

CSC2556

Lecture 10

Game Theory II: Prices of Anarchy and Stability

Prices of Anarchy and Stability

Price of Anarchy and Stability

- If players play a Nash equilibrium instead of “socially optimum”, how bad can it be?
- **Objective function**: sum of utilities/costs
- **Price of Anarchy (PoA)**: compare the optimum to the **worst** Nash equilibrium
- **Price of Stability (PoS)**: compare the optimum to the **best** Nash equilibrium

Price of Anarchy and Stability

- Price of Anarchy (PoA)

$$\frac{\text{Max social utility}}{\text{Min social utility in any NE}}$$

- Price of Stability (PoS)

$$\frac{\text{Max social utility}}{\text{Max social utility in any NE}}$$

Costs → flip:
Nash equilibrium
divided by optimum

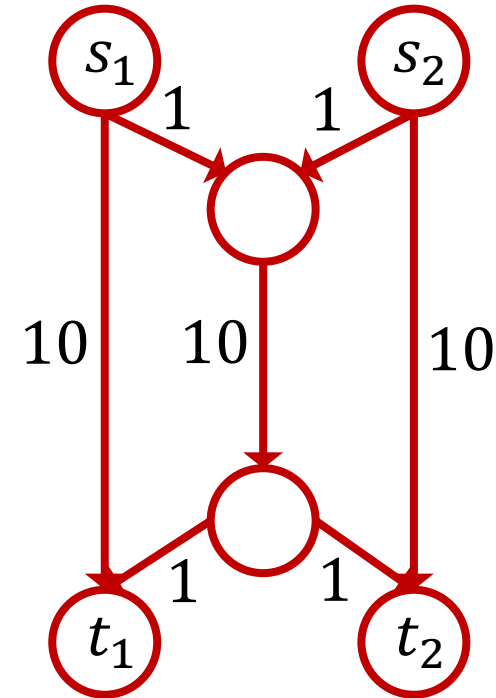
Revisiting Stag-Hunt

Hunter 2 \ Hunter 1	Stag	Hare
Stag	(4, 4)	(0, 2)
Hare	(2, 0)	(1, 1)

- Optimum social utility = $4+4 = 8$
- Three equilibria:
 - (Stag, Stag) : Social utility = 8
 - (Hare, Hare) : Social utility = 2
 - (Stag:1/3 - Hare:2/3, Stag:1/3 - Hare:2/3)
 - Social utility = $(1/3)*(1/3)*8 + (1-(1/3))*(1/3)*2 =$ Btw 2 and 8
- Price of stability? Price of anarchy?

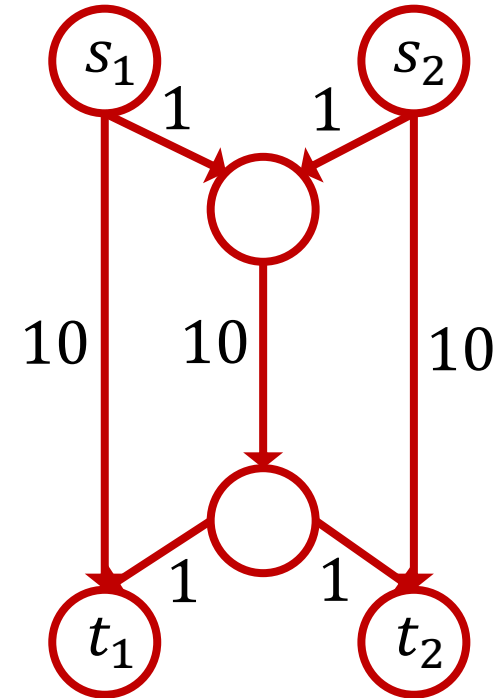
Cost Sharing Game

- n players on directed weighted graph G
- Player i
 - Wants to go from s_i to t_i
 - Strategy set $S_i = \{\text{directed } s_i \rightarrow t_i \text{ paths}\}$
 - Denote his chosen path by $P_i \in S_i$
- Each edge e has cost c_e (weight)
 - Cost is split among all players taking edge e
 - That is, among all players i with $e \in P_i$



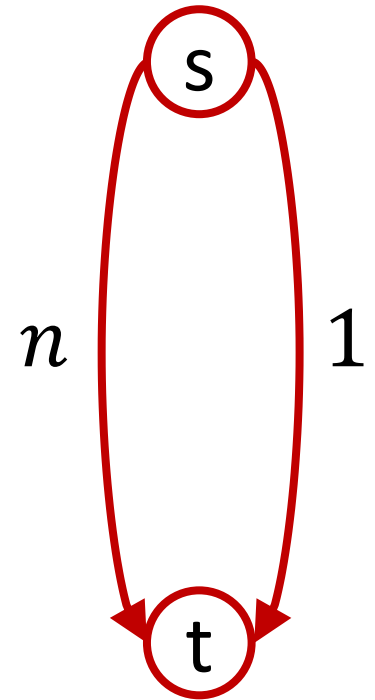
Cost Sharing Game

- Given strategy profile \vec{P} , cost $c_i(\vec{P})$ to player i is sum of his costs for edges $e \in P_i$
- Social cost $C(\vec{P}) = \sum_i c_i(\vec{P})$
 - Note that $C(\vec{P}) = \sum_{e \in E(\vec{P})} c_e$, where $E(\vec{P}) = \{\text{edges taken in } \vec{P} \text{ by at least one player}\}$
- In the example on the right:
 - What if both players take the direct paths?
 - What if both take the middle paths?
 - What if only one player takes the middle path while the other takes the direct path?



Cost Sharing: Simple Example

- Example on the right: n players
- Two pure NE
 - All taking the n -edge: social cost = n
 - All taking the 1 -edge: social cost = 1
 - Also the social optimum
- In this game, price of anarchy $\geq n$
- We can show that for all cost sharing games, price of anarchy $\leq n$



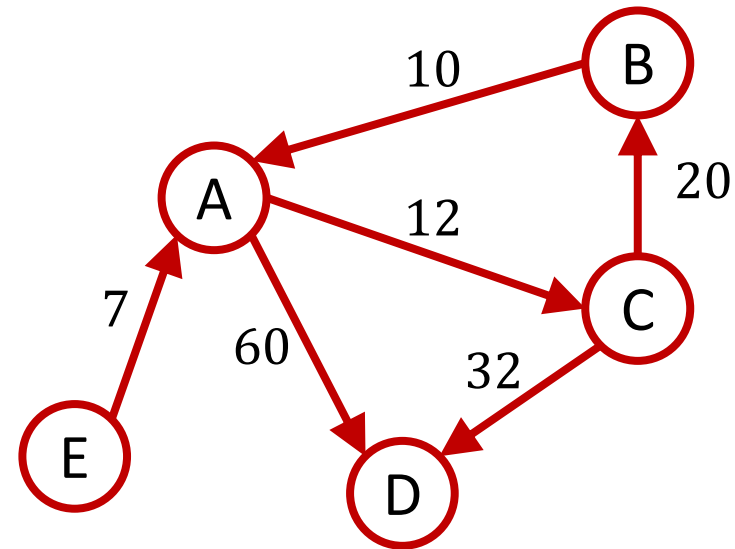
Cost Sharing: PoA

- **Theorem:** The price of anarchy of a cost sharing game is at most n .
- **Proof:**
 - Suppose the social optimum is $(P_1^*, P_2^*, \dots, P_n^*)$, in which the cost to player i is c_i^* .
 - Take any NE with cost c_i to player i .
 - Let c_i' be his cost if he switches to P_i^* .
 - NE $\Rightarrow c_i' \geq c_i$ (Why?)
 - But : $c_i' \leq n \cdot c_i^*$ (Why?)
 - $c_i \leq n \cdot c_i^*$ for each i \Rightarrow no worse than $n \times$ optimum



Cost Sharing

- Price of anarchy
 - All cost-sharing games: $\text{PoA} \leq n$
 - \exists example where $\text{PoA} = n$
- Price of stability? Later...
- Both examples we saw had pure Nash equilibria
 - What about more complex games, like the one on the right?



10 players: $E \rightarrow C$
27 players: $B \rightarrow D$
19 players: $C \rightarrow D$

Good News

- **Theorem:** All cost sharing games admit a pure Nash equilibrium.
- **Proof:**
 - Via a “potential function” argument.

Step 1: Define Potential Fn

- Potential function: $\Phi : \prod_i S_i \rightarrow \mathbb{R}_+$
 - For all pure strategy profiles $\vec{P} = (P_1, \dots, P_n) \in \prod_i S_i, \dots$
 - all players i , and ...
 - all alternative strategies $P'_i \in S_i$ for player i ...

$$c_i(P'_i, \vec{P}_{-i}) - c_i(\vec{P}) = \Phi(P'_i, \vec{P}_{-i}) - \Phi(\vec{P})$$

- When a single player changes his strategy, the change in *his* cost is equal to the change in the potential function
 - Do not care about the changes in the costs to others

Step 2: Potential $F^n \rightarrow$ pure Nash Eq

- All games that admit a potential function have a pure Nash equilibrium. **Why?**
 - Think about \vec{P} that minimizes the potential function.
 - What happens when a player deviates?
 - If his cost decreases, the potential function value must also decrease.
 - \vec{P} already minimizes the potential function value.
- Pure strategy profile minimizing potential function is a pure Nash equilibrium.

Step 3: Potential F^n for Cost-Sharing

- Recall: $E(\vec{P}) = \{\text{edges taken in } \vec{P} \text{ by at least one player}\}$
- Let $n_e(\vec{P})$ be the number of players taking e in \vec{P}

$$\Phi(\vec{P}) = \sum_{e \in E(\vec{P})} \sum_{k=1}^{n_e(\vec{P})} \frac{c_e}{k}$$

- Note: The cost of edge e to each player taking e is $c_e/n_e(\vec{P})$. But the potential function includes all fractions: $c_e/1, c_e/2, \dots, c_e/n_e(\vec{P})$.

Step 3: Potential F^n for Cost-Sharing

$$\Phi(\vec{P}) = \sum_{e \in E(\vec{P})} \sum_{k=1}^{n_e(\vec{P})} \frac{c_e}{k}$$


- Why is this a potential function?
 - If a player changes path, he pays $\frac{c_e}{n_e(\vec{P})+1}$ for each new edge e , gets back $\frac{c_f}{n_f(\vec{P})}$ for each old edge f .
 - This is precisely the change in the potential function too.
 - So $\Delta c_i = \Delta \Phi$.




Potential Minimizing Eq.


- There could be multiple pure Nash equilibria
 - Pure Nash equilibria are “local minima” of the potential function.
 - *A single player* deviating should not decrease the function value.
- Is the *global minimum* of the potential function a special pure Nash equilibrium?

Potential Minimizing Eq.

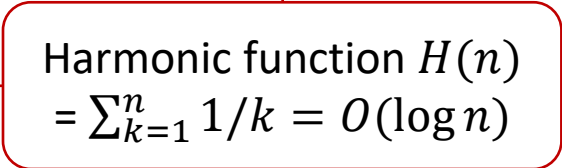



$$\sum_{e \in E(\vec{P})} c_e \leq \Phi(\vec{P}) = \sum_{e \in E(\vec{P})} \sum_{k=1}^{n_e(\vec{P})} \frac{c_e}{k} \leq \sum_{e \in E(\vec{P})} c_e * \sum_{k=1}^n \frac{1}{k}$$

 Social cost







$$\forall \vec{P}, C(\vec{P}) \leq \Phi(\vec{P}) \leq C(\vec{P}) * H(n)$$

 Harmonic function $H(n) = \sum_{k=1}^n 1/k = O(\log n)$



$$C(\vec{P}^*) \leq \Phi(\vec{P}^*) \leq \Phi(OPT) \leq C(OPT) * H(n)$$



 Potential minimizing eq.



 Social optimum

Potential Minimizing Eq.

- Potential minimizing equilibrium gives $O(\log n)$ approximation to the social optimum
 - Price of stability is $O(\log n)$
 - This is tight as there exists an example where the price of stability is $\Omega(\log n)$
 - Compare this to the price of anarchy, which can be n