CSC2556

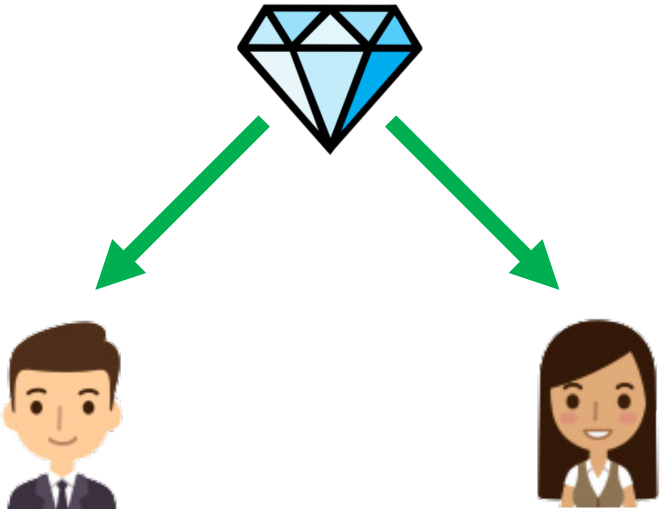
Lecture 9

Fair Division 2: Allocating Indivisible Goods

Indivisible Goods

Indivisible Goods

- Goods which cannot be shared among players
  - E.g., house, painting, car, jewelry, …
- Problem: Envy-free allocations may not exist!



Model

- Set of n agents $N = \{1, \dots, n\}$
- Set of m indivisible goods M
- Valuation function of agent i is $V_i: 2^M \to \mathbb{R}_{\geq 0}$
  - Additive: $V_i(S) = \sum_{g \in S} V_i(\{g\})$
  - We write $v_{i,g}$ to denote $V_i(\{g\})$ for simplicity
- Allocation $A = (A_1, \dots, A_m)$ is a partition of M
  - $\cup_i A_i = M$ and $A_i \cap A_j = \emptyset, \forall i, j$
  - For *partial* allocations, we drop the $\cup_i A_i = M$ requirement

Indivisible Goods

				
	8	7	20	5
	9	11	12	8
	9	10	18	3

Indivisible Goods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EF1

- Envy-freeness up to one good (EF1):

$$\forall i, j \in N, \exists g \in A_j : V_i(A_i) \geq V_i\left(A_j \setminus \{g\}\right)$$

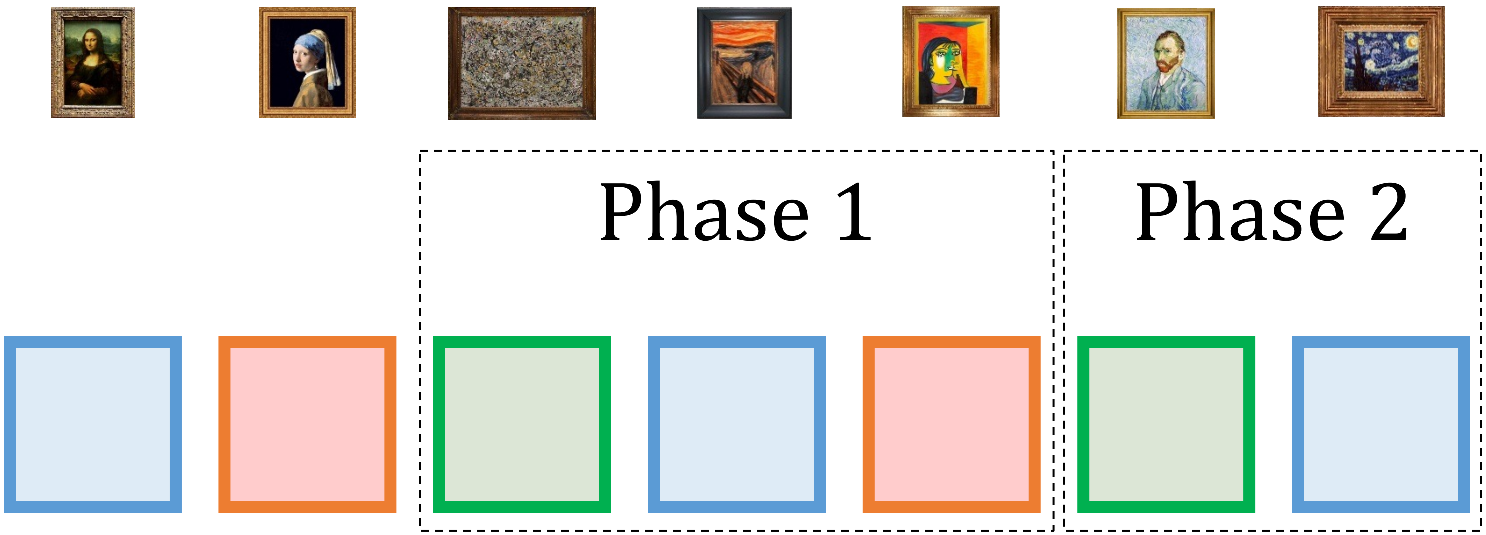
  - Technically, we need either this or $A_j = \emptyset$.

- In words…
  - "If i envies j, there must be some good in j's bundle such that removing it would make i envy-free of j."

- Question: Does there always exist an EF1 allocation?

EF1

- Yes, a simple round-robin procedure guarantees EF1
  - Order the agents arbitrarily (say $1,2, \ldots, n$)
  - In a cyclic fashion, agents arrive one-by-one and pick the item they like the most among the ones left



EF1 + PO

- Pareto optimality (PO)
  - An allocation A is Pareto optimal if there is no other allocation B such that $v_i(B_i) \geq v_i(A_i)$ for all i and the inequality is strict for at least one i

- Sadly, round-robin does not always return a PO allocation
  - There exist instances in which, by reallocating items at the end, we can make all agents strictly happier

- Question: Does there always exist an allocation that is both EF1 and PO simultaneously?

EF1+PO?

- Nash welfare to the rescue!
- Theorem [Caragiannis et al. '16]
  - The allocation $\text{argmax}_A \prod_{i \in N} V_i(A_i)$ is EF1 + PO.
- Note
  - Maximization is over integral allocations
  - Recall that in cake-cutting, the *fractional* allocation maximizing Nash welfare was EF + PO.

Integral Nash Allocation

				
	8	7	20	5
	9	11	12	8
	9	10	18	3

20 * 8 * (9+10) = 3040

				
	8	7	20	5
	9	11	12	8
	9	10	18	3

(8+7) * 8 * 18 = 2160

				
	8	7	20	5
	9	11	12	8
	9	10	18	3

$$8 * (12+8) * 10 = 1600$$

				
	8	7	20	5
	9	11	12	8
	9	10	18	3

$20 * (11+8) * 9 = 3420$

				
	8	7	20	5
	9	11	12	8
	9	10	18	3

EF1+PO?

What is wrong in these arguments?

- Proof that A maximizing $\prod_i v_i(A_i)$ is EF1 + PO
  - PO is obvious
    - Suppose for contradiction that there is an allocation B such that $v_i(B_i) \geq v_i(A_i)$ for each i and $v_i(B_i) > v_i(A_i)$ for at least one i
    - Then, $\prod_i v_i(B_i) \geq \prod_i v_i(A_i)$, which is a contradiction
  - EF1 requires a bit more work
    - Fix any agents i, j and consider moving good g from A_j to A_i
    - $v_i(A_i \cup \{g\}) \cdot v_j(A_j \setminus \{g\}) \leq v_i(A_i) \cdot v_j(A_j)$
    - $1 - \frac{v_j(g)}{v_j(A_j)} \leq 1 - \frac{v_i(g)}{v_i(A_i \cup \{g\})} \leq 1 - \frac{v_i(g)}{v_i(A_i \cup \{g^*\})} \Rightarrow \frac{v_j(g)}{v_j(A_j)} \geq \frac{v_i(g)}{v_i(A_i \cup \{g^*\})} \quad \forall g \in A_j$
      - Here, $g^* \in A_j$ is the good liked the most by i
    - Summing over all $g \in A_j$, we get $v_i(A_i \cup \{g^*\}) \geq v_i(A_j)$, which means i doesn't envy j up to good g^*

EF1+PO?

- Edge case
  - It may be possible that all allocations have zero Nash welfare
    - For example, allocate two goods between three agents
    - Allocating both goods to a single agent can violate EF1
    - Allocating the goods to the "wrong agents" can violate PO

  - Requires a slight modification of the rule in this edge case
    - Step 1: Choose a subset of agents $S \subseteq N$ with largest $|S|$ such that it is possible to give a positive utility to each agent in S simultaneously
    - Step 2: Choose $\text{argmax}_A \prod_{i \in S} V_i(A_i)$

Computation

- For indivisible goods, finding a Nash-optimal allocation is strongly NP-hard
  - That is, remains NP-hard even if all values in the matrix are bounded

- Open Question:
  - Can we compute *some* EF1+PO allocation in polynomial time?
  - [Barman et al., '17]:
    - There exists a pseudo-polynomial time algorithm for finding an EF1+PO allocation
      - Time is polynomial in n, m, and $\max_{i,g} v_{i,g}$
      - Already better than the time complexity of computing a Nash-optimal allocation

EFX

- Envy-freeness up to any good (EFX)
  - $\forall i, j \in N, \forall g \in A_j : V_i(A_i) \geq V_i(A_j \setminus \{g\})$
  - In words, i shouldn't envy j if she removes *any* good from j's bundle
- EFX ⇒ EF1
  - Recall EF1: $\forall i, j \in N, \exists g \in A_j : V_i(A_i) \geq V_i(A_j \setminus \{g\})$
  - In words, i shouldn't envy j if she removes *some* good from j's bundle
- Question: Does there always exist EFX allocation?
  - Still open

Stronger Fairness

- The difference between EF1 and EFx:
  - Suppose there are two players and three goods with values as follows.

	A	B	C
P1	5	1	10
P2	0	1	10

  - If you give {A} → P1 and {B,C} → P2, it's EF1 but not EFx.
    - EF1 because if P1 removes C from P2's bundle, all is fine.
    - Not EFx because removing B doesn't eliminate envy.
  - Instead, {A,B} → P1 and {C} → P2 would be EFx.

EFX

- It is easy to show that an EFX allocation always exists when…
  - Agents have identical valuations (i.e. $V_i = V_j$ for all i, j)
  - Agents have binary valuations (i.e. $v_{i,g} \in \{0,1\}$ for all i, g)
  - There are $n = 2$ agents with general additive valuations

- But answering this question in general (or even in some other special cases) has proved to be surprisingly difficult!

EFX: Recent Progress

- Partial allocations
  - [Caragiannis et al., '19]: There exists a partial EFX allocation A that has at least half of the optimal Nash welfare
  - [Ray Chaudhury et al., '19]: There exists a partial EFX allocation A such that for the set of unallocated goods U, $|U| \leq n-1$ and $V_i(A_i) \geq V_i(U)$ for all i
- Restricted number of agents
  - [Ray Chaudhury et al., '20]: There exists a complete EFX allocation with $n = 3$ agents
- Restricted valuations
  - [Amanatidis et al., '20]: Maximizing Nash welfare achieves EFX when there exist a, b such that $v_{i,g} \in \{a, b\}$ for all i, g

MMS

- Maximin Share Guarantee (MMS):
  - Generalization of “cut and choose” for n players
  - MMS value of agent i =
    - The highest value that agent i can get…
    - If *she* divides the goods into n bundles…
    - But receives the worst bundle according to her valuation
  - Let $\mathcal{P}_n(M)$ = family of partitions of M into n bundles

$$MMS_i = \max_{(B_1,\ldots,B_n)\in\mathcal{P}_n(M)} \min_{k\in\{1,\ldots,n\}} V_i(B_k)\,.$$

  - Allocation A is α-MMS if $V_i(A_i) \geq \alpha \cdot MMS_i$ for all i

MMS

- [Procaccia & Wang, '14]
  - There exists an instance in which no MMS allocation exists
  - A $^2/_3$ - MMS allocation always exists
- [Amanatidis et al., '17]
  - A $(^2/_3 - \epsilon)$ - MMS allocation can be computed in polynomial time
- [Ghodsi et al. '17]
  - A $^3/_4$ - MMS allocation always exists
  - A $(^3/_4 - \epsilon)$ - MMS allocation can be computed in polynomial time
- [Garg & Taki, '20]
  - A $^3/_4$ - MMS allocation can be computed in polynomial time
  - A $(^3/_4 + {}^1/_{12n})$ - MMS allocation always exists

Allocating Bads

- Costs instead of utilities
  - $c_{i,b}$ = cost of player i for bad b
    - $C_i(S) = \sum_{b \in S} c_{i,b}$
  - EF: $\forall i, j \;\; C_i(A_i) \le C_i(A_j)$
  - PO: There is no allocation B such that $C_i(B_i) \le C_i(A_i)$ for all i and at least one inequality is strict

- Divisible bads
  - An EF + PO allocation always exists
  - However, we can no longer just maximize the product (of what?)
  - Open question: Can we compute an EF+PO allocation of divisible bads in polynomial time?

Allocating Bads

- Indivisible bads
  - EF1: $\forall i, j \;\; \exists b \in A_i \;\; C_i(A_i \backslash \{b\}) \leq C_i(A_j)$
  - EFX: $\forall i, j \;\; \forall b \in A_i \; C_i(A_i \backslash \{b\}) \leq C_i(A_j)$
  - Open Question 1:
    - Does there always exist an EF1 + PO allocation?
  - Open Question 2:
    - Does there always exist an EFX allocation?
  - Many more open problems for allocating bads