CSC2556

Lecture 7

Cake-Cutting (continued)
Indivisible Goods

Cake-Cutting (Continued)

Other Desiderata

- There are two more properties that we often desire from an allocation.

- Pareto optimality (PO)
  - Notion of efficiency
  - Informally, it says that there should be no “obviously better” allocation

- Strategyproofness (SP)
  - No player should be able to gain by misreporting her valuation

Strategyproofness (SP)

- For deterministic mechanisms
  - "Strategyproof": No player should be able to increase her *utility* by misreporting her valuation, irrespective of what other players report.

- For randomized mechanisms
  - "Strategyproof-in-expectation": No player should be able to increase her *expected utility* by misreporting.
  - For simplicity, we'll call this strategyproofness, and assume we mean "in expectation" if the mechanism is randomized.

Strategyproofness (SP)

- Deterministic
  - Bad news!
  - Theorem [Menon & Larson '17] : No deterministic SP mechanism is (even approximately) proportional.

- Randomized
  - Good news!
  - Theorem [Chen et al. '13, Mossel & Tamuz '10]: There is a randomized SP mechanism that always returns an envy-free allocation.

Perfect Partition

- Theorem [Lyapunov '40]:
  - There always exists a "perfect partition" $(B_1, \ldots, B_n)$ of the cake such that $V_i\left(B_j\right) = {}^1/_n$ for every $i, j \in [n]$.
  - Every agent values every bundle equally.

- Theorem [Alon '87]:
  - There exists a perfect partition that only cuts the cake at $poly(n)$ points.
  - In contrast, Lyapunov's proof is non-constructive, and might need an unbounded number of cuts.

Perfect Partition

- Q: Can you use an algorithm for computing a perfect partition as a black-box to design a randomized SP-in-expectation+EF mechanism?
  - Yes! Compute a perfect partition, and assign the n bundles to the n players uniformly at random.
  - Why is this EF?
    - Every agent values every bundle at $1/n$.
  - Why is this SP-in-expectation?
    - Because an agent is assigned a random bundle, her expected utility is $1/n$, irrespective of what she reports.

Pareto Optimality (PO)

- Definition
  - We say that an allocation $A = (A_1, \dots, A_n)$ is PO if there is no alternative allocation $B = (B_1, \dots, B_n)$ such that
  1. Every agent is at least as happy: $V_i(B_i) \geq V_i(A_i), \forall i \in N$
  2. Some agent is strictly happier: $V_i(B_i) > V_i(A_i), \exists i \in N$

  - I.e., an allocation is PO if there is no "better" allocation.

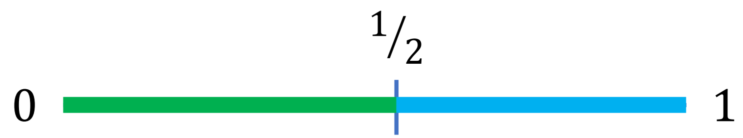
- Q: Is it PO to give the entire cake to player 1?
- A: Not necessarily. But yes if player 1 values "every part of the cake positively".

PO + EF

- Theorem [Weller '85]:
  - There always exists an allocation of the cake that is both envy-free and Pareto optimal.
- One way to achieve PO+EF:
  - Nash-optimal allocation: $\text{argmax}_A \prod_{i \in N} V_i(A_i)$
  - Obviously, this is PO. The fact that it is EF is non-trivial.
  - This is named after John Nash.
    - Nash social welfare = product of utilities
    - Different from utilitarian social welfare = sum of utilities

Nash-Optimal Allocation

- Example:
  - Green player has value 1 distributed over $[0, {}^2/_3]$
  - Blue player has value 1 distributed over $[0,1]$
  - Without loss of generality (why?) suppose:
    - Green player gets x fraction of $[0, {}^2/_3]$
    - Blue player gets the remaining $1 - x$ fraction of $[0, {}^2/_3]$ AND all of $[{}^2/_3 , 1]$.
  - Green's utility = x,   blue's utility = $(1 - \mathrm{x}) \cdot \frac{2}{3} + \frac{1}{3} = \frac{3-2x}{3}$
  - Maximize: $x \cdot \frac{3-2x}{3} \Rightarrow x = {}^3/_4$ (${}^3/_4$ fraction of ${}^2/_3$ is ${}^1/_2$).

Allocation 

Green has utility $\frac{3}{4}$

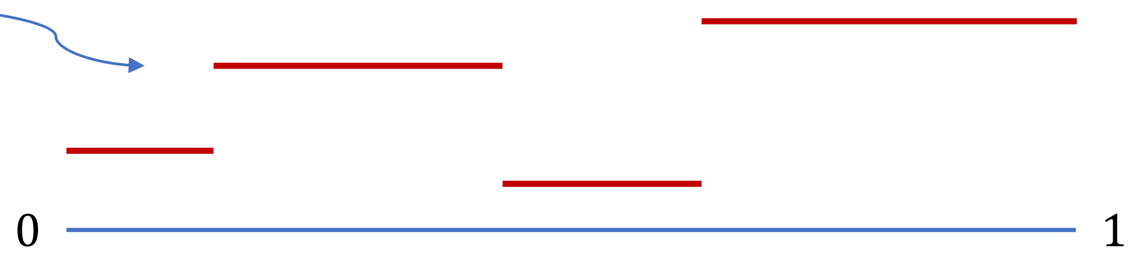
Blue has utility $\frac{1}{2}$

Problem with Nash Solution

- Difficult to compute in general
  - I believe it should require an unbounded number of queries in the Robertson-Webb model. But I can't find such a result in the literature.

- Theorem [Aziz & Ye '14]:
  - For *piecewise constant* valuations, the Nash-optimal solution can be computed in polynomial time.

The density function of a piecewise constant valuation looks like this



Interlude: Homogeneous Divisible Goods

- Suppose there are m homogeneous divisible goods
  - Each good can be divided fractionally between the agents
- Let $x_{i,g}$ = fraction of good g that agent i gets
  - Homogeneous = agent doesn't care which "part"
    - E.g., CPU or RAM
- Special case of cake-cutting
  - Line up the goods on $[0,1]$ → piecewise uniform valuations

Interlude: Homogeneous Divisible Goods

- Nash-optimal solution:

  Maximize $\sum_i \log U_i$

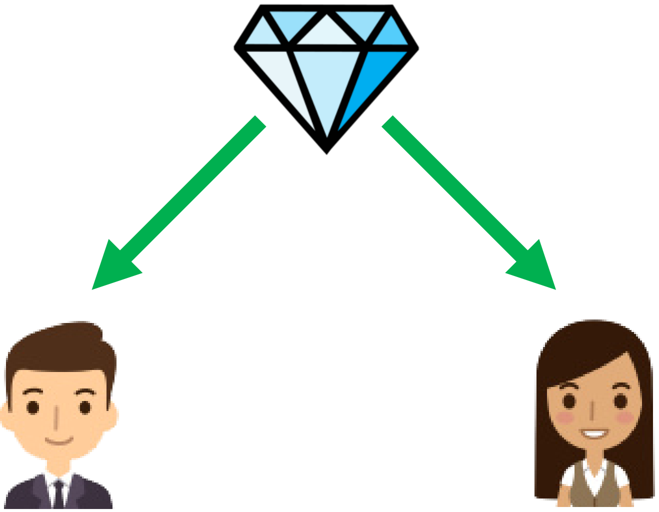
  $$U_i = \Sigma_g \, x_{i,g} * v_{i,g} \qquad \forall i$$
  $$\Sigma_i \, x_{i,g} = 1 \qquad \forall g$$
  $$x_{i,g} \in [0,1] \qquad \forall i, g$$

- Gale-Eisenberg Convex Program
  - Polynomial time solvable

Indivisible Goods

Indivisible Goods

- Goods which cannot be shared among players
  - E.g., house, painting, car, jewelry, …
- Problem: Envy-free allocations may not exist!



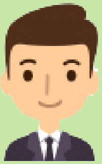
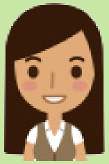
Indivisible Goods: Setting

				
	8	7	20	5
	9	11	12	8
	9	10	18	3

Given such a matrix of numbers, assign each good to a player.

We assume additive values. So, e.g., V  $(\{$  ,  $\}) = 8 + 7 = 15$

Indivisible Goods

				
	8	7	20	5
	9	11	12	8
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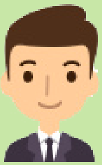
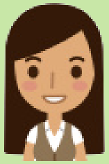
Indivisible Goods

				
	8	7	20	5
	9	11	12	8
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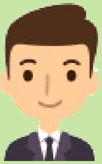
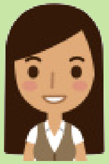
Indivisible Goods

				
	8	7	20	5
	9	11	12	8
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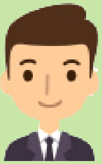
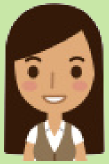
Indivisible Goods

				
	8	7	20	5
	9	11	12	8
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Indivisible Goods

- Envy-freeness up to one good (EF1):

$$\forall i, j \in N, \exists g \in A_j : V_i(A_i) \geq V_i\left(A_j \setminus \{g\}\right)$$

  - Technically, we need either this or $A_j = \emptyset$.
  - "If i envies j, there must be some good in j's bundle such that removing it would make i envy-free of j."

- Does there always exist an EF1 allocation?

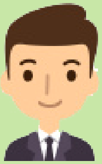
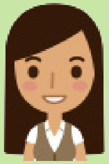
EF1

- Yes! We can use Round Robin.
  - Agents take turns in cyclic order: $1,2, \dots, n, 1,2, \dots, n, \dots$
  - In her turn, an agent picks the good she likes the most among the goods still not picked by anyone.
- Observation: This always yields an EF1 allocation.
  - Informal proof on the board.
- Sadly, on some instances, this returns an allocation that is not Pareto optimal.

EF1+PO?

- Nash welfare to rescue!
- Theorem [Caragiannis et al. '16]:
  - The allocation $\text{argmax}_A \prod_{i \in N} V_i(A_i)$ is EF1 + PO.
  - Note: This maximization is over only "integral" allocations that assign each good to some player in whole.
  - Note: Subtle tie-breaking if all allocations have zero Nash welfare.
    - Step 1: Choose a subset of players $S \subseteq N$ with largest $|S|$ such that it is possible to give a positive utility to every player in S simultaneously.
    - Step 2: Choose $\text{argmax}_A \prod_{i \in S} V_i(A_i)$

Integral Nash Allocation

				
	8	7	20	5
	9	11	12	8
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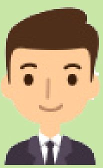
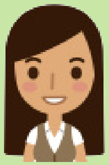
20 * 8 * (9+10) = 3040

				
	8	7	20	5
	9	11	12	8
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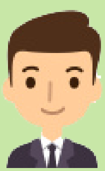
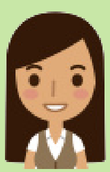
(8+7) * 8 * 18 = 2160

				
	8	7	20	5
	9	11	12	8
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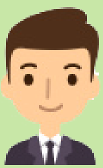
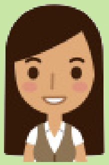
8 * (12+8) * 10 = 1600

				
	8	7	20	5
	9	11	12	8
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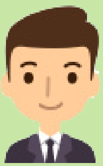
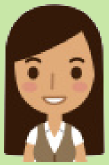
20 * (11+8) * 9 = 3420

				
	8	7	20	5
	9	11	12	8
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Computation

- For indivisible goods, Nash-optimal solution is strongly NP-hard to compute
  - That is, remains NP-hard even if all values in the matrix are bounded

- Open Question: If our goal is EF1+PO, is there a different polynomial time algorithm?
  - Not sure. But a recent paper gives a pseudo-polynomial time algorithm for EF1+PO
    - Time is polynomial in n, m, and $\max_{i,g} V_i(\{g\})$.

Other Fairness Notions

- Maximin Share Guarantee (MMS):
  - Generalization of “cut and choose” for n players
  - MMS value of player i =
    - The highest value player i can get…
    - If *she* divides the goods into n bundles…
    - But receives the worst bundle for her (“worst case guarantee”)
  - Let $\mathcal{P}_n(M)$ denote the family of partitions of the set of goods M into n bundles.

$$MMS_i = \max_{(B_1,\dots,B_n)\in\mathcal{P}_n(M)} \min_{k\in\{1,\dots,n\}} V_i(B_k).$$

  - An allocation is α-MMS if every player i receives value at least $\alpha * MMS_i$.

Other Fairness Notions

- Maximin Share Guarantee (MMS)
  - [Procaccia, Wang '14]:
    There is an example in which no MMS allocation exists.
  - [Procaccia, Wang '14]:
    A $^2/_3$ - MMS allocation always exists.
  - [Ghodsi et al. '17]:
    A $^3/_4$ - MMS allocation always exists.
  - [Caragiannis et al. '16]:
    The Nash-optimal solution is $\frac{2}{1+\sqrt{4n-3}}$ −MMS, and this is the best possible guarantee.

Stronger Fairness

- Open Question: Does there always exist an EFx allocation?
- EF1: $\forall i, j \in N, \exists g \in A_j : \; V_i(A_i) \geq V_i\left(A_j \setminus \{g\}\right)$
  - Intuitively, i doesn't envy j if she gets to remove her most valued item from j's bundle.
- EFx: $\forall i, j \in N, \forall g \in A_j : \; V_i(A_i) \geq V_i\left(A_j \setminus \{g\}\right)$
  - Note: Need to quantify over g such that $V_i(\{g\}) > 0$.
  - Intuitively, i doesn't envy j even if she removes her least positively valued item from j's bundle.

Stronger Fairness

- The difference between EF1 and EFx:
  - Suppose there are two players and three goods with values as follows.

	A	B	C
P1	5	1	10
P2	0	1	10

  - If you give {A} → P1 and {B,C} → P2, it's EF1 but not EFx.
    - EF1 because if P1 removes C from P2's bundle, all is fine.
    - Not EFx because removing B doesn't eliminate envy.
  - Instead, {A,B} → P1 and {C} → P2 would be EFx.

Allocation of Bads

- Negative utilities (costs instead of values)
  - Let $c_{i,b}$ be the cost of player i for bad b.
    - $C_i(S) = \sum_{b \in S} c_{i,b}$
  - EF: $\forall i, j \;\; C_i(A_i) \le C_i(A_j)$
  - PO: There should be no alternative allocation in which no player has more cost, and some player has less cost.

- Divisible bads
  - EF + PO allocation always exists, like for divisible goods.
    - One way to achieve is through "Competitive Equilibria" (CE).
    - For divisible goods, Nash-optimal allocation is the unique CE.
    - For bads, there are exponentially many CE.

Allocation of Bads

- Indivisible bads
  - EF1: $\forall i, j \;\; \exists b \in A_i \;\; c_i(A_i \setminus \{b\}) \le c_i(A_j)$
  - EFx: $\forall i, j \;\; \forall b \in A_i \; c_i(A_i \setminus \{b\}) \le c_i(A_j)$
    - Note: Again, we need to restrict to b such that $c_{i,b} > 0$
  - Open Question 1:
    - Does an EF1 + PO allocation always exist?
  - Open Question 2:
    - Does an EFx allocation always exist?
  - More open questions related to relaxations of proportionality