CSC2556

Lecture 2

Voting II

Credit for many visuals: Ariel D. Procaccia

Recap

- Voting
  - n voters, m alternatives
  - Each voter i expresses a ranked preference $\succ_i$
  - Voting rule f
    - Takes as input the collection of preferences $\vec{\succ}$
    - Returns a single alternative

- A plethora of voting rule
  - Plurality, Borda count, STV, Kemeny, Copeland, maximin, …

Condorcet Winner

- Definition
  - Alternative x defeats y in a pairwise election if a *strict* majority of voters prefer x to y
  - Alternative x is a Condorcet winner if it defeats every other alternative in a pairwise election
- Question
  - Can there be two Condorcet winners?
- Condorcet paradox
  - No Condorcet winner when the majority preference is cyclic

1	2	3
a	b	c
b	c	a
c	a	b

Majority Preference

$$a \succ b$$
$$b \succ c$$
$$c \succ a$$

Condorcet Consistency

- Condorcet consistency
  - A voting rule is Condorcet consistent if it selects the Condorcet winner whenever one exists
  - On preference profiles where there is no Condorcet winner, it is free to output any winner

- Among the rules we saw so far…
  - ***NOT*** **Condorcet consistent:** all positional scoring rules (plurality, Borda, …), plurality with runoff, STV
  - **Condorcet consistent:** Kemeny (Why?)

Majority Consistency

- Majority consistency
  - If a strict majority of voters rank alternative x first, then x must be the winner.

- Question: What is the relation between majority consistency and Condorcet consistency?

  1. Majority consistency $\Rightarrow$ Condorcet consistency
  2. Condorcet consistency $\Rightarrow$ Majority consistency
  3. Equivalent
  4. Incomparable

Condorcet Consistency

- Copeland
  - Score(x) = # alternatives x beats in pairwise elections
  - Select x^* with the maximum score
  - Condorcet consistent (Why?)

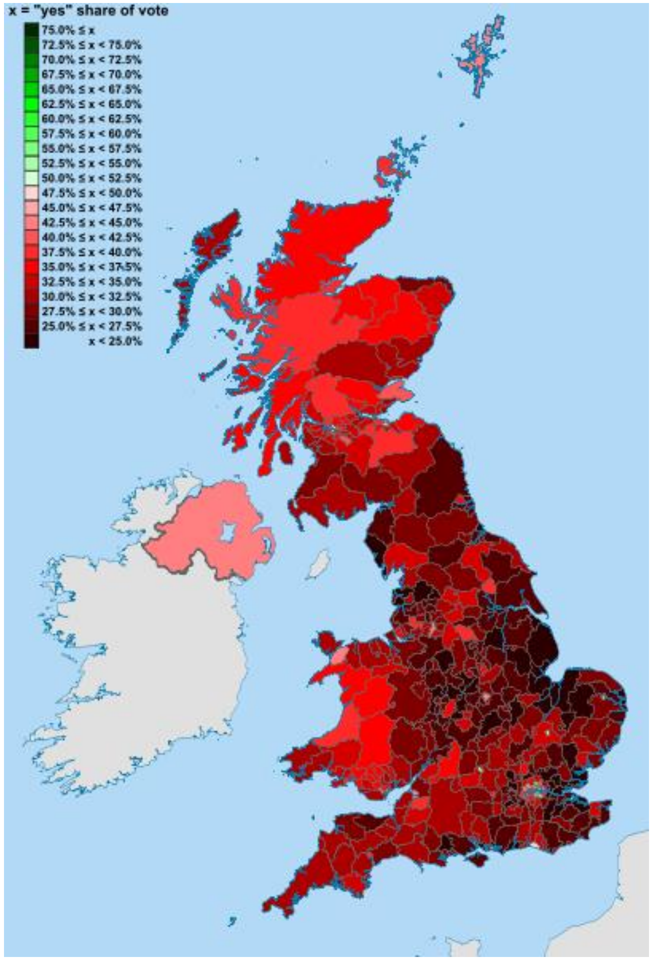
- Maximin
  - Score(x) = $\min_y n_{x \succ y}$
  - Select x^* with the maximum score
  - Also Condorcet consistent (Why?)

Which rule to use?

- We just introduced infinitely many rules
  - (Recall positional scoring rules…)
- How do we know which is the "right" rule to use?
  - Various approaches
  - Axiomatic, statistical, utilitarian, …
- How do we ensure good incentives without using money?
  - Bad luck! [Gibbard-Satterthwaite, next lecture]

Is Social Choice Practical?

- UK referendum: Choose between plurality and STV for electing MPs
- Academics agreed STV is better...
- ...but STV seen as beneficial to the hated Nick Clegg
- Hard to change political elections!



ROBOVOTE

Voting:
For the People,
By the People

- Voting can be useful in day-to-day activities
- On such a platform, easy to deploy the rules that we believe are the best

Incentives

- Can a voting rule incentivize voters to truthfully report their preferences?

- Strategyproofness
  - A voting rule is strategyproof if a voter cannot submit a false preference and get a more preferred alternative (under her true preference) elected, irrespective of the preferences of other voters
  - Formally, a voting rule f is strategyproof if for every preference profile $\vec{\succ}$, voter i, and preference $\succ_i'$, we have

$$f(\vec{\succ}) \succcurlyeq_i f(\vec{\succ}_{-i}, \succ_i')$$

  - Question: What is the relation between $f(\vec{\succ})$ and $f(\vec{\succ}_{-i}, \succ_i')$ according to $\succcurlyeq_i'$?

Strategyproofness

- None of the rules we saw are strategyproof!
- Example: Borda Count
  - In the true profile, b wins
  - Voter 3 can make a win by pushing b to the end

Winner
b

1	2	3
b	b	a
a	a	b
c	c	c
d	d	d

→

1	2	3
b	b	a
a	a	c
c	c	d
d	d	b

Winner
a

Borda's Response to Critics

My scheme is intended only for honest men!

Random 18th century French dude

Strategyproofness

- Are there any strategyproof rules?
  - Sure
- Dictatorial voting rule
  - The winner is always the most preferred alternative of voter i
- Constant voting rule
  - The winner is always the same
- Not satisfactory (for most cases)

Dictatorship

Constant function

Three Properties

- Strategyproof: Already defined. No voter has an incentive to misreport.

- Onto: Every alternative can win under some preference profile.

- Nondictatorial: There is no voter i such that $f(\overrightarrow{\succ})$ is always the alternative most preferred by voter i.

Gibbard-Satterthwaite

- Theorem: For $m \geq 3$, no deterministic social choice function is strategyproof, onto, and nondictatorial simultaneously ☹

- Proof: We will prove this for $n = 2$ voters.
  - Step 1: Show that SP ⇒ "strong monotonicity" [Assignment]
  - Strong Monotonicity (SM): If $f(\overrightarrow{\succ}) = a$, and $\overrightarrow{\succ}'$ is such that $\forall i \in N, x \in A$: $a \succ_i x \Rightarrow a \succ_i' x$, then $f(\overrightarrow{\succ}') = a$.
    - If, for each i, the set of alternatives defeated by a in $\succ_i'$ is a superset of what it defeats in $\succ_i$, then if it was winning under $\overrightarrow{\succ}$, it should also win under $\overrightarrow{\succ}'$

Gibbard-Satterthwaite

- Theorem: For $m \geq 3$, no deterministic social choice function is strategyproof, onto, and nondictatorial simultaneously ☹

- Proof: We will prove this for $n = 2$ voters.
  - Step 2: Show that SP + onto ⇒ "Pareto optimality" [Assignment]
  - Pareto Optimality (PO): If $a \succ_i b$ for all $i \in N$, then $f(\overrightarrow{\succ}) \neq b$.
    - If there is a different alternative a that *everyone* prefers to b, then b should not be the winner.

Gibbard-Satterthwaite

- Proof for n=2: Consider problem instance $I(a, b)$

$I(a,b)$

$\succ_1$	$\succ_2$
a	b
b	a

$\succ_1$	$\succ_2'$
a	b
b	
	a

$\succ_1''$	$\succ_2''$
a	A N Y
A N Y	

$f(\succ_1, \succ_2) \in \{a, b\}$
- ➢ PO

Say $f(\succ_1, \succ_2) = a$

$f(\succ_1, \succ_2') = a$

- PO: $f(\succ_1, \succ_2') \in \{\mathrm{a}, \mathrm{b}\}$
- SP: $f(\succ_1, \succ_2') \neq b$

$f(\succ'') = \mathrm{a}$
- ➢ SM

Gibbard-Satterthwaite

- Proof for n=2:
  - If f outputs a on instance $I(a, b)$, voter 1 can get a elected whenever she puts a first.
    - In other words, voter 1 becomes dictatorial for a.
    - Denote this property by the notation $D(1, a)$.
  - If f outputs b on $I(a, b)$
    - Voter 2 becomes dictatorial for b, i.e., we have $D(2, b)$.
- For every (a, b), f either satisfies the property $D(1, a)$ or the property $D(2, b)$.
  - We're not done! (Why?)

Gibbard-Satterthwaite

- Proof for n=2:
  - Fix a^* and b^*. Suppose $D(1, a^*)$ holds.
  - Then, we show that voter 1 is a dictator.
    - That is, $D(1, c)$ also holds for every $c \neq a^*$
  - Take $c \neq a^*$. Because $|A| \geq 3$, there exists $d \in A \backslash \{a^*, c\}$
  - Consider $I(c, d)$; f sastisifies either $D(1, c)$ or $D(2, d)$
  - But $D(2, d)$ is incompatible with $D(1, a^*)$
    - Who would win if voter 1 puts a^* first and voter 2 puts d first?
  - Thus, we have $D(1, c)$, as required ■

Circumventing G-S

- Restricted preferences (later in the course)
  - Not allowing all possible preference profiles
  - Example: single-peaked preferences
    - Alternatives are on a line (say 1D political spectrum)
    - Voters are also on the same line
    - Voters prefer alternatives that are closer to them
- Use of money (later in the course)
  - Require payments from voters that depend on the preferences they submit
  - Prevalent in auctions

Circumventing G-S

- Randomization (later in this lecture)
- Equilibrium analysis
  - How will strategic voters act under a voting rule that is not strategyproof?
  - Will they reach an "equilibrium" where each voter is happy with the (possibly false) preference she is submitting?
- Restricting information required for manipulation
  - Can voters successfully manipulate if they don't know the votes of the other voters?

Circumventing G-S

- Computational complexity
  - We need to use a rule that is the rule is manipulable
  - Can we make it NP-hard for voters to manipulate? [Bartholdi et al., SC&W 1989]
  - NP-hardness can be a good thing!
- f-MANIPULATION problem (for a given voting rule f)
  - Input: Manipulator i, alternative p, votes of other voters (non-manipulators)
  - Output: Can the manipulator cast a vote that makes p uniquely win under f?

Example: Borda

- Can voter 3 make a win?
  - Yes

1	2	3
b	b	
a	a	
c	c	
d	d	

→

1	2	3
b	b	a
a	a	c
c	c	d
d	d	b

A Greedy Algorithm

- Goal:
  - The manipulator wants to make alternative p win uniquely

- Algorithm:
  - Rank p in the first place
  - While there are unranked alternatives:
    - If there is an alternative that can be placed in the next spot without preventing p from winning, place this alternative.
    - Otherwise, return false.

Example: Borda

1	2	3
b	b	a
a	a	
c	c	
d	d	

1	2	3
b	b	a
a	a	b
c	c	
d	d	

1	2	3
b	b	a
a	a	c
c	c	
d	d	

1	2	3
b	b	a
a	a	c
c	c	b
d	d	

1	2	3
b	b	a
a	a	c
c	c	d
d	d	

1	2	3
b	b	a
a	a	c
c	c	d
d	d	b

Example: Copeland

1	2	3	4	5
a	b	e	e	a
b	a	c	c	
c	d	b	b	
d	e	a	a	
e	c	d	d	

Preference profile

	a	b	c	d	e
a	-	2	3	5	3
b	3	-	2	4	2
c	2	2	-	3	1
d	0	0	1	-	2
e	2	2	3	2	-

Pairwise elections

Example: Copeland

1	2	3	4	5
a	b	e	e	a
b	a	c	c	c
c	d	b	b	
d	e	a	a	
e	c	d	d	

Preference profile

	a	b	c	d	e
a	-	2	3	5	3
b	3	-	2	4	2
c	2	3	-	4	2
d	0	0	1	-	2
e	2	2	3	2	-

Pairwise elections

Example: Copeland

1	2	3	4	5
a	b	e	e	a
b	a	c	c	c
c	d	b	b	d
d	e	a	a	
e	c	d	d	

Preference profile

	a	b	c	d	e
a	-	2	3	5	3
b	3	-	2	4	2
c	2	3	-	4	2
d	0	1	1	-	3
e	2	2	3	2	-

Pairwise elections

Example: Copeland

1	2	3	4	5
a	b	e	e	a
b	a	c	c	c
c	d	b	b	d
d	e	a	a	e
e	c	d	d	

Preference profile

	a	b	c	d	e
a	-	2	3	5	3
b	3	-	2	4	2
c	2	3	-	4	2
d	0	1	1	-	3
e	2	3	3	2	-

Pairwise elections

Example: Copeland

1	2	3	4	5
a	b	e	e	a
b	a	c	c	c
c	d	b	b	d
d	e	a	a	e
e	c	d	d	b

Preference profile

	a	b	c	d	e
a	-	2	3	5	3
b	3	-	2	4	2
c	2	3	-	4	2
d	0	1	1	-	3
e	2	3	3	2	-

Pairwise elections