

CSC2556

Lecture 3

Approaches to Voting

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Approaches to Voting

- What does an approach give us?
 - A way to compare voting rules
 - Hopefully a “uniquely optimal voting rule”
- Axiomatic Approach
- Distance Rationalizability
- Statistical Approach
- Utilitarian Approach
- ...

Axiomatic Approach

- Axiom: requirement that the voting rule should behave in a certain way
- **Goal:** define a set of reasonable axioms, and search for voting rules that satisfy them together
 - **Ultimate hope:** a unique voting rule satisfies the set of axioms simultaneously!
 - **What often happens:** no voting rule satisfies the axioms together 😞

Axiomatic Approach

- Weak axioms, satisfied by all popular voting rules
- **Unanimity**: If all voters have the same top choice, that alternative is the winner.

$$(top(\succ_i) = a \forall i \in N) \Rightarrow f(\vec{\succ}) = a$$

➤ An even weaker version requires all rankings to be identical

- **Pareto optimality**: If all voters prefer a to b , then b is not the winner.

$$(a \succ_i b \forall i \in N) \Rightarrow f(\vec{\succ}) \neq b$$

- **Q**: *What is the relation between these axioms?*
 - *Pareto optimality \Rightarrow Unanimity*

Axiomatic Approach

- **Anonymity:** Permuting votes does not change the winner (i.e., voter identities don't matter).
 - E.g., these two profiles must have the same winner:
{voter 1: $a \succ b \succ c$, voter 2: $b \succ c \succ a$ }
{voter 1: $b \succ c \succ a$, voter 2: $a \succ b \succ c$ }
- **Neutrality:** Permuting alternative names just permutes the winner.
 - E.g., say a wins on {voter 1: $a \succ b \succ c$, voter 2: $b \succ c \succ a$ }
 - We permute all names: $a \rightarrow b$, $b \rightarrow c$, and $c \rightarrow a$
 - New profile: {voter 1: $b \succ c \succ a$, voter 2: $c \succ a \succ b$ }
 - Then, the new winner must be b .

Axiomatic Approach

- Neutrality is tricky
 - For deterministic rules, it is inconsistent with anonymity!
 - Imagine {voter 1: $a \succ b$, voter 2: $b \succ a$ }
 - Without loss of generality, say a wins
 - Imagine a different profile: {voter 1: $b \succ a$, voter 2: $a \succ b$ }
 - Neutrality: We just exchanged $a \leftrightarrow b$, so winner is b .
 - Anonymity: We just exchanged the votes, so winner stays a .
 - Typically, we only require neutrality for...
 - Randomized rules: E.g., a rule could satisfy both by choosing a and b as the winner with probability $\frac{1}{2}$ each, on both profiles
 - Deterministic rules that return a set of tied winners: E.g., a rule could return $\{a, b\}$ as tied winners on both profiles.

Axiomatic Approach

- Stronger but more subjective axioms
- **Majority consistency:** If a majority of voters have the same top choice, that alternative wins.

$$\left(|\{i: \text{top}(\succ_i) = a\}| > \frac{n}{2} \right) \Rightarrow f(\vec{\succ}) = a$$

- **Condorcet consistency:** If a defeats every other alternative in a pairwise election, a wins.

$$\left(|\{i: a \succ_i b\}| > \frac{n}{2}, \forall b \neq a \right) \Rightarrow f(\vec{\succ}) = a$$

Axiomatic Approach

- **Recall:** Condorcet consistency \Rightarrow Majority consistency
- All positional scoring rules violate Condorcet consistency.
- Most positional scoring rules also violate majority consistency.
 - Plurality satisfies majority consistency.

Axiomatic Approach

- **Consistency:** If a is the winner on two profiles, it must be the winner on their union.

$$f(\vec{\succ}_1) = a \wedge f(\vec{\succ}_2) = a \Rightarrow f(\vec{\succ}_1 + \vec{\succ}_2) = a$$

- Example: $\vec{\succ}_1 = \{a \succ b \succ c\}$, $\vec{\succ}_2 = \{a \succ c \succ b, b \succ c \succ a\}$
- Then, $\vec{\succ}_1 + \vec{\succ}_2 = \{a \succ b \succ c, a \succ c \succ b, b \succ c \succ a\}$

- **Theorem [Young '75]:**
 - Subject to mild requirements, a voting rule is consistent if and only if it is a positional scoring rule!

Axiomatic Approach

- **Weak monotonicity:** If a is the winner, and a is “pushed up” in some votes, a remains the winner.
 - $f(\succ) = a \Rightarrow f(\succ') = a$, where
 - $b \succ_i c \Leftrightarrow b \succ'_i c, \forall i \in N, b, c \in A \setminus \{a\}$ (Order of others preserved)
 - $a \succ_i b \Rightarrow a \succ'_i b, \forall i \in N, b \in A \setminus \{a\}$ (a only improves)
- In contrast, strong monotonicity requires $f(\succ') = a$ even if \succ' only satisfies the 2nd condition
 - Too strong; only satisfied by dictatorial or non-onto rules [GS Theorem]

Axiomatic Approach

- **Weak monotonicity:** If a is the winner, and a is “pushed up” in some votes, a remains the winner.
 - $f(\vec{>}) = a \Rightarrow f(\vec{>}') = a$, where
 - $b \succ_i c \Leftrightarrow b \succ'_i c, \forall i \in N, b, c \in A \setminus \{a\}$ (Order of others preserved)
 - $a \succ_i b \Rightarrow a \succ'_i b, \forall i \in N, b \in A \setminus \{a\}$ (a only improves)
- Weak monotonicity is satisfied by most voting rules
 - Popular exceptions: STV, plurality with runoff
 - But this helps STV be hard to manipulate
 - **Theorem [Conitzer-Sandholm '06]:** “Every weakly monotonic voting rule is easy to manipulate on average.”

Axiomatic Approach

- STV violates weak monotonicity

7 voters	5 voters	2 voters	6 voters
a	b	b	c
b	c	c	a
c	a	a	b

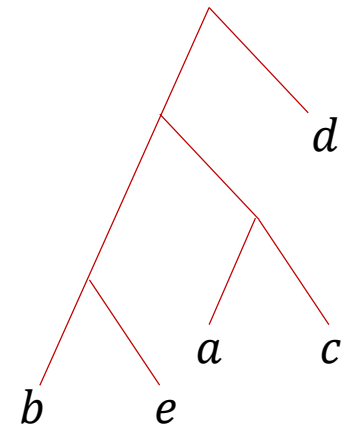
- First c , then b eliminated
- Winner: a

7 voters	5 voters	2 voters	6 voters
a	b	a	c
b	c	b	a
c	a	c	b

- First b , then a eliminated
- Winner: c

Axiomatic Approach

- **Pareto optimality:** If $a \succ_i b$ for all voters i , then $f(\vec{\succ}) \neq b$.
- Relatively weak requirement
 - Some rules that throw out alternatives early may violate this.
 - Example: voting trees
 - Alternatives move up by defeating opponent in pairwise election
 - d may win even if all voters prefer b to d if b loses to e early, and e loses to c



Axiomatic Approach

- Arrow's Impossibility Theorem
 - Applies to social welfare functions (profile \rightarrow ranking)
 - **Independence of Irrelevant Alternatives (IIA)**: If the preferences of all voters between a and b are unchanged, the social preference between a and b should not change
 - **Pareto optimality**: If all prefer a to b , then the social preference should be $a \succ b$
 - **Theorem**: IIA + Pareto optimality \Rightarrow dictatorship.
- Interestingly, automated theorem provers can also prove Arrow's and GS impossibilities!

Axiomatic Approach

- One can think of polynomial time computability as an axiom
 - Two rules that attempt to make the pairwise comparison graph acyclic are NP-hard to compute:
 - Kemeny's rule: invert edges with minimum total weight
 - Slater's rule: invert minimum number of edges
 - Both rules can be implemented by straightforward integer linear programs
 - For small instances (say, up to 20 alternatives), NP-hardness isn't a practical concern.

Statistical Approach

- According to Condorcet [1785]:
 - The purpose of voting is not merely to balance subjective opinions; it is a collective quest for the truth.
 - Enlightened voters try to judge which alternative best serves society.
- Modern motivation due to human computation systems
 - EteRNA: Select 8 RNA designs to synthesize so that the truly most stable design is likely one of them



Statistical Approach

- Traditionally well-explored for choosing a ranking
- For $m = 2$, the majority choice is most likely the true choice under any reasonable model.
- For $m \geq 3$: Condorcet suggested an approach, but the writing was too ambiguous to derive a well-defined voting rule.

Statistical Approach

- Young's interpretation of Condorcet's approach:
 - Assume there is a ground truth ranking σ^*
 - Each voter i makes a noisy observation σ_i
 - The observations are i.i.d. given the ground truth
 - $\Pr[\sigma|\sigma^*] \propto \varphi^{d(\sigma,\sigma^*)}$
 - d = Kendall-tau distance = #pairwise disagreements
 - Interesting tidbit: Normalization constant is independent of σ^*
$$\sum_{\sigma} \varphi^{d(\sigma,\sigma^*)} = 1 \cdot (1 + \varphi) \cdot \dots \cdot (1 + \varphi + \dots + \varphi^{m-1})$$
 - Which ranking is most likely to be the ground truth (maximum likelihood estimate – MLE)?
 - The ranking that Kemeny's rule returns!

Statistical Approach

- The approach yields a **uniquely optimal voting rule**, but relies on a **very specific distribution**
 - Other distributions will lead to different MLE rankings.
 - Reasonable if sufficient data is available to estimate the distribution well
 - Else, we may want robustness to a wide family of possible underlying distributions [Caragiannis et al. '13, '14]
- A connection to the axiomatic approach
 - A voting rule can be MLE for *some* distribution only if it satisfies consistency. **(Why?)**
 - Maximin violates consistency, and therefore can never be MLE!

Implicit Utilitarian Approach

- **Utilities:** Voters have underlying numerical utilities
 - Utility of voter i for alternative $a = u_i(a)$
 - Normalization: $\sum_a u_i(a) = 1$ for all voters i
 - Given utility vector \vec{u} , $sw(a, \vec{u}) = \sum_i u_i(a)$
 - **Goal:** choose $a^* \in \operatorname{argmax}_a sw(a, \vec{u})$
- **Preferences:** Voters only report ranked preferences consistent with their utilities
 - $u_i(a) > u_i(b) \Rightarrow a \succ_i b$
 - Preference profile: $\vec{\succ}$
 - Cannot maximize welfare given only partial information

Implicit Utilitarian Approach

- **Modified goal:** Achieve the best worst-case approximation to social welfare
- **Distortion** of voting rule f

$$\max_{\vec{u}} \frac{\max_a \text{sw}(a, \vec{u})}{\text{sw}(f(\vec{>}), \vec{u})}$$

- Here, $\vec{>}$ are the preferences cast by voters when their utilities are \vec{u}
- If f is randomized, we need $E[\text{sw}(f(\vec{>}), \vec{u})]$

Utilitarian Approach

- **Pros:**

- Uses minimal subjective assumptions
- Yields a uniquely optimal voting rule
 - One can define the distortion of f on a given input \succrightarrow by taking the worst case over all \vec{u} which would generate \succrightarrow
 - Optimal voting rule minimizes the distortion on every \succrightarrow individually

- **Cons:**

- The optimal rule does not have an intuitive formula that humans can comprehend
- In some scenarios, the optimal rule is difficult to compute

Choosing One Alternative

- **Theorem** [Caragiannis et al. '16]:
Given ranked preferences, the optimal **deterministic** voting rule has $\Theta(m^2)$ distortion.
- **Proof:**
 - **Lower bound:** Construct a profile on which every deterministic voting rule has $\Omega(m^2)$ distortion.
 - **Upper bound:** Show *some* deterministic voting rule that has $O(m^2)$ distortion on every profile.

Choosing One Alternative

- **Proof (lower bound):**

- Consider the profile on the right

- If the rule chooses a_m :

- Infinite distortion. **WHY?**

- If the rule chooses a_i for $i < m$:

- Construct a bad utility profile \vec{u} as follows

- Voters in column i have utility $1/m$ for every alternative

- All other voters have utility $1/2$ for their top two alternatives

- $sw(a_i, \vec{u}) = \frac{n}{m-1} \cdot \frac{1}{m}$, $sw(a_m, \vec{u}) \geq \frac{n - n/(m-1)}{2}$

- Distortion = $\Omega(m^2)$

$n/(m-1)$ voters per column			
a_1	a_2	...	a_{m-1}
a_m	a_m	...	a_m
\vdots	\vdots	\vdots	\vdots

Choosing One Alternative

- **Proof (upper bound):**

- Simply using plurality achieves $O(m^2)$ distortion.

- WHY?

- Suppose plurality winner is a .

- At least n/m voters prefer a the most, and thus have utility at least $1/m$ for a .

- $sw(a, \vec{u}) \geq n/m^2$

- $sw(a^*, \vec{u}) \leq n$ for every alternative a^*

- $O(m^2)$ distortion

Implicit Utilitarian Voting

- Plurality is as good as any other deterministic voting rule!
- Alternatively:
 - If we must choose an alternative deterministically, ranked preferences provide no more useful information than top-place votes do, in the worst case.
- There's more hope if we're allowed to randomize.

Choosing One Alternative

- **Theorem** [Boutilier et al. '12]:
Given ranked preferences, the optimal **randomized** voting rule has distortion $O(\sqrt{m} \cdot \log^* m)$, $\Omega(\sqrt{m})$.
- **Proof:**
 - **Lower bound:** Construct a profile on which every randomized voting rule $\Omega(\sqrt{m})$ distortion.
 - **Upper bound:** Show *some* randomized voting rule that has $O(\sqrt{m} \cdot \log^* m)$ distortion
 - We'll do the much simpler $O(\sqrt{m \log m})$ distortion

Choosing One Alternative

- **Proof (lower bound):**

- Consider a similar profile:

- \sqrt{m} special alternatives
- Voting rule must choose one of them (say a^*) w.p. at most $1/\sqrt{m}$

n/\sqrt{m} voters per column			
a_1	a_2	...	$a_{\sqrt{m}}$
⋮	⋮	⋮	⋮

- Bad utility profile \vec{u} :

- All voters ranking a^* first give utility 1 to a^*
- All other voters give utility $1/m$ to each alternative
- $\frac{n}{\sqrt{m}} \leq sw(a^*, \vec{u}) \leq \frac{2n}{\sqrt{m}}$
- $sw(a, \vec{u}) \leq n/m$ for every other a .
- **Distortion lower bound:** $\sqrt{m}/3$ (proof on the board!)

Choosing One Alternative

- **Proof (upper bound):**

- Given profile \vec{y} , define the harmonic score $sc(a, \vec{y})$:
 - Each voter gives $1/k$ points to her k^{th} most preferred alternative
 - Take the sum of points across voters
 - $sw(a, \vec{u}) \leq sc(a, \vec{y})$ (WHY?)
 - $\sum_a sc(a, \vec{y}) = n \cdot \sum_{k=1}^m 1/k = n H_m \leq n \cdot (\ln m + 1)$
- **Golden rule:**
 - W.p. $1/2$: Choose every a w.p. proportional to $sc(a, \vec{y})$
 - W.p. $1/2$: Choose every a w.p. $1/m$ (uniformly at random)
- Distortion $\leq 2\sqrt{m \cdot (\ln m + 1)}$ (proof on the board!)

Optimal vs Near-Optimal Rules

- The distortion is often bad for large m
 - E.g., $\Theta(m^2)$ for deterministic rules.
 - But one can argue that the optimal alternative which minimizes distortion represents *some* meaningful aggregation of information.
- How difficult is it to find the *optimal* alternative?
 - Polynomial time computable for both deterministic (via a direct formula) and randomized (via a non-trivial LP) cases

Input Format

- What if we ask about underlying numerical utilities in a format other than ranking?
- Threshold approval votes
 - Voting rule selects a threshold τ , asks each voter i , for each alternative a , whether $u_i(a) \geq \tau$
 - $O(\log m)$ distortion!
- Food for thought
 - What is the tradeoff between the number of bits of information elicited and the distortion achieved?
 - What is the best input format for a given number of bits?

Implicit Utilitarian Approach

- Extensions
 - Selecting a subset of alternatives or a ranking
 - Lack of an obvious objective function
 - Has been studied for some natural objective functions [Caragiannis et al. '16, ongoing work]
 - Participatory budgeting [Benade et al. '17]
 - Graph problems
 - **Project idea:** Replace numbers with rankings in any problem!

- Deployed

