages, and a directed (u, v) edge means u has a link to v .
- Questions:
 - What properties do we want from such a rule?
 - What rule satisfies these properties?

PageRank

- Here is the **PageRank Algorithm**:
 - Start from any node in the graph.
 - At each iteration, choose an outgoing edge of the current node, uniformly at random among all its outgoing edges.
 - Move to the neighbor node on that edge.
 - In the limit of $T \rightarrow \infty$ iterations, measure the fraction of time the “random walk” visits each node.
 - Rank the nodes by these “stationary probabilities”.
- Google uses (a version of) this algorithm
 - It's seems a reasonable algorithm.
 - What nice axioms might it satisfy?

Axioms

- Axiom 1 (Isomorphism)
 - Permuting node names permutes the final ranking.
- Axiom 2 (Vote by Committee)
 - Voting through intermediate fake nodes cannot change the ranking.
- Axiom 3 (Self Edge)
 - v adding a self edge cannot change the ordering of the *other* nodes.
- Axiom 4 (Collapsing)
 - Merging identically voting nodes cannot change the ordering of the *other* nodes.
- Axiom 5 (Proxy)
 - If a set of nodes with equal score vote for v through a proxy, it should not be different than voting directly.



PageRank

- **Theorem [Altman and Tennenholtz, 2005]:**
The PageRank algorithm satisfies these five axioms, and is the unique algorithm to satisfy all five axioms.
- That is, any algorithm that satisfies all five axioms must output the ranking returned by PageRank on every single graph.