# CSC304 Lecture 12

## Ending Mechanism Design w/ Money: Recap revenue maximization & Myerson's auction

## Begin Mechanism Design w/o Money: Facility Location

# Recap

- Single-item auction with 1 seller, $n$ buyers
- Buyer $i$ has value $v_i$ drawn from cdf $F_i$ (pdf $f_i$)
- Virtual value function: $\varphi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$
- Myerson's theorem: E[Revenue] = $E[\sum_i \varphi_i(v_i) * x_i]$
  - Maximize revenue = maximize virtual welfare subject to monotonic allocation rule

# Recap

- When all $F_i$'s are regular
  - Monotonicity is automatic

- Allocation: Give to agent $i$ with maximum $\varphi_i(v_i)$ if $\varphi_i(v_i) \geq 0$
  - When the maximum $\varphi_i(v_i)$ is negative, not selling the item is better (zero virtual welfare > negative virtual welfare)

- Payment: Charge
  $$v_i^* = \min\{v_i' : \varphi_i(v_i') \geq \max(0, \varphi_j(v_j))\ \forall j \neq i\}$$
  - Least possible value for which the agent still gets the item
  - If virtual value drops below any other virtual value or below 0, the agent loses the item

# Recap

- Special case: All $F_i = F =$ Regular
  - VCG with reserve price $\varphi^{-1}(0)$

- Allocation: Give the item to agent $i$ with the maximum value $v_i$ but only if $v_i \geq \varphi^{-1}(0)$
  - Equivalent to $\varphi(v_i) \geq 0$

- Payment: $\max\left(\varphi^{-1}(0), \max_{j \neq i} v_j\right)$
  - Least possible value for which the agent still gets the item
  - The agent loses the item as soon as his value goes below either the 2nd highest bid or the reserve price

# Approx. Optimal Auctions

- When $F_i$'s are complex, the virtual valuation function is complex too
  - The optimal auction is unintuitive
  - Two simple auctions that achieve good revenue

- Theorem [Hartline & Roughgarden, 2009]:
  For independent regular distributions, VCG with bidder-specific reserve prices can guarantee 50% of the optimal revenue.

# Approx. Optimal Auctions

- Still relies on knowing bidders' distributions
  - Can break down if the true distribution is different than the assumed distribution

- Theorem [Bulow and Klemperer, 1996]:
  For i.i.d. bidder valuations,
  $E$[Revenue of VCG with $n + 1$ bidders] $\geq$
  $E$[Optimal revenue with $n$ bidders]

- "Spend effort in getting one more bidder than in figuring out the optimal auction"

# Simple Proof

- (n+1)-bidder VCG has the maximum expected revenue among all (n+1)-bidder DSIC auctions that always allocate the item
  - Revenue Equivalence Theorem

- Consider the following (n+1)-bidder DSIC auction
  - Run $n$-bidder Myerson on first $n$ bidders. If the item is unallocated, give it to agent $n + 1$ for free.
  - As much expected revenue as $n$-bidder Myerson auction
  - No more expected revenue than (n+1)-bidder VCG

- QED!

# Optimizing Revenue is Hard

- Beyond single-parameter settings, the optimal auctions become even trickier

- Example: Two items, a single bidder with i.i.d. values for both items
  - Q: Shouldn't the optimal auction just sell each item individually using Myerson's auction?
  - A: No! Putting a take-it-or-leave-it offer on the two items bundled together can increase revenue!

- Slow progress on optimal auctions, but fast progress on simple and approximately optimal auctions

# Mechanism Design Without Money

# Lack of Money

- Mechanism design with money:
  - VCG can implement the welfare maximizing outcome because it can charge payments
- Mechanism design without money:
  - Suppose you want to give away a single item, but cannot charge any payments
  - Impossible to get meaningful information about valuations from strategic agents
  - How would you maximize welfare as much as possible?

# Lack of Money

- One possibility: Give the item to each of $n$ bidders with probability $1/n$.
- Does not maximize welfare
  - It's impossible to maximize welfare without money
- Achieves an $n$-approximation of maximum welfare
  - $\max_v \frac{\max_i v_i}{(1/n)\sum_i v_i} \le n$ (What is this?)
- Can't do better than $n$-approximation

# MD w/o Money Theme

1. Define the problem: agents, outcomes, valuations
2. Define the goal (e.g., maximizing social welfare)
3. Check if the goal can be achieved using a strategyproof mechanism
   - "strategyproof" = DSIC
4. If not, find the strategyproof mechanism that provides the best approximation ratio
   - Approximation ratio is similar to price of anarchy (PoA)

# Facility Location

- Set of agents $N$
- Each agent $i$ has a true location $x_i \in \mathbb{R}$
- Mechanism $f$
  - Takes as input reports $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$
  - Returns a location $y \in \mathbb{R}$ for the new facility
- Cost to agent $i$ : $c_i(y) = |y - x_i|$
- Social cost $C(y) = \sum_i c_i(y) = \sum_i |y - x_i|$

# Facility Location

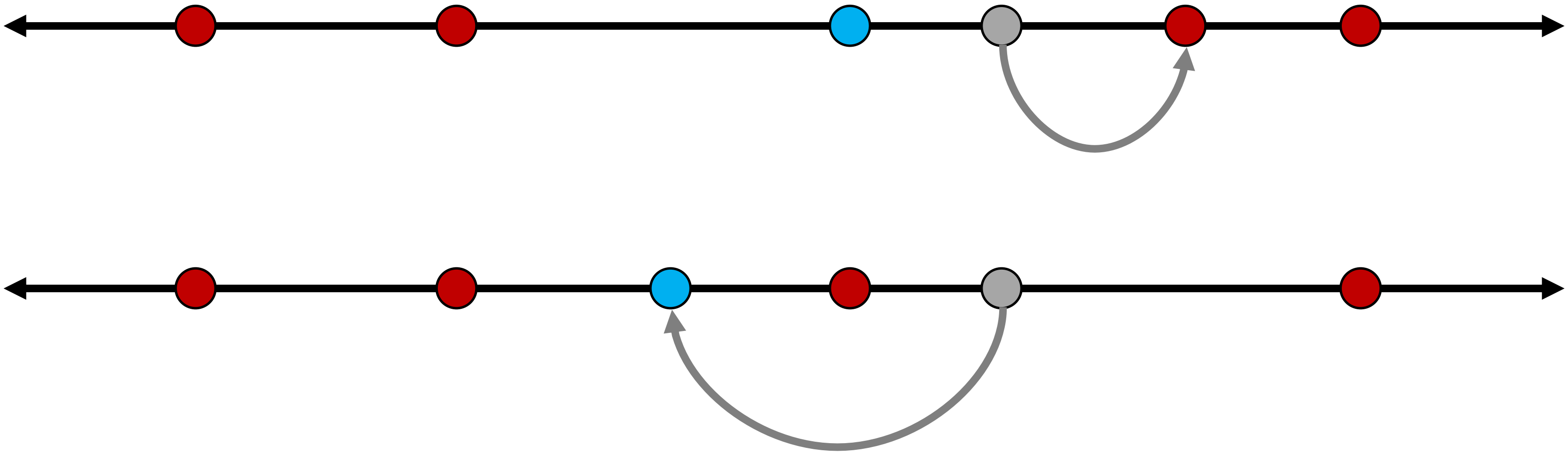
- Social cost $C(y) = \sum_i c_i(y) = \sum_i |y - x_i|$
- Q: Ignoring incentives, what choice of $y$ would minimize the social cost?
- A: The median location $\text{med}(x_1, \dots, x_n)$
  - $n$ is odd → the unique “(n+1)/2”th smallest value
  - $n$ is even → “n/2”th or “(n/2)+1”st smallest value
  - Why?

# Facility Location

- Social cost $C(y) = \sum_i c_i(y) = \sum_i |y - x_i|$
- Median is optimal (i.e., 1-approximation)
- What about incentives?
  - Median is also strategyproof (SP)!
  - Irrespective of the reports of other agents, agent $i$ is best off reporting $x_i$

# Median is SP

No manipulation can help



# Max Cost

- A different objective function $C(y) = \max_i |y - x_i|$
- Q: Again ignoring incentives, what value of $y$ minimizes the maximum cost?
- A: The midpoint of the leftmost ($\min_i x_i$) and the rightmost ($\max_i x_i$) locations (WHY?)
- Q: Is this optimal rule strategyproof?
- A: No! (WHY?)

# Max Cost

- $C(y) = \max_i |y - x_i|$
- We want to use a strategyproof mechanism.
- Question: What is the approximation ratio of median for maximum cost?
  1. $\in [1,2)$
  2. $\in [2,3)$
  3. $\in [3,4)$
  4. $\in [4, \infty)$

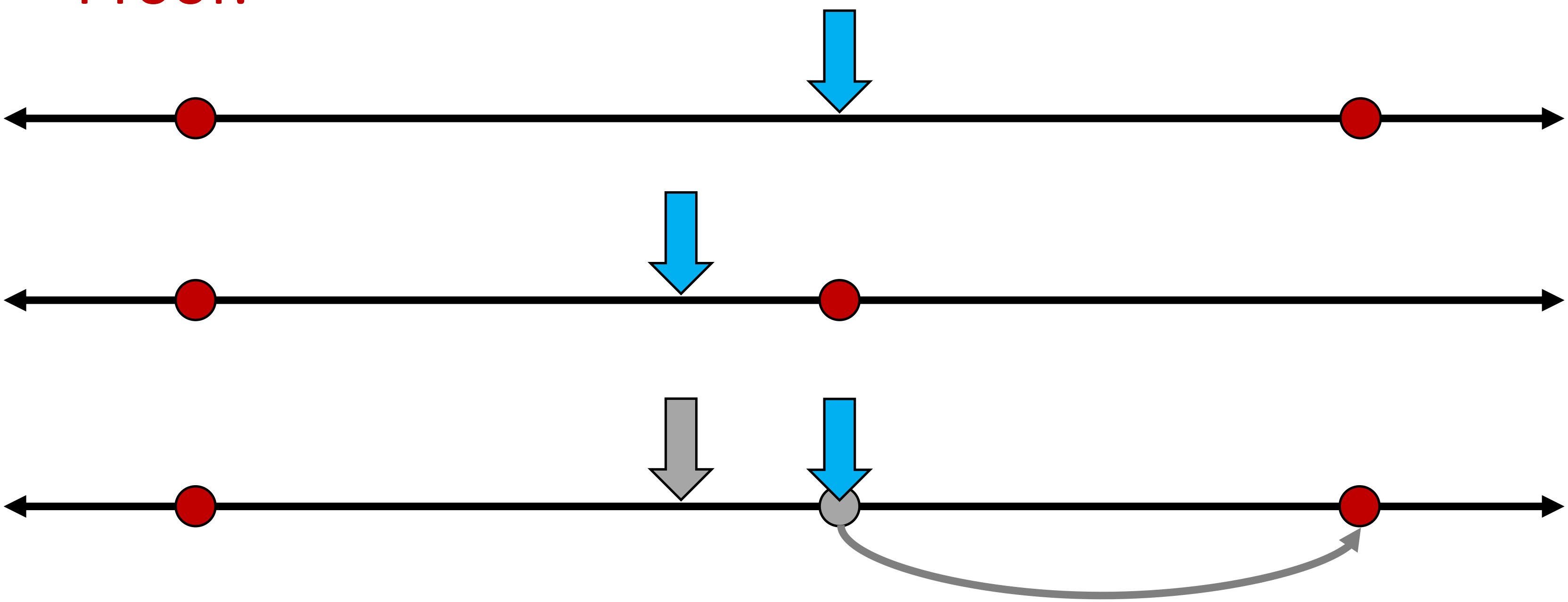
# Max Cost

- Answer: 2-approximation

- Other SP mechanisms that are 2-approximation
  - Leftmost: Choose the leftmost reported location
  - Rightmost: Choose the rightmost reported location
  - Dictatorship: Choose the location reported by agent 1
  - …

# Max Cost

- Theorem [Procaccia & Tennenholtz, '09]
  No deterministic SP mechanism has approximation ratio $< 2$ for maximum cost.
- Proof:



# Max Cost + Randomized

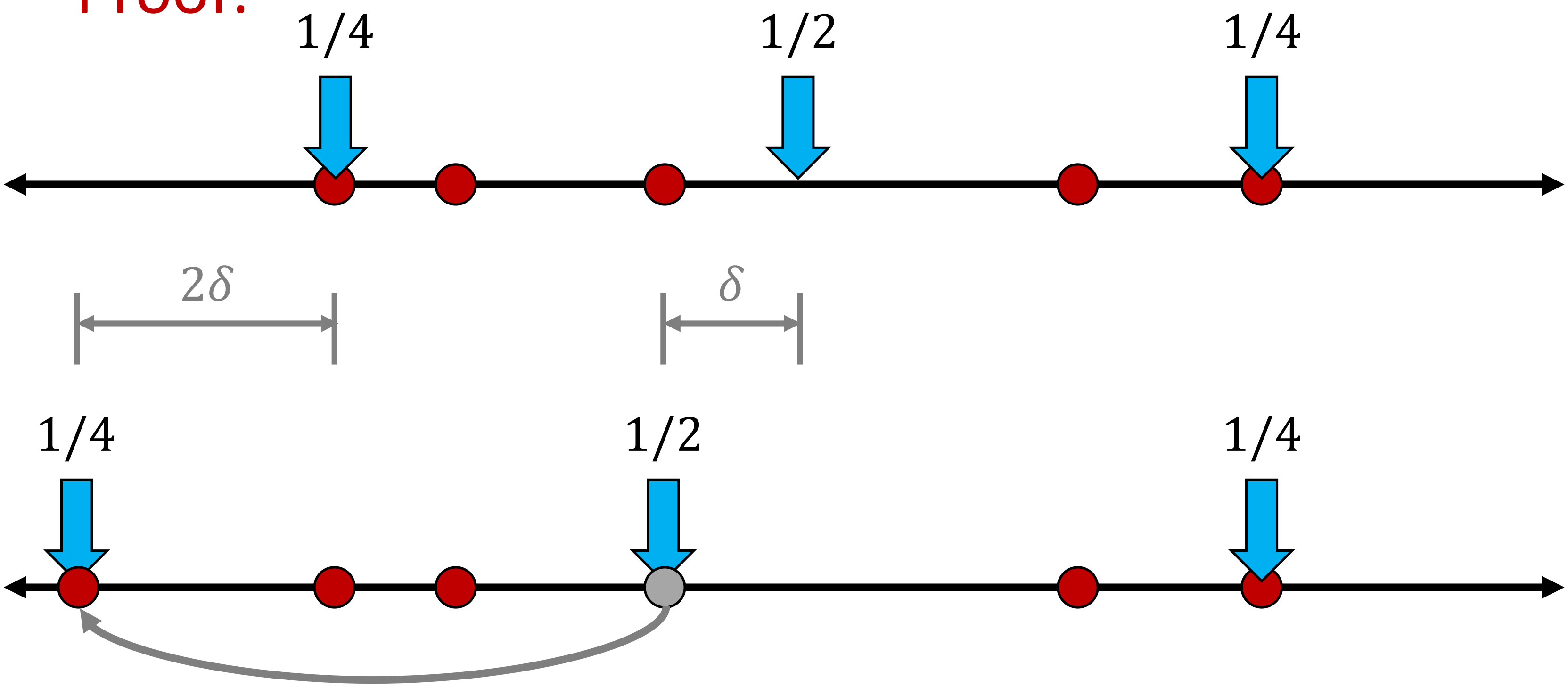
- The Left-Right-Middle (LRM) Mechanism
  - Choose $\min_i x_i$ with probability ¼
  - Choose $\max_i x_i$ with probability ¼
  - Choose $(\min_i x_i + \max_i x_i)/2$ with probability ½

- Question: What is the approximation ratio of LRM for maximum cost?

- At most $\frac{(1/4)*2C+(1/4)*2C+(1/2)*C}{C} = \frac{3}{2}$

# Max Cost + Randomized

- Theorem [Procaccia & Tennenholtz, '09]:
  The LRM mechanism is strategyproof.
- Proof:



# Max Cost + Randomized

- Exercise for you!
  Try showing that no randomized SP mechanism can achieve approximation ratio $< 3/2$

- Suggested outline
  - Consider two agents with $x_1 = 0$ and $x_2 = 1$
  - Show that one of them has expected cost at least ½
  - What happens if that agent moves 1 unit farther from the other agent?