CSC2556

Lecture 12

Game Theory III:
Congestion Games, Braess' Paradox, Zero-Sum Games

Congestion Games & Braess' Paradox

Congestion Games

- Generalize cost sharing games
- n players, m resources (e.g., edges)
- Each player i chooses a set of resources P_i (e.g., $s_i \rightarrow t_i$ paths)
- When n_j player use resource j, each of them get a cost $f_j(n_j)$
- Cost to player is the sum of costs of resources used

Congestion Games

- Theorem [Rosenthal 1973]: Every congestion game is a potential game.

- Potential function:

$$\Phi(\vec{P}) = \sum_{j \in E(\vec{P})} \sum_{k=1}^{n_j(\vec{P})} f_j(k)$$

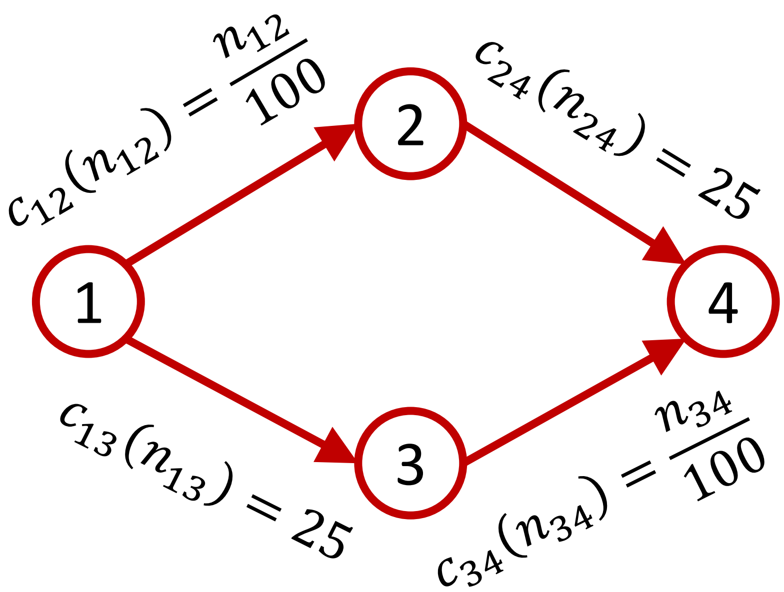
- Theorem [Monderer and Shapley 1996]: Every potential game is equivalent to a congestion game.

The Braess' Paradox

- In cost sharing, f_j is decreasing
  - The more people use a resource, the less the cost to each.
- f_j can also be increasing
  - Road network, each player going from home to work
  - Uses a sequence of roads
  - The more people on a road, the greater the congestion, the greater the delay (cost)
- Can lead to unintuitive phenomena

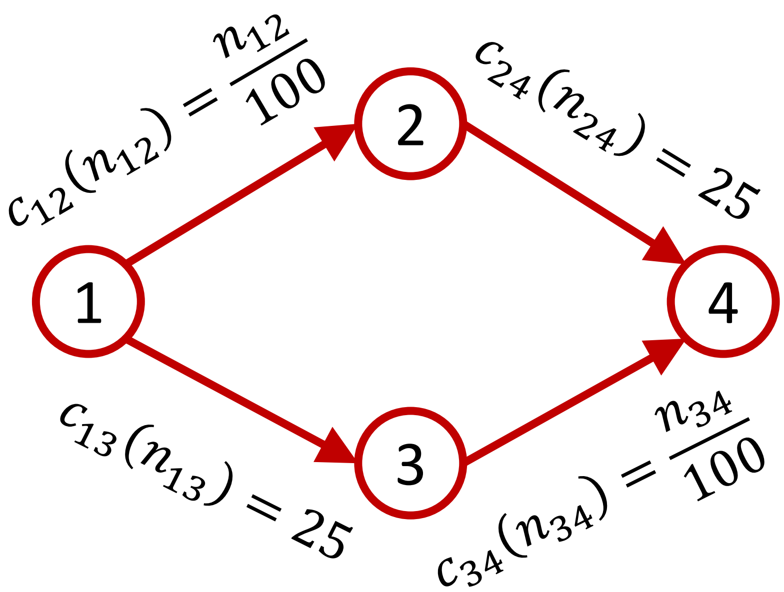
The Braess' Paradox

- Due to Parkes and Seuken:
  - 2000 players want to go from 1 to 4
  - $1 \rightarrow 2$ and $3 \rightarrow 4$ are "congestible" roads
  - $1 \rightarrow 3$ and $2 \rightarrow 4$ are "constant delay" roads



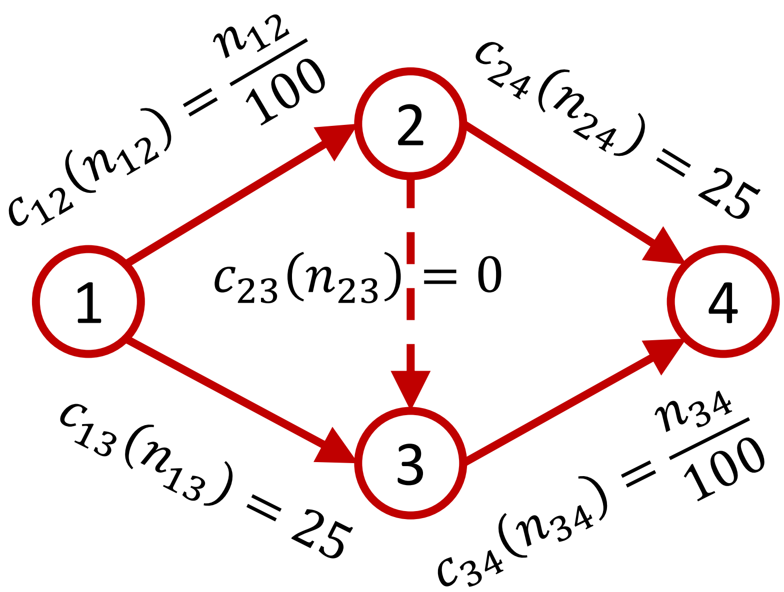
The Braess' Paradox

- Pure Nash equilibrium?
  - 1000 take $1 \to 2 \to 4$, 1000 take $1 \to 3 \to 4$
  - Each player has cost $10 + 25 = 35$
  - Anyone switching to the other creates a greater congestion on it, and faces a higher cost



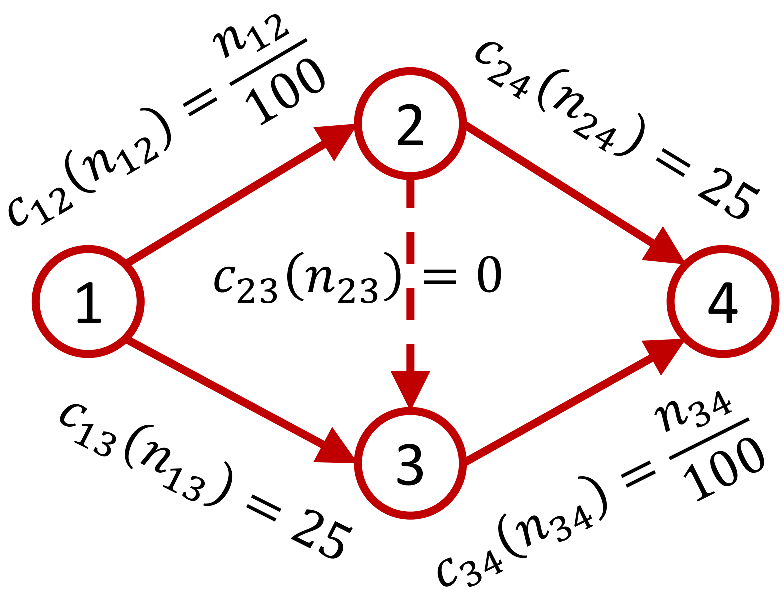
The Braess' Paradox

- What if we add a zero-cost connection $2 \to 3$?
  - Intuitively, adding more roads should only be helpful
  - In reality, it leads to a greater delay for everyone in the unique equilibrium!



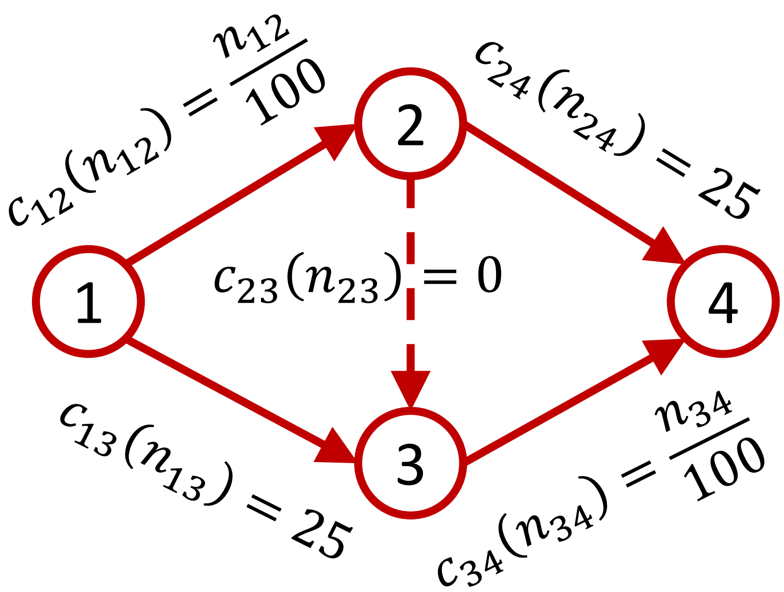
The Braess' Paradox

- Nobody chooses $1 \rightarrow 3$ as $1 \rightarrow 2 \rightarrow 3$ is better irrespective of how many other players take it
- Similarly, nobody chooses $2 \rightarrow 4$
- Everyone takes $1 \rightarrow 2 \rightarrow 3 \rightarrow 4$, faces delay = 40!



The Braess' Paradox

- In fact, what we showed is:
  - In the new game, $1 \to 2 \to 3 \to 4$ is a strictly dominant strategy for each firm!



Zero-Sum Games

Zero-Sum Games

- Total reward is constant in all outcomes (w.l.o.g. 0)
- Focus on two-player zero-sum games (2p-zs)
  - "The more I win, the more you lose"
  - Chess, tic-tac-toe, rock-paper-scissor, …

P2 / P1	Rock	Paper	Scissor
Rock	(0 , 0)	(-1 , 1)	(1 , -1)
Paper	(1 , -1)	(0 , 0)	(-1 , 1)
Scissor	(-1 , 1)	(1 , -1)	(0 , 0)

Zero-Sum Games

- Reward for P2 = - Reward for P1
  - Only need a single matrix A : reward for P1
  - P1 wants to maximize, P2 wants to minimize

P1 \ P2	Rock	Paper	Scissor
Rock	**0**	**-1**	**1**
Paper	**1**	**0**	**-1**
Scissor	**-1**	**1**	**0**

Rewards in Matrix Form

- Reward for P1 when…
  - P1 uses mixed strategy x_1, and
  - P2 uses mixed strategy x_2, is
  - $x_1^T\, A\, x_2$ (where x_1 and x_2 are column vectors)

Maximin/Minimax Strategy

- Worst-case approach of P1:
  - Let's say I use strategy x_1.
  - In the worst case, P2 finds out what I'm doing and chooses x_2 to minimize my reward (i.e., maximize his reward).
  - So, the best I can guarantee myself in this worst case is:

$$V_1^* = \max_{x_1} \min_{x_2} x_1^T * A * x_2$$

  - A maximizer x_1^* is a maximin strategy for P1

Maximin/Minimax Strategy

- P1's best worst-case guarantee:

$$V_1^* = \max_{x_1} \min_{x_2} x_1^T * A * x_2$$

- P2's best worst-case guarantee:

$$V_2^* = \min_{x_2} \max_{x_1} x_1^T * A * x_2$$

  - P2's minimax strategy x_2^* minimizes this

- Claim: $V_1^* \leq V_2^*$
  - Consider what would happen if they both play their "safe" strategies at the same time

The Minimax Theorem

- Jon von Neumann [1928]

- Theorem: For any 2p-zs game,
  - $V_1^* = V_2^* = V^*$ (called the minimax value of the game)
  - Set of Nash equilibria =

    $\{(\mathrm{x}_1^*, \mathrm{x}_2^*) : \mathrm{x}_1^* =$ any maximin for P1, $\mathrm{x}_2^* =$ any minimax for P2$\}$

- Corollary: x_1^* is best response to x_2^* and vice-versa.

The Minimax Theorem

- Jon von Neumann [1928]

  "As far as I can see, there could be no theory of games … without that theorem …

  I thought there was nothing worth publishing until the Minimax Theorem was proved"

- Indeed, much more compelling and predictive than Nash equilibria in general-sum games (which came much later).

Computing Nash Equilibria

- General-sum games: Computing a Nash equilibrium is PPAD-complete even with just two players.
  - Trivia: Another notable PPAD-complete problem is finding a three-colored point in Sperner's Lemma.

- 2p-zs games: Polynomial time using linear programming
  - Polynomial in #actions of the two players: m_1 and m_2

Computing Nash Equilibria

Maximize v

Subject to

$(x_1^T A)_j \geq v,\ j \in \{1, \dots, m_2\}$

$x_1(1) + \cdots + x_1(m_1) = 1$

$x_1(i) \geq 0, i \in \{1, \dots, m_1\}$

Minimax Theorem in Real Life?

Minimax Theorem in Real Life?

Kicker \ Goalie	L	R
L	0.58	0.95
R	0.93	0.70

Kicker
Maximize v
Subject to
$0.58p_L + 0.93p_R \geq v$
$0.95p_L + 0.70p_R \geq v$
$p_L + p_R = 1$
$p_L \geq 0, p_R \geq 0$

Goalie
Minimize v
Subject to
$0.58q_L + 0.95q_R \leq v$
$0.93q_L + 0.70q_R \leq v$
$q_L + q_R = 1$
$q_L \geq 0, q_R \geq 0$

Minimax Theorem in Real Life?

Kicker \ Goalie	L	R
L	0.58	0.95
R	0.93	0.70

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Professionals Play Minimax

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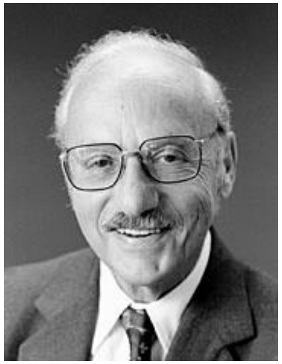
Minimax Theorem

- Implies Yao's minimax principle
- Equivalent to linear programming duality

John von Neumann



George Dantzig

von Neumann and Dantzig

George Dantzig loves to tell the story of his meeting with John von Neumann on October 3, 1947 at the Institute for Advanced Study at Princeton. Dantzig went to that meeting with the express purpose of describing the linear programming problem to von Neumann and asking him to suggest a computational procedure. He was actually looking for methods to benchmark the simplex method. Instead, he got a 90-minute lecture on Farkas Lemma and Duality (Dantzig's notes of this session formed the source of the modern perspective on linear programming duality). Not wanting Dantzig to be completely amazed, von Neumann admitted:

"I don't want you to think that I am pulling all this out of my sleeve like a magician. I have recently completed a book with Morgenstern on the theory of games. What I am doing is conjecturing that the two problems are equivalent. The theory that I am outlining is an analogue to the one we have developed for games.“

- (Chandru & Rao, 1999)