# CSC304 Lecture 16

## Voting 3: Axiomatic, Statistical, and Utilitarian Approaches to Voting

# Announcements

- Assignment 2 was due today at 3pm
- If you have grace credits left (check MarkUs), you could take up to two more days, and submit by Wed 3pm
- On Wednesday, we will go over solutions to A2 problems in class
  - We'll do a Piazza poll to find the most popular questions, and solve them first

# Recap

- We introduced a plethora of voting rules
  - Plurality
  - Borda
  - Veto
  - $k$-Approval
  - STV
  - Plurality with runoff
  - Kemeny
  - Copeland
  - Maximin
- Which is the right way to aggregate preferences?
  - GS Theorem: There is no good strategyproof voting rule.
  - For now, let us forget about incentives. Let us focus on how to aggregate given truthful votes.

# Recap

- Set of voters $N = \{1, \dots, n\}$
- Set of alternatives $A$, $|A| = m$
- Voter $i$ has a preference ranking $\succ_i$ over the alternatives

| 1 | 2 | 3 |
|---|---|---|
| a | c | b |
| b | a | a |
| c | b | c |

- Preference profile $\vec{\succ}$ = collection of all voter rankings
- Voting rule (social choice function) $f$
  - Takes as input a preference profile $\vec{\succ}$
  - Returns an alternative $a \in A$

# Axiomatic Approach

- An axiom is a desideratum in which we require a voting rule to behave in a specific way.

- Goal: define a set of reasonable axioms, and search for voting rules that satisfy them
  - Ultimate hope: find that a unique voting rule satisfies the axioms we are interested in!

- Sadly, we often find the opposite.
  - Many combinations of reasonable axioms cannot be satisfied by any voting rule.
  - E.g., the GS theorem (nondictatorship, ontoness, strategyproofness), Arrow's theorem (will see), …

# Axiomatic Approach

- Weak axioms, satisfied by all popular voting rules
- Unanimity: If all voters have the same top choice, that alternative is the winner.

$$(top(\succ_i) = a\ \forall i \in N) \Rightarrow f(\overrightarrow{\succ}) = a$$

  - An even weaker version requires all rankings to be identical
- Pareto optimality: If all voters prefer $a$ to $b$, then $b$ is not the winner.

$$(a \succ_i b\ \forall i \in N) \Rightarrow f(\overrightarrow{\succ}) \neq b$$

- Q: *What is the relation between these axioms?*
  - *Pareto optimality* $\Rightarrow$ *Unanimity*

# Axiomatic Approach

- **Anonymity:** Permuting votes does not change the winner (i.e., voter identities don't matter).
  - E.g., these two profiles must have the same winner:
    {voter 1: $a \succ b \succ c$, voter 2: $b \succ c \succ a$}
    {voter 1: $b \succ c \succ a$, voter 2: $a \succ b \succ c$}
- **Neutrality:** Permuting alternative names just permutes the winner.
  - E.g., say $a$ wins on {voter 1: $a \succ b \succ c$, voter 2: $b \succ c \succ a$}
  - We permute all names: $a \rightarrow b$, $b \rightarrow c$, and $c \rightarrow a$
  - New profile: {voter 1: $b \succ c \succ a$, voter 2: $c \succ a \succ b$}
  - Then, the new winner must be $b$.

# Axiomatic Approach

- Neutrality is tricky
  - As we have it now, it is inconsistent with anonymity!
    - Imagine {voter 1: $a \succ b$, voter 2: $b \succ a$}
    - Without loss of generality, say $a$ wins
    - Imagine a different profile: {voter 1: $b \succ a$, voter 2: $a \succ b$}
      - Neutrality: We just exchanged $a \leftrightarrow b$, so winner is $b$.
      - Anonymity: We just exchanged the votes, so winner stays $a$.
  - Typically, we only require neutrality for…
    - Randomized rules: E.g., a rule could satisfy both by choosing $a$ and $b$ as the winner with probability ½ each, on both profiles
    - Deterministic rules that return a set of tied winners: E.g., a rule could return $\{a, b\}$ as tied winners on both profiles.

# Axiomatic Approach

- Stronger but more subjective axioms
- Majority consistency: If a majority of voters have the same top choice, that alternative wins.

$$\left(|\{i: top(\succ_i) = a\}| > \frac{n}{2}\right) \Rightarrow f(\vec{\succ}) = a$$

- Condorcet consistency: If $a$ defeats every other alternative in a pairwise election, $a$ wins.

$$\left(|\{i: a \succ_i b\}| > \frac{n}{2}, \forall b \neq a\right) \Rightarrow f(\vec{\succ}) = a$$

- Q: What is the relation between these two?
  - *Condorcet consistency ⇒ Majority consistency*

# Axiomatic Approach

- Majority consistency: If a majority of voters have the same top choice, that alternative wins.
- Condorcet consistency: If $a$ defeats every other alternative in a pairwise election, $a$ wins.

- Question: Which of these does *plurality* satisfy?
  - A. Both.
  - B. Only majority consistency.
  - C. Only Condorcet consistency.
  - D. Neither.

# Axiomatic Approach

- **Majority consistency:** If a majority of voters have the same top choice, that alternative wins.
- **Condorcet consistency:** If $a$ defeats every other alternative in a pairwise election, $a$ wins.

- **Question:** Which of these does *Borda count* satisfy?
  - A. Both.
  - B. Only majority consistency.
  - C. Only Condorcet consistency.
  - D. Neither.

# Axiomatic Approach

- Majority consistency: If a majority of voters have the same top choice, that alternative wins.
- Condorcet consistency: If $a$ defeats every other alternative in a pairwise election, $a$ wins.

- Fun fact about Condorcet consistency
  - Most rules that "focus on positions" (positional scoring rules, STV, plurality with runoff) violate it
  - Most rules that "focus on pairwise comparisons" (Kemeny, Copeland, Maximin) satisfy it

# Axiomatic Approach

- Is even the weaker axiom majority consistency a reasonable one to expect?

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| a | a | a | b | b |
| b | b | b | | |
| | | | | |
| | | | | |
| | | | | |
| | | | | |
| | | | | |
| | | | | |
| | | | a | a |

Piazza Poll: Do you think we should require that the voting rule must output $a$ irrespective of how tall the gray region is?

# Axiomatic Approach

- Consistency: If $a$ is the winner on two profiles, it must be the winner on their union.

$$f\left(\overrightarrow{\succ}_1\right) = a \wedge f\left(\overrightarrow{\succ}_2\right) = a \Rightarrow f\left(\overrightarrow{\succ}_1 + \overrightarrow{\succ}_2\right) = a$$

  - Example: $\overrightarrow{\succ}_1 = \{a \succ b \succ c\}$, $\overrightarrow{\succ}_2 = \{a \succ c \succ b, b \succ c \succ a\}$
  - Then, $\overrightarrow{\succ}_1 + \overrightarrow{\succ}_2 = \{a \succ b \succ c, a \succ c \succ b, b \succ c \succ a\}$

- Do you think consistency must be satisfied?
  - Young [1975] showed that subject to mild requirements, a voting rule is consistent if and only if it is a positional scoring rule!
  - Thus, plurality with runoff, STV, Kemeny, Copeland, Maximin, etc are *not* consistent.

# Axiomatic Approach

- Weak monotonicity: If $a$ is the winner, and $a$ is "pushed up" in some votes, $a$ remains the winner.
  - $f(\overrightarrow{\succ}) = a \Rightarrow f(\overrightarrow{\succ}') = a$ if
    1. $b \succ_i c \Leftrightarrow b \succ_i' c, \forall i \in N,\ b, c \in A \backslash \{a\}$
       "Order among other alternatives preserved in all votes"
    2. $a \succ_i b \Rightarrow a \succ_i' b, \forall i \in N,\ b \in A \backslash \{a\}$ ($a$ only improves)
       "In every vote, $a$ still defeats all the alternatives it defeated"
- Contrast: strong monotonicity requires $f(\overrightarrow{\succ}') = a$ even if $\overrightarrow{\succ}'$ only satisfies the 2nd condition
  - It is thus too strong. Equivalent to strategyproofness!
  - Only satisfied by dictatorial/non-onto rules [GS theorem]

# Axiomatic Approach

- Weak monotonicity: If $a$ is the winner, and $a$ is "pushed up" in some votes, $a$ remains the winner.
  - $f(\vec{\succ}) = a \Rightarrow f(\vec{\succ}') = a$, where
    - $b \succ_i c \Leftrightarrow b \succ_i' c, \forall i \in N,\ b, c \in A \backslash \{a\}$ (Order of others preserved)
    - $a \succ_i b \Rightarrow a \succ_i' b, \forall i \in N,\ b \in A \backslash \{a\}$ ($a$ only improves)

- Weak monotonicity is satisfied by most voting rules
  - Only exceptions (among rules we saw): STV and plurality with runoff
  - But this helps STV be hard to manipulate
    - [Conitzer & Sandholm 2006]: "Every weakly monotonic voting rule is easy to manipulate on average."

# Axiomatic Approach

- STV violates weak monotonicity

| 7 voters | 5 voters | 2 voters | 6 voters |
|---|---|---|---|
| a | b | b | c |
| b | c | c | a |
| c | a | a | b |

| 7 voters | 5 voters | 2 voters | 6 voters |
|---|---|---|---|
| a | b | a | c |
| b | c | b | a |
| c | a | c | b |

- First $c$, then $b$ eliminated
- Winner: $a$

- First $b$, then $a$ eliminated
- Winner: $c$

Good news: The material in the slides that follow is *not* part of the syllabus.

- It is to give you a flavor of other interesting results/ approaches in voting.

Bad news: That's because I'm going to go over it really fast!

# Axiomatic Approach

- Arrow's Impossibility Theorem
  - Applies to social welfare functions (want a consensus ranking)
  - Independence of Irrelevant Alternatives (IIA): If the preferences of all voters between $a$ and $b$ are unchanged, the social preference between $a$ and $b$ should not change
    - Criticized to be too strong
  - Theorem: IIA cannot be achieved together with Pareto optimality (if all prefer $a$ to $b$, social preference should be $a \succ b$) unless the rule is a dictatorship.
  - Arrow's theorem set the foundations for the axiomatic approach to voting

# Statistical Approach

- Assume that there is a ground truth ranking $\sigma^*$

- Votes $\{\succ_i\}$ are generated i.i.d. from a distribution parametrized by $\sigma^*$
  - Formally, there is a probability distribution $\Pr[\cdot\,|\sigma]$ for every ranking $\sigma$
  - $\Pr[\succ|\sigma]$ denotes the probability of drawing a vote $\succ$ given that the ground truth is $\sigma$

- Use maximum likelihood estimate (MLE) of the ground truth
  - Given $\vec{\succ}$, return $\text{argmax}_\sigma\left(\Pr[\vec{\succ}\,|\sigma] = \prod_{i=1}^{n} \Pr[\succ_i\,|\sigma]\right)$

# Statistical Approach

- Example: Mallows' model
  - Recall: Kendall-tau distance $d$ between two rankings is the #pairs of alternatives whose comparisons they differ on
  - Malllows' model: $\Pr[\succ |\sigma] \propto \varphi^{d(\succ,\sigma)}$, where
    - $\varphi \in (0,1]$ is the "noise parameter"
    - $\varphi \to 0$ means the distribution becomes accurate ($\Pr[\sigma|\sigma] \to 1$)
    - $\varphi = 1$ represents the uniform distribution
    - Normalization constant $Z_\varphi = \sum_{\succ} \varphi^{d(\succ,\sigma)}$ does not depend on $\sigma$
  - The greater the distance from the ground truth, the smaller the probability

# Statistical Approach

- Example: Mallows' model
  - What is the MLE ranking for Mallows' model?

$$\max_{\sigma} \prod_{i=1}^{n} \Pr[\succ_i |\sigma] = \max_{\sigma} \prod_{i=1}^{n} \frac{\varphi^{d(\succ_i,\sigma)}}{Z_\varphi} = \max_{\sigma} \frac{\varphi^{\sum_{i=1}^{n} d(\succ_i,\sigma)}}{Z_\varphi}$$

  - The MLE ranking minimizes $\sum_{i=1}^{n} d(\succ_i, \sigma)$
  - This is precisely the Kemeny ranking!
- Statistical approach yields a unique rule, but is specific to the assumed distribution of votes

# Utilitarian Approach

- Assume that voters have numerical utilities $\{v_i(a)\}$
- Their votes reflect comparisons of utilities:

$$a \succ_i b \Leftrightarrow v_i(a) \geq v_i(b)$$

- Goal:
  - Select $a^*$ with the maximum social welfare $\sum_i v_i(a^*)$
  - Cannot achieve this if we just know comparisons of utilities
    - Select $a^*$ that gives the best worst-case approximation of welfare (ratio of maximum social welfare to social welfare achieved)

$$\min_a \max_{\{v_i \text{ consistent with } \succ_i\}} \frac{\max_b \sum_i v_i(b)}{\sum_i v_i(a)}$$

# Utilitarian Approach

- Pros: Uses minimal subjective assumptions and yet yields a unique voting rule
- Cons: Difficult to compute and unintuitive to humans

- This approach is currently deployed on RoboVote
  - It has been extended to select a set of alternatives
  - My ongoing work: use it to select a consensus ranking
    - Results in a large, nonconvex, quadratically constrained quadratic program