CSC304 Lecture 11

Mechanism Design w/ Money: Revenue maximization; Myerson's auction

Announcements

- Returning graded midterm
  - Was only able to keep my promise due to wonderful TAs
- Delighted by your performance!
  - Given that the midterm was relatively hard
- Coming up: 4-5 questions of homework 2

Welfare vs Revenue

- In the auction setting…
  - We choose an outcome a based on agent valuations $\{v_i\}$
  - And charge payments p_i to each agent i

- In welfare maximization, we want to maximize $\sum_i v_i(a)$
  - VCG = DSIC + maximizes welfare on every single instance
  - Beautiful!

- In revenue maximization, we want to maximize $\sum_i p_i$
  - We can still use DSIC mechanisms (revelation principle). BUT…

Welfare vs Revenue

- Different DSIC mechanisms are better for different instances.

- Example: 1 item, 1 bidder, unknown value v
  - DSIC = fix a price r, let the agent decide to "take it" ($v \geq r$) or "leave it" ($v < r$)
  - Maximize welfare → set $r = 0$
  - Maximize revenue → $r = ?$
    - Different r are better for different v

- Must analyze in a Bayesian setting

Single-Item Auction Framework

- n bidders
- Value v_i of bidder i is drawn from distribution F_i with density f_i and support $[0, v_{max}]$
- Principal knows $\{F_i\}$, and wants to maximize $E[\sum_i p_i]$
  - Expectation over each v_i drawn i.i.d. from F_i
  - Principal wants to use a DSIC mechanism
    - IC part is without loss of generality (revelation principle)
    - Will see that can't do better using BNIC mechanisms

Single Item, Single Bidder

- Revisiting 1 item, 1 bidder
- Value $v \sim F$
- Want to post a price r on the item

- Revenue from price $r \;\rightarrow\; r \cdot \left(1 - F(r)\right)$ (Why?)

- Awesome! Select $r^* = \text{argmax}_r\, r \cdot \left(1 - F(r)\right)$
  - “Monopoly price”
  - Note: r^* depends on F, but not on $v \;\Rightarrow\;$ DSIC

Single Item, Single Bidder

- Suppose the bidder's value is drawn from the uniform distribution $U[0,1]$.
- Recall: E[Revenue] from price r is $r \cdot \left(1 - F(r)\right)$

- Q: What is the optimal posted price?
- Q: What is the corresponding optimal revenue?

- Compare this to the revenue of VCG, which is 0

An Aside

- In welfare maximization, we are bound to always selling the item
- In revenue maximization, we are willing to risk leaving the item unsold
  - If the item is not sold, you get 0 revenue
  - But if sold, you can get more than 2^{nd} highest bid
- Subject to always selling the item, VCG actually has the highest revenue
  - Revenue equivalence: “same allocation ⇒ same payment”

Single Item, Two Bidders

- $v_1, v_2 \sim U[0,1]$
- VCG revenue = 2nd highest bid = $\min(v_1, v_2)$
  - $E[\min(v_1, v_2)] = 1/3$
- A possible improvement: "VCG with reserve price"
  - Reserve price r.
  - Highest bidder gets the item if bid more than r
  - Pays $\max(r, 2^{\text{nd}} \text{ highest bid})$

Single Item, Two Bidders

- Reserve prices are ubiquitous
  - E.g., opening bids in eBay auctions
  - Guarantee a certain revenue to the auctioneer if item is sold
- $E[\text{revenue}] = E[\max(r, \min(v_1, v_2))]$
  - Maximize over r?
- What about other DSIC mechanisms? What if there are more bidders? Other distributions?

Single-Parameter Environments

- Roger B. Myerson solved revenue optimal auctions in "single-parameter environments"
- Proposed a simple auction that maximizes expected revenue

Single-Parameter Environments

- Each agent i has a private value $v_i \sim F_i$,
  - Value if the agent is "served"
  - Example: single-item auction → win the item
  - Example: combinatorial auction + single-minded bidder → get the desired set
  - Can potentially allow agents to be "fractionally" served

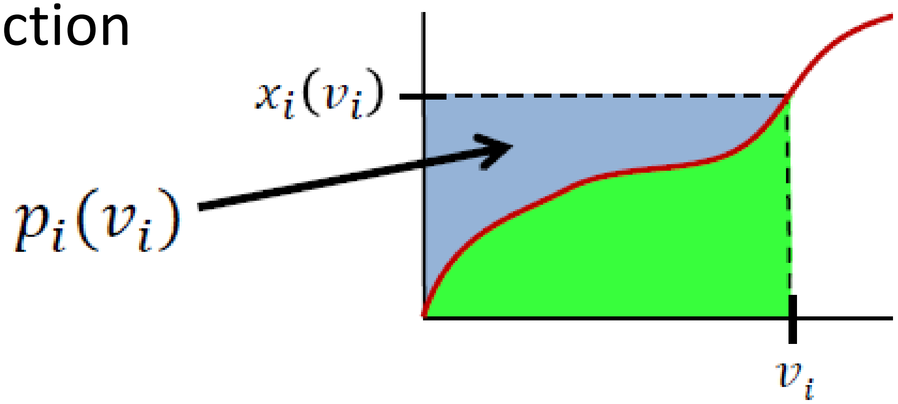
- Fixing bids of other agents…
  - Let $x_i(v_i) =$ fraction served when reporting v_i
    - When fractional serving not allowed, this is in $\{0,1\}$
  - Let $p_i(v_i) =$ payment charged when reporting v_i

Myerson's Lemma

- Myerson's Lemma:
  For a single-parameter environment, a strategy profile is in BNE under a mechanism if and only if
  1. $x_i(v_i)$ is monotone non-decreasing
  2. $p_i(v_i) = v_i \cdot x_i(v_i) - \int_0^{v_i} x_i(z)dz + p_i(0)$

     (typically, $p_i(0) = 0$)

- Intuition similar to 2nd price auction
  - For every "δ" allocation, pay the lowest value that would have won it



Myerson's Lemma

- Note: allocation determines unique payments

$$p_i(v_i) = v_i \cdot x_i(v_i) - \int_0^{v_i} x_i(z)dz + p_i(0)$$

- A corollary: revenue equivalence
  - If two mechanisms use the same allocation x_i, they "essentially" have the same expected revenue
- Another corollary: optimal revenue auctions
  - Optimizing revenue = optimizing some function of allocation (easier to analyze)

Myerson's Theorem

- "Expected Revenue = Expected Virtual Welfare"
  - Recall: $p_i(v_i) = v_i \cdot x_i(v_i) - \int_0^{v_i} x_i(z)dz + p_i(0)$
  - Take expectation over draw of valuations + lots of calculus

$$E_{\{v_i \sim F_i\}}[\Sigma_i \, p_i(v_i)] = E_{\{v_i \sim F_i\}}[\Sigma_i \, \varphi_i(v_i) \cdot x_i(v_i) \,]$$

- $\varphi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ is called the virtual value of bidder i
- Virtual welfare = sum of virtual values*allocations

Myerson's Auction

- Need the allocation x_i to be monotonic
- E[revenue] = E[virtual welfare]

- Myerson's auction: "The auction that maximizes (expected) revenue is the one whose allocation maximizes the virtual welfare subject to monotonicity"

- Let's apply this to some examples!

Example

- 2 bidders, 1 item, values drawn i.i.d. from $U[0,1]$
  - $\varphi(v) = v - \frac{1-F(v)}{f(v)} = v - \frac{1-v}{1} = 2v - 1$
  - Note: virtual value can be negative!!

- Given bids (v_1, v_2), …
  - Maximize $x_1 \cdot (2v_1 - 1) + x_2 \cdot (2v_2 - 1)$
  - Subject to $x_1, x_2 \in \{0,1\}$ and $x_1 + x_2 \leq 1$

Optimal Auction Example

- Maximize $x_1 \cdot (2v_1 - 1) + x_2 \cdot (2v_2 - 1)$
  - $x_1, x_2 \in \{0,1\}$ and $x_1 + x_2 \leq 1$
- Prove on the board:
  - Allocation:
    - If ∃ bidder with value ≥ ½, give to the highest bidder.
    - If both have value < ½, neither gets the item.
  - Payment if item sold = $\max(½, \text{lesser bid})$
- Precisely VCG with reserve price ½

Optimal Auctions

- Theorem: For a single item and n bidders whose valuations are drawn i.i.d., the optimal auction is VCG with reserve price $\varphi^{-1}(0)$.
  - Note: Reserve price is independent of #bidders!

- *Wait!* We didn't check for monotonicity of allocation!

- It turns out that for "nice" distributions, maximizing virtual welfare already gives a monotonic allocation rule!

Special Distributions

- Regular Distributions:
  A distribution F is regular if its virtual value function $v - (1 - F(v))/f(v)$ is non-decreasing.

- Lemma: If all F_i's are regular, the virtual welfare maximizing rule is monotone.

- Monotone Hazard Rate (MHR):
  A distribution F has monotone hazard rate if $(1 - F(v))/f(v)$ is non-increasing.
  - Important special case (MHR $\Rightarrow$ Regular)

Special Distributions

- Not crazy assumptions
  - Many practical distributions are MHR: e.g., uniform, exponential, Gaussian.
  - Some important distributions are not MHR, but still regular: e.g., power-law distributions.

Optimal Single-Item Auction

- Allocation: Give the item to agent i with highest $\varphi_i(v_i)$ if that is non-negative
- Payment: "lowest bid that still would have won"
  - Follows from $p_i(v_i) = v_i \cdot x_i(v_i) - \int_0^{v_i} x_i(z)dz + p_i(0)$

- All F_i's are equal to F and regular:
  - r^* = monopoly price of F
  - Item goes to the highest bidder if bid more than r^*
  - Payment charged is $\max(r^*, \text{2nd highest bid})$,
  - VCG with reserve price r^*!

Extensions

- Irregular distributions:
  - E.g., multi-modal or extremely heavy tail distributions
  - Need to add the monotonicity constraint
  - Turns out, we can “iron” irregular distributions to make them regular, and use standard Myerson’s framework
- Relaxing DSIC to BNIC
  - Myerson’s mechanism has optimal revenue among all DSIC mechanisms
  - Turns out, it also has optimal revenue among the much larger class of BNIC mechanisms!

Approx. Optimal Auctions

- For i.i.d. regular distributions, the optimal auction is simple (VCG with reserve price)

- For unequal distributions, it can be very complex
  - In practice, we prefer simple auctions that bidders can understand, but still want approximately optimal revenue

- Theorem [Hartline & Roughgarden, 2009]:
  For independent regular distributions, VCG with bidder-specific reserve prices is a 2-approximation of the optimal revenue.

Approximately Optimal

- Still relies on knowing bidders' distributions
  - Dangerous! Guarantees can break down if the true distribution is different from the assumed distribution

- Theorem [Bulow and Klemperer, 1996]:
  For i.i.d. bidder valuations,
  E[Revenue of VCG with $n + 1$ bidders] $\geq$
  E[Optimal revenue with n bidders]

- "Spend effort in getting one more bidder than in figuring out the optimal auction"

Simple proof

- VCG with $n+1$ bidders has the maximum revenue among all $n+1$ bidder DSIC auctions that always allocate the item [via revenue equivalence]

- Consider the auction: "Run n-bidder Myerson on first n bidders. If the item is unallocated, give it to agent $n+1$ for free."
  - $n+1$ bidder DSIC auction
  - As much revenue as n-bidder Myerson auction

Optimizing Revenue is Hard

- Slow progress beyond single-parameter setting
  - Even with just two items and one bidder with i.i.d. values for both items, the optimal auction DOES NOT run Myerson's auction on individual items!
  - "Take-it-or-leave-it" offers for the two items bundled might increase revenue

- But nowadays, the focus is on simple, approximately optimal auctions instead of complicated, optimal auctions.