CSC2556

Lecture 10

Game Theory I: Nash equilibria

Request

- Please fill out course evaluation.
  - You should have received the link in your email
- HW1 marks will be out this evening
- Feedback on proposals will be provided by mid next week
- Project presentations are cancelled
  - Grading will be based on proposals and reports
- Project meeting sign-up sheet is out

Game Theory

Game Theory

- How do rational, self-interested agents act in a given environment?
- Each agent has a set of possible actions
- Rules of the game:
  - Rewards for the agents as a function of the actions taken by all agents

- Noncooperative games
  - No external trusted agency, no legal agreements

Normal Form Games

- A set of players $\mathrm{N} = \{1, \dots, n\}$
- Each player i has an action set S_i, chooses $s_i \in S_i$
- $\mathcal{S} = S_1 \times \cdots \times S_n$.
- Action profile $\vec{s} = (s_1, \dots, s_n) \in \mathcal{S}$
- Each player i has a utility function $u_i: \mathcal{S} \to \mathbb{R}$
  - Given the action profile $\vec{s} = (s_1, \dots, s_n)$, each player i gets a reward $u_i(s_1, \dots, s_n)$

Normal Form Games

Prisoner's dilemma $\quad S = \{\text{Silent}, \text{Betray}\}$

John's Actions / Sam's Actions	Stay Silent	Betray
Stay Silent	(-1 , -1)	(-3 , 0)
Betray	(0 , -3)	(-2 , -2)

$u_{Sam}(Betray, Silent)$ $\quad$ $u_{John}(Betray, Silent)$

s_{Sam} $\quad$ s_{John}

Player Strategies

- Pure strategy
  - Deterministic choice of an action, e.g., “Betray”
- Mixed strategy
  - Randomized choice of an action, e.g., “Betray with probability 0.3, and stay silent with probability 0.7”

Dominant Strategies

- For player i, s_i dominates s_i' if s_i is "better than" s_i', *irrespective of other players' strategies*.
- Two variants: weak and strict domination
  - $u_i(s_i, \vec{s}_{-i}) \geq u_i(s_i', \vec{s}_{-i}), \forall \vec{s}_{-i}$
  - Strict inequality for some $\vec{s}_{-i}$ ← Weak domination
  - Strict inequality for all $\vec{s}_{-i}$ ← Strict domination
- s_i is a strictly (or weakly) dominant strategy for player i if it strictly (or weakly) dominates every other strategy

Dominant Strategies

- Q: How does this relate to strategyproofness?
- A: Strategyproofness means "truth-telling should be a weakly dominant strategy for every player".

Example: Prisoner's Dilemma

- Recap:

John's Actions / Sam's Actions	Stay Silent	Betray
Stay Silent	**(-1 , -1)**	**(-3 , 0)**
Betray	**(0 , -3)**	**(-2 , -2)**

- Each player strictly wants to
  - Betray if the other player will stay silent
  - Betray if the other player will betray

- Betray = strictly dominant strategy for each player

Iterated Elimination

- What if there are no dominant strategies?
  - No single strategy dominates every other strategy
  - But some strategies might still be dominated

- Assuming everyone knows everyone is rational…
  - Can remove their dominated strategies
  - Might reveal a newly dominant strategy

- Eliminating only strictly dominated vs eliminating weakly dominated

Iterated Elimination

- Toy example:
  - Microsoft vs Startup
  - Enter the market or stay out?

Microsoft \ Startup	Enter	Stay Out
Enter	**(2 , -2)**	**(4 , 0)**
Stay Out	**(0 , 4)**	**(0 , 0)**

- Q: Is there a dominant strategy for startup?
- Q: Do you see a rational outcome of the game?

Iterated Elimination

- "Guess 2/3 of average"
  - Each student guesses a real number between 0 and 100 (inclusive)
  - The student whose number is the closest to 2/3 of the average of all numbers wins!

- Piazza Poll: What would you do?

Nash Equilibrium

- If we find dominant strategies, or a unique outcome after iteratively eliminating dominated strategies, it *may* be considered the rational outcome of the game.
- What if this is not the case?

Students \ Professor	Attend	Be Absent
Attend	(3 , 1)	(-1 , -3)
Be Absent	(-1 , -1)	(0 , 0)

Nash Equilibrium

- Instead of hoping to find strategies that players would play *irrespective of what other players play,* we want to find strategies that players would play *given what other players play.*

- Nash Equilibrium
  - A strategy profile $\vec{s}$ is in Nash equilibrium if s_i is the best action for player i given that other players are playing $\vec{s}_{-i}$

$$u_i(s_i, \vec{s}_{-i}) \geq u_i(s_i', \vec{s}_{-i}), \forall s_i'$$

Recap: Prisoner's Dilemma

Sam's Actions \ John's Actions	Stay Silent	Betray
Stay Silent	(-1 , -1)	(-3 , 0)
Betray	(0 , -3)	(-2 , -2)

- Nash equilibrium?

- (Dominant strategies)

Recap: Microsoft vs Startup

Microsoft \ Startup	Enter	Stay Out
Enter	(2 , -2)	(4 , 0)
Stay Out	(0 , 4)	(0 , 0)

- Nash equilibrium?
- (Iterated elimination of strongly dominated strategies)

Recap: Attend or Not

Students \ Professor	Attend	Be Absent
Attend	(3 , 1)	(-1 , -3)
Be Absent	(-1 , -1)	(0 , 0)

- Nash equilibria?

- Lack of predictability

Example: Rock-Paper-Scissor

P1 / P2	Rock	Paper	Scissor
Rock	(0 , 0)	(-1 , 1)	(1 , -1)
Paper	(1 , -1)	(0 , 0)	(-1 , 1)
Scissor	(-1 , 1)	(1 , -1)	(0 , 0)

- Pure Nash equilibrium?

Nash's Beautiful Result

- Theorem: Every normal form game admits a mixed-strategy Nash equilibrium.

- What about Rock-Paper-Scissor?

P1 / P2	Rock	Paper	Scissor
Rock	(0 , 0)	(-1 , 1)	(1 , -1)
Paper	(1 , -1)	(0 , 0)	(-1 , 1)
Scissor	(-1 , 1)	(1 , -1)	(0 , 0)

Indifference Principle

- If the mixed strategy of player i in a Nash equilibrium has support T_i, the expected payoff of player i from each $s_i \in T_i$ must be identical.

- Derivation of rock-paper-scissor on the board.

Stag-Hunt

Hunter 1 / Hunter 2	Stag	Hare
Stag	(4 , 4)	(0 , 2)
Hare	(2 , 0)	(1 , 1)

- Game
  - Stag requires both hunters, food is good for 4 days for each hunter.
  - Hare requires a single hunter, food is good for 2 days
  - If they both catch the same hare, they share.

- Two pure Nash equilibria: (Stag,Stag), (Hare,Hare)

Stag-Hunt

Hunter 1 / Hunter 2	Stag	Hare
Stag	(4 , 4)	(0 , 2)
Hare	(2 , 0)	(1 , 1)

- Two pure Nash equilibria: (Stag,Stag), (Hare,Hare)
  - Other hunter plays "Stag" → "Stag" is best response
  - Other hunter plays "Hare" → "Hare" is best reponse

- What about mixed Nash equilibria?

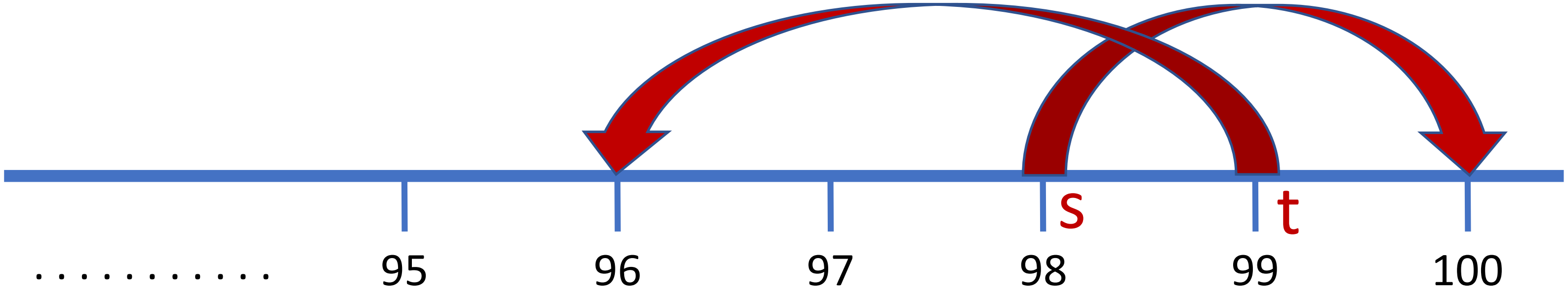
Stag-Hunt

Hunter 1 / Hunter 2	Stag	Hare
Stag	(4 , 4)	(0 , 2)
Hare	(2 , 0)	(1 , 1)

- Symmetric: $s \rightarrow \{\text{Stag w.p. } p, \text{ Hare w.p. } 1 - p\}$
- Indifference principle:
  - *Given the other hunter plays s, equal $\mathbb{E}$[reward] for Stag and Hare*
  - $\mathbb{E}[\text{Stag}] = p * 4 + (1 - p) * 0$
  - $\mathbb{E}[\text{Hare}] = p * 2 + (1 - p) * 1$
  - Equate the two $\Rightarrow p = 1/3$

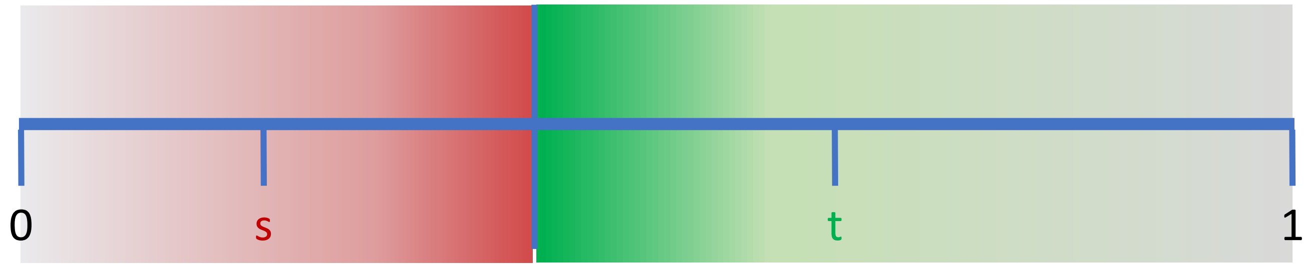
Extra Fun 1: Cunning Airlines

- Two travelers lose their luggage.
- Airline agrees to refund up to $100 to each.
- Policy: Both travelers would submit a number between 2 and 99 (inclusive).
  - If both report the same number, each gets this value.
  - If one reports a lower number (s) than the other (t), the former gets s+2, the latter gets s-2.



Extra Fun 2: Ice Cream Shop

- Two brothers, each wants to set up an ice cream shop on the beach ([0,1]).
- If the shops are at s, t (with $s \leq t$)
  - The brother at s gets $\left[0, \frac{s+t}{2}\right]$, the other gets $\left[\frac{s+t}{2}, 1\right]$



Nash Equilibria: Critique

- Noncooperative game theory provides a framework for analyzing rational behavior.

- But it relies on many assumptions that are often violated in the real world.

- Due to this, human actors are observed to play Nash equilibria in some settings, but play something far different in other settings.

Nash Equilibria: Critique

- Assumptions:
  - Rationality is common knowledge.
    - All players are rational.
    - All players know that all players are rational.
    - All players know that all players know that all players are rational.
    - … [Aumann, 1976]
    - Behavioral economics
  - Rationality is perfect = “infinite wisdom”
    - Computationally bounded agents
  - Full information about what other players are doing.
    - Bayes-Nash equilibria

Nash Equilibria: Critique

- Assumptions:
  - No binding contracts.
    - Cooperative game theory
  - No player can commit first.
    - Stackelberg games (will study this in a few lectures)
  - No external help.
    - Correlated equilibria
  - Humans reason about randomization using expectations.
    - Prospect theory

Nash Equilibria: Critique

- Also, there are often multiple equilibria, and no clear way of "choosing" one over another.

- For many classes of games, finding a single equilibrium is provably hard.
  - Cannot expect humans to find it if your computer cannot.

Nash Equilibria: Critique

- Conclusion:
  - For human agents, take it with a grain of salt.
  - For AI agents playing against AI agents, perfect!