

CSC2556

Game Theory 1: Nash Equilibria

Game Theory

Game Theory

- How do **rational self-interested** agents act in a given environment?
- Environment modeled as a game
 - Each agent or player has a set of possible actions
 - Rules of the game dictate the rewards for the agents as a function of the actions taken by all the players
 - My reward also depends on what action you take
 - Therefore, I must reason about what action you'll take as well
- Non-cooperative games
 - No external trusted agency, no legally binding agreements

Normal Form Games

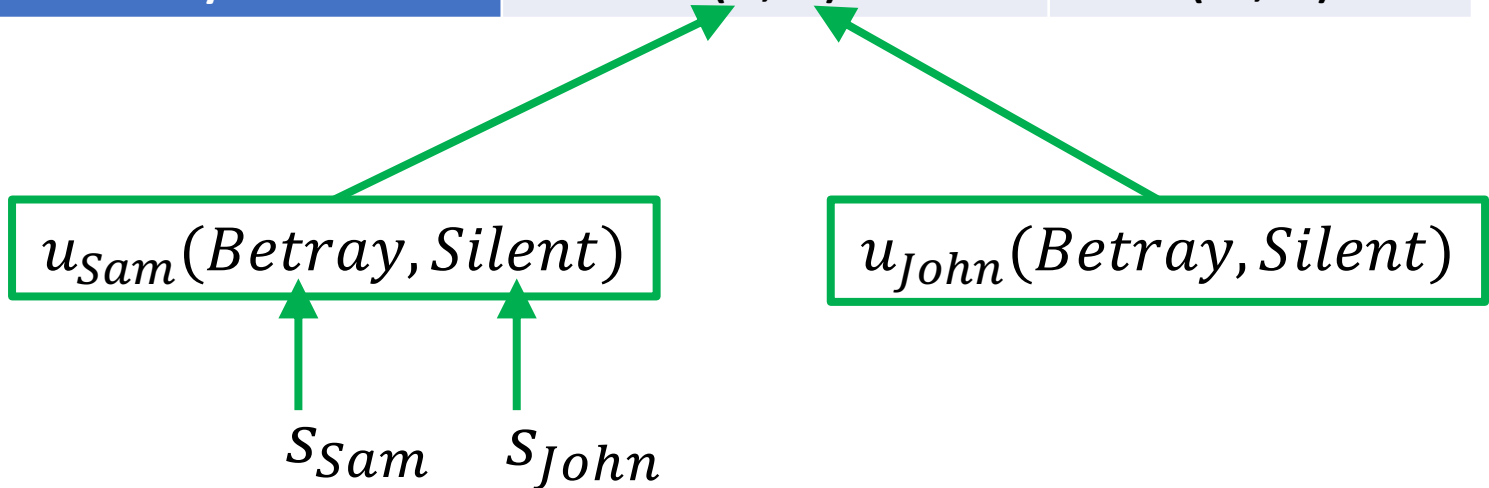
- A set of **players** $N = \{1, \dots, n\}$
- Each player i chooses an **action** $a_i \in A_i$
 - **Action profile** $\vec{a} = (a_1, \dots, a_n) \in \mathcal{A} = A_1 \times \dots \times A_n$
 - $\vec{a}_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$
- Each player i has a **utility function** $u_i : \mathcal{A} \rightarrow \mathbb{R}$
 - Given the action profile $\vec{a} = (a_1, \dots, a_n)$, each player i gets reward $u_i(a_1, \dots, a_n)$
- Note that the utility to player i depends on the action chosen by the other players too

Normal Form Games

Prisoner's dilemma

$$S = \{\text{Silent}, \text{Betray}\}$$

		John's Actions	
		Stay Silent	Betray
Sam's Actions	Stay Silent	$(-1, -1)$	$(-3, 0)$
	Betray	$(0, -3)$	$(-2, -2)$



Strategies

- **Pure strategy**

- Choose an action deterministically, e.g., “*betray*”

- **Mixed strategy**

- Choose an action in a randomized fashion, e.g., “*stay silent* with probability 0.3, and *betray* with probability 0.7” (call this s^*)
- We compute expected utilities when each player’s action is sampled from her mixed strategy independently of the other players
- **Example:** Say both Sam and John adopt s^* :

$$\begin{aligned} E[u_{sam}(s^*, s^*)] &= 0.3 \times 0.3 \times u_{sam}(\text{Silent}, \text{Silent}) \\ &\quad + 0.3 \times 0.7 \times u_{sam}(\text{Silent}, \text{Betray}) \\ &\quad + 0.7 \times 0.3 \times u_{sam}(\text{Betray}, \text{Silent}) \\ &\quad + 0.7 \times 0.7 \times u_{sam}(\text{Betray}, \text{Betray}) \end{aligned}$$

Domination Among Strategies

- Consider two strategies s_i, s'_i of player i
- Informally, s_i “dominates” s'_i if s_i is “better than” s'_i , *irrespective of the other players’ strategies*
- **Weak vs strict domination**
 - Both require: $u_i(s_i, \vec{s}_{-i}) \geq u_i(s'_i, \vec{s}_{-i}), \forall \vec{s}_{-i}$
 - Weak domination requires: Strict inequality for **some** \vec{s}_{-i}
 - Strict domination requires: Strict inequality for **all** \vec{s}_{-i}

Dominant Strategies

- Dominant strategies

- s_i is a strictly (resp. weakly) dominant strategy for player i if it strictly (resp. weakly) dominates every other strategy
- Strictly/weakly dominating every other *pure* strategy is sufficient (Why?)
- Can a player have two strictly/weakly dominant strategies?

- How does this relate to strategyproofness?

- “Truth-telling should be at least as good as any other strategy, regardless of what the other players do”
- Basically, strategyproofness requires truth-telling to be weakly dominant (except it doesn’t require it to be strictly better for *some* combination of strategies of the other players)

Example: Prisoner's Dilemma

- Recap:

		John's Actions	
		Stay Silent	Betray
Sam's Actions	Stay Silent	$(-1, -1)$	$(-3, 0)$
	Betray	$(0, -3)$	$(-2, -2)$

- Each player strictly wants to
 - Betray if the other player will stay silent
 - Betray if the other player will betray
- Betraying strictly dominates staying silent
 - So betraying is a strictly dominant strategy for each player

Solution Concept 1:

- If each player i has a strictly/weakly dominant strategy s_i^* , then the realized strategy profile would be (s_1^*, \dots, s_n^*)

Iterated Elimination

- What if there are no dominant strategies?
 - No single strategy dominates every other strategy
 - But some strategies might still be dominated
- Assuming everyone knows everyone is rational...
 - Can remove their dominated strategies
 - Might reveal a newly dominant strategy
- Eliminating only strictly dominated vs eliminating weakly dominated

Iterated Elimination

- Toy example:
 - Microsoft vs Startup
 - Enter the market or stay out?

		Startup	
		Enter	Stay Out
Microsoft	Enter	(2, -2)	(4, 0)
	Stay Out	(0, 4)	(0, 0)

- Q: Is there a dominant strategy for startup?
- Q: Do you see a rational outcome of the game?

Iterated Elimination

- “Guess $2/3$ of average”
 - Each student guesses a real number between 0 and 100 (inclusive)
 - The student whose number is the closest to $2/3$ of the average of all numbers wins!
- Poll: What would you do?

Solution Concept 2:

- If iterated elimination of strictly dominated strategies leads to a single strategy profile, then that would be the realized strategy profile

Nash Equilibrium

- What if not all players have a dominant strategy and iterated elimination does not help predict the outcome of the game either?

		Professor	
		Attend	Be Absent
Students	Attend	(3 , 1)	(-1 , -3)
	Be Absent	(-1 , -1)	(0 , 0)

Nash Equilibrium

- Instead of hoping to find strategies that players would play *irrespective of what other players play*, we find strategies that players would play *given what the other players are playing*

- **Nash Equilibrium**

- A strategy profile \vec{s} is in Nash equilibrium if s_i is the best action for player i given that other players are playing \vec{s}_{-i}

$$u_i(s_i, \vec{s}_{-i}) \geq u_i(s'_i, \vec{s}_{-i}), \forall s'_i$$

- Pure NE: All strategies are pure
- Mixed NE: At least one strategy is mixed

Recap: Prisoner's Dilemma

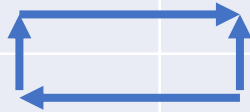
		John's Actions	
		Stay Silent	Betray
Sam's Actions	Stay Silent	$(-1, -1)$	$(-3, 0)$
	Betray	$(0, -3)$	$(-2, -2)$



- Nash equilibrium?
- (Dominant strategies)

Recap: Microsoft vs Startup

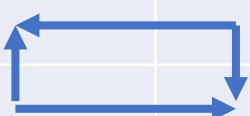
		Startup	
		Enter	Stay Out
Microsoft	Enter	(2, -2)	(4, 0)
	Stay Out	(0, 4)	(0, 0)



- Nash equilibrium?
- (Iterated elimination of strongly dominated strategies)

Recap: Attend or Not

		Professor	
		Attend	Be Absent
Students	Attend	(3, 1)	(-1, -3)
	Be Absent	(-1, -1)	(0, 0)



- Nash equilibria?
- Lack of predictability

Example: Rock-Paper-Scissor

P2 \ P1	Rock	Paper	Scissor
Rock	(0, 0)	(-1, 1)	(1, -1)
Paper	(1, -1)	(0, 0)	(-1, 1)
Scissor	(-1, 1)	(1, -1)	(0, 0)

- Pure Nash equilibrium?

Nash's Beautiful Result

- **Theorem:** Every normal form game admits a mixed-strategy Nash equilibrium.
- What about Rock-Paper-Scissor?

P2 \ P1	Rock	Paper	Scissor
Rock	(0 , 0)	(-1 , 1)	(1 , -1)
Paper	(1 , -1)	(0 , 0)	(-1 , 1)
Scissor	(-1 , 1)	(1 , -1)	(0 , 0)

Indifference Principle

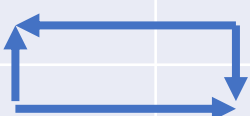
- Let \vec{s} be a Nash equilibrium
 - Let s_i be a mixed strategy with support T_i
 - Then, the expected payoff of player i from each $a_i \in T_i$ must be identical and at least as much as the expected payoff from any $a'_i \notin T_i$
-
- Derivation of rock-paper-scissor on the board.

Complexity

- **Theorem [DGP'06, CD'06]**
 - The problem of computing a Nash equilibrium of a given game is PPAD-complete even with two players.

Stag-Hunt

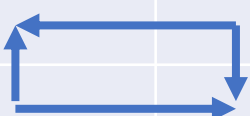
		Hunter 1	
		Stag	Hare
Hunter 2	Stag	(4, 4)	(0, 2)
	Hare	(2, 0)	(1, 1)



- Game
 - Stag requires both hunters, food is good for 4 days for each hunter.
 - Hare requires a single hunter, food is good for 2 days
 - If they both catch the same hare, they share.
- Two pure Nash equilibria: (Stag,Stag), (Hare,Hare)

Stag-Hunt

		Hunter 1	
		Stag	Hare
Hunter 2	Stag	(4, 4)	(0, 2)
	Hare	(2, 0)	(1, 1)

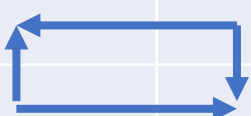


- Two pure Nash equilibria: (Stag,Stag), (Hare,Hare)
 - Other hunter plays “Stag” → “Stag” is best response
 - Other hunter plays “Hare” → “Hare” is best reponse

- What about mixed Nash equilibria?

Stag-Hunt

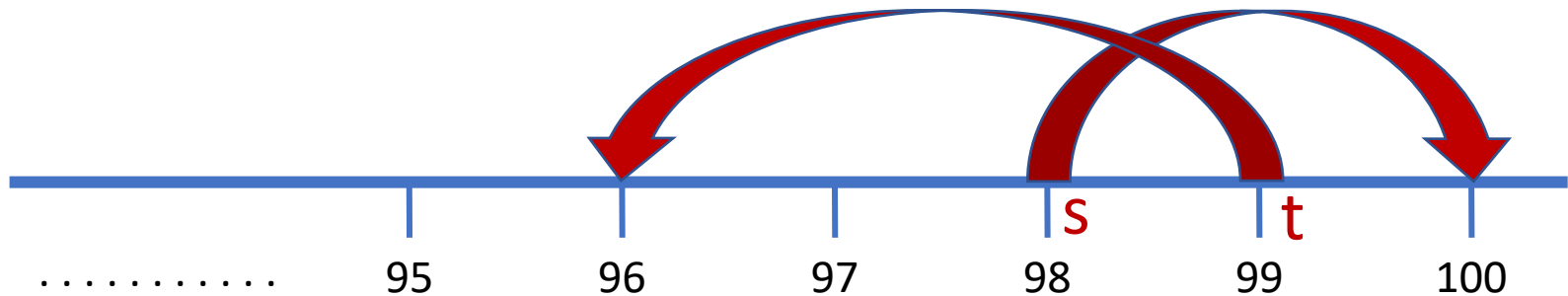
		Hunter 1	
		Stag	Hare
Hunter 2	Stag	(4, 4)	(0, 2)
	Hare	(2, 0)	(1, 1)



- Symmetric: $s \rightarrow \{\text{Stag w.p. } p, \text{Hare w.p. } 1 - p\}$
- Indifference principle:
 - Given the other hunter plays s , equal $\mathbb{E}[\text{reward}]$ for Stag and Hare
 - $\mathbb{E}[\text{Stag}] = p * 4 + (1 - p) * 0$
 - $\mathbb{E}[\text{Hare}] = p * 2 + (1 - p) * 1$
 - Equate the two $\Rightarrow p = 1/3$

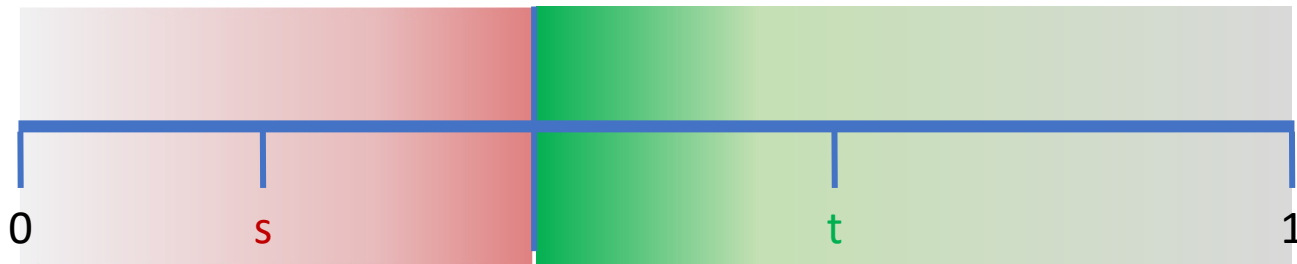
Extra Fun 1: Cunning Airlines

- Two travelers lose their luggage.
- Airline agrees to refund up to \$100 to each.
- Policy: Both travelers would submit a number between 2 and 99 (inclusive).
 - If both report the same number, each gets this value.
 - If one reports a lower number (s) than the other (t), the former gets $s+2$, the latter gets $s-2$.



Extra Fun 2: Ice Cream Shop

- Two brothers, each wants to set up an ice cream shop on the beach $([0,1])$.
- If the shops are at s, t (with $s \leq t$)
 - The brother at s gets $\left[0, \frac{s+t}{2}\right]$, the other gets $\left[\frac{s+t}{2}, 1\right]$



Nash Equilibria: Critique

- Noncooperative game theory provides a framework for analyzing rational behavior.
- But it relies on many assumptions that are often violated in the real world.
- Due to this, human actors are observed to play Nash equilibria in some settings, but play something far different in other settings.

Nash Equilibria: Critique

- Assumptions:
 - Rationality is common knowledge.
 - All players are rational.
 - All players know that all players are rational.
 - All players know that all players know that all players are rational.
 - ... [Aumann, 1976]
 - Behavioral economics
 - Rationality is perfect = “infinite wisdom”
 - Computationally bounded agents
 - Full information about what other players are doing.
 - Bayes-Nash equilibria

Nash Equilibria: Critique

- Assumptions:
 - No binding contracts.
 - Cooperative game theory
 - No player can commit first.
 - Stackelberg games (will study this in a few lectures)
 - No external help.
 - Correlated equilibria
 - Humans reason about randomization using expectations.
 - Prospect theory

Nash Equilibria: Critique

- Also, there are often multiple equilibria, and no clear way of “choosing” one over another.
- For many classes of games, finding a single equilibrium is provably hard.
 - Cannot expect humans to find it if your computer cannot.

Nash Equilibria: Critique

- Conclusion:
 - For human agents, take it with a grain of salt.
 - For AI agents playing against AI agents, perfect!

