CSC2556

Lecture 4

Impartial Selection; PageRank; Facility Location

Announcements

- Proposal tentatively due around the end of Feb
  - But it will help to decide the topic earlier, and start working.

- I'll put up a list of possible project ideas (in case you cannot find something related to your research)
  - Will also be available to have more meetings during the next two months to help select projects

Impartial Selection

Impartial Selection

- "How can we select k people out of n people?"
  - Applications: electing a student representation committee, selecting k out of n grant applications to fund using peer review, …

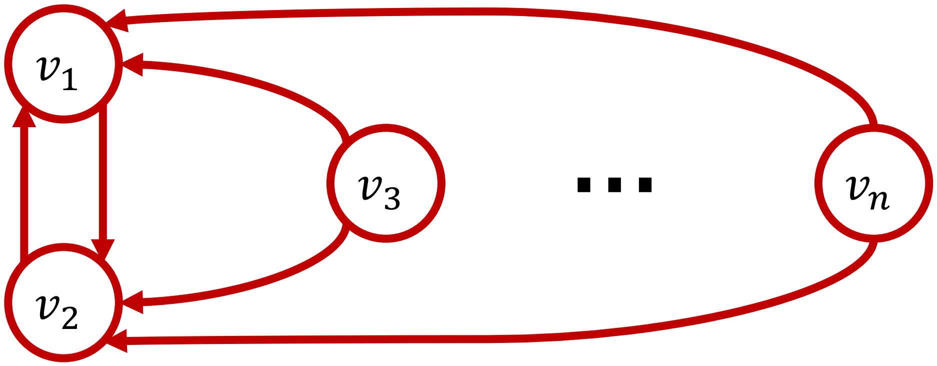
- Model
  - Input: a *directed* graph $G = (V, E)$
  - Nodes $V = \{v_1, \dots, v_n\}$ are the n people
  - Edge $e = (v_i, v_j) \in E$: v_i supports/approves of v_j
    - We do not allow or ignore self-edges (v_i, v_i)
  - Output: a subset $V' \subseteq V$ with $|V'| = k$
  - $k \in \{1, \dots, n-1\}$ is given

Impartial Selection

- Impartiality: A k-selection rule f is *impartial* if $v_i \in f(G)$ does not depend on the outgoing edges of v_i
  - v_i cannot manipulate his outgoing edges to get selected
  - Q: But the definition says v_i can neither go from $v_i \notin f(G)$ to $v_i \in f(G)$, nor from $v_i \in f(G)$ to $v_i \notin f(G)$. Why?

- Societal goal: maximize the sum of in-degrees of selected agents $\sum_{v \in f(G)} |in(v)|$
  - $in(v)$ = set of nodes that have an edge to v
  - $out(v)$ = set of nodes that v has an edge to
  - Note: OPT will pick the k nodes with the highest indegrees

Optimal $\neq$ Impartial



- An optimal 1-selecton rule must select v_1 or v_2
- The other node can remove his edge to the winner, and make sure the optimal rule selects him instead
- This violates impartiality

Goal: Approximately Optimal

- α-approximation: We want a k-selection system that always returns a set with total indegree at least α times the total indegree of the optimal set

- Q: For $k = 1$, what about the following rule?

  Rule: "Select the lowest index vertex in $out(v_1)$.
  If $out(v_1) = \emptyset$, select v_2."

  - A. Impartial + constant approximation
  - B. Impartial + bad approximation
  - C. Not impartial + constant approximation
  - D. Not impartial + bad approximation

No Finite Approximation ☹

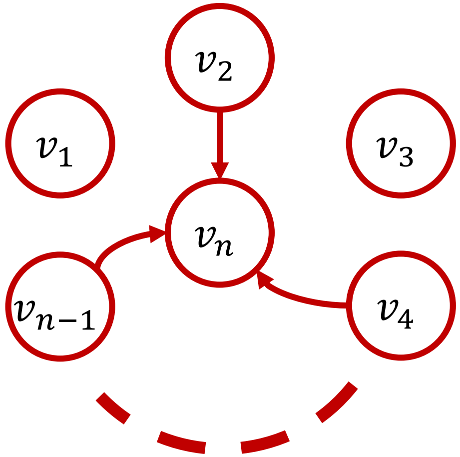
- Theorem [Alon et al. 2011]
  For every $k \in \{1, \dots, n-1\}$, there is no impartial k-selection rule with a finite approximation ratio.
- Proof:
  - For small k, this is trivial. E.g., consider $k = 1$.
    - What if G has two nodes v_1 and v_2 that point to each other, and there are no other edges?
    - For finite approximation, the rule must choose either v_1 or v_2
    - Say it chooses v_1. If v_2 now removes his edge to v_1, the rule must choose v_2 for any finite approximation.
    - Same argument as before. But applies to any "finite approximation rule", and not just the optimal rule.

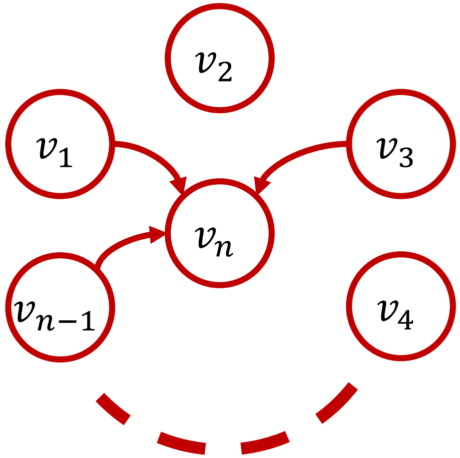
No Finite Approximation ☹

- Theorem [Alon et al. 2011]
  For every $k \in \{1, \dots, n-1\}$, there is no impartial k-selection rule with a finite approximation ratio.
- Proof:
  - Proof is more intricate for larger k. Let's do $k = n - 1$.
    - $k = n - 1$: given a graph, "eliminate" a node.
  - Suppose for contradiction that there is such a rule f.
  - W.l.o.g., say v_n is eliminated in the empty graph.
  - Consider a family of graphs in which a subset of $\{v_1, \dots, v_{n-1}\}$ have edges to v_n.

No Finite Approximation ☹

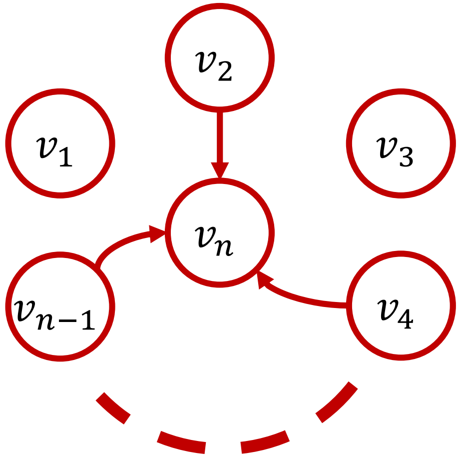
- **Proof ($k = n - 1$ continued):**
  - Consider *star graphs* in which a non-empty subset of $\{v_1, \ldots, v_{n-1}\}$ have edge to v_n, and there are no other edges
    - Represented by bit strings $\{0,1\}^{n-1} \backslash \{\vec{0}\}$
  - v_n cannot be eliminated in any star graph
    - Otherwise we have infinite approximation
  - f maps $\{0,1\}^{n-1} \backslash \{\vec{0}\}$ to $\{1, \ldots, n-1\}$
    - "Who will be eliminated?"
  - Impartiality: $f(\vec{x}) = i \Leftrightarrow f(\vec{x} + \vec{e}_i) = i$
    - $\vec{e}_i$ has 1 at i^{th} coordinate, 0 elsewhere
    - In words, i cannot prevent elimination by adding or removing his edge to v_n

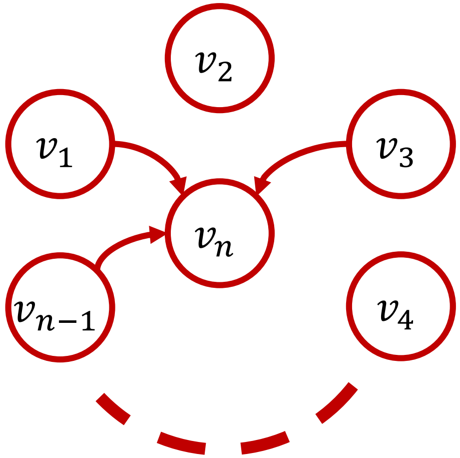




No Finite Approximation ☹

- Proof ($k = n - 1$ continued):
  - $f: \{0,1\}^{n-1} \setminus \{\vec{0}\} \rightarrow \{1, \ldots, n-1\}$
  - $f(\vec{x}) = i \Leftrightarrow f(\vec{x} + \vec{e}_i) = i$
    - $\vec{e}_i$ has 1 only in i^{th} coordinate
  - Pairing implies…
    - The number of strings on which f outputs i is even, for every i.
    - Thus, total number of strings in the domain must be even too.
    - But total number of strings is $2^{n-1} - 1$ (odd)
  - So impartiality must be violated for some pair of $\vec{x}$ and $\vec{x} + \vec{e}_i$





Back to Impartial Selection

- Question: So what *can* we do to select impartially?
- Answer: Randomization!
  - Impartiality now requires that the probability of an agent being selected be independent of his outgoing edges.
- Examples: Randomized Impartial Mechanisms
  - Choose k nodes uniformly at random
    - Sadly, this still has arbitrarily bad approximation.
    - Imagine having k special nodes with indegree $n - 1$, and all other nodes having indegree 0.
    - Mechanism achieves $(k/n) * OPT \Rightarrow$ approximation = n/k
    - Good when k is comparable to n, but bad when k is small.

Random Partition

- Idea:
  - What if we partition V into V_1 and V_2, and select k nodes from V_1 based only on edges coming to them from V_2?

- Mechanism:
  - Assign each node to V_1 or V_2 i.i.d. with probability ½
  - Choose $V_i \in \{V_1, V_2\}$ at random
  - Choose k nodes from V_i that have most incoming edges from nodes in V_{3-i}

Random Partition

- Analysis:
  - Goal: approximate I = # edges incoming to OPT.
    - I_1 = # edges $V_2 \to OPT \cap V_1$, I_2 = # edges $V_1 \to OPT \cap V_2$
  - Note: $E[I_1 + I_2] = I/2$. (WHY?)
  - W.p. ½, we pick k nodes in V_1 with the most incoming edges from V_2 ⇒ # incoming edges $\geq I_1$ (WHY?)
    - $|OPT \cap V_1| \leq k$; $OPT \cap V_1$ has I_1 incoming edges from V_2
  - W.p. ½, we pick k nodes in V_2 with the most incoming edges from V_1 ⇒ # incoming edges $\geq I_2$
  - E[#incoming edges] $\geq E\left[\left(\frac{1}{2}\right) \cdot I_1 + \left(\frac{1}{2}\right) \cdot I_2\right] = \frac{I}{4}$

Random Partition

- Generalization
  - Divide into ℓ parts, and pick k/ℓ nodes from each part based on incoming edges from all other parts.

- Theorem [Alon et al. 2011]:
  - $\ell = 2$ gives a 4-approximation.
  - For $k \geq 2$, $\ell \sim k^{1/3}$ gives $1 + O\left(\frac{1}{k^{1/3}}\right)$ approximation.

Better Approximations

- Alon et al. [2011] conjectured that for randomized impartial 1-selection…
  - (For which their mechanism is a 4-approximation)
  - It should be possible to achieve a 2-approximation.
  - Recently proved by Fischer & Klimm [2014]
  - Permutation mechanism:
    - Select a random permutation $(\pi_1, \pi_2, \ldots, \pi_n)$ of the vertices.
    - Start by selecting $y = \pi_1$ as the "current answer".
    - At any iteration t, let $y \in \{\pi_1, \ldots, \pi_t\}$ be the current answer.
    - From $\{\pi_1, \ldots, \pi_t\} \backslash \{y\}$, if there are more edges to π_{t+1} than to y, change the current answer to $y = \pi_{t+1}$.

Better Approximations

- 2-approximation is tight.
  - In an n-node graph, fix u and v, and suppose no other nodes have any incoming/outgoing edges.
  - Three cases: only $u \to v$ edge, only $v \to u$, or both.
    - The best impartial mechanism selects u and v with probability ½ in every case, and achieves 2-approximation.
- But this is because $n - 2$ nodes are not voting!
  - What if every node must have an outgoing edge?
  - Fischer & Klimm [2014]:
    - Permutation mechanism gives between ${}^{12}/_{7}$ and ${}^{3}/_{2}$ approximation.
    - No mechanism gives better than $4/3$ approximation.

PageRank Axiomatization

PageRank

- An extension of the impartial selection problem
  - Instead of selecting k nodes, we want to *rank* all nodes
- The PageRank Problem: Given a directed graph, rank all nodes by their "importance".
  - Think of the web graph, where nodes are webpages, and a directed (u, v) edge means u has a link to v.
- Questions:
  - What properties do we want from such a rule?
  - What rule satisfies these properties?

PageRank

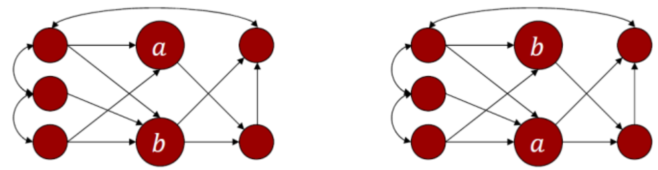
- Here is the PageRank Algorithm:
  - Start from any node i in the graph.
  - At each iteration, choose an outgoing edge $i \rightarrow j$, uniformly at random among all outgoing edges of i.
  - Move to the neighbor node j.
  - In the long run, measure the fraction of time the random walk visits each node.
  - Rank the nodes by these "stationary probabilities".
- Google uses (a version of) this algorithm

PageRank

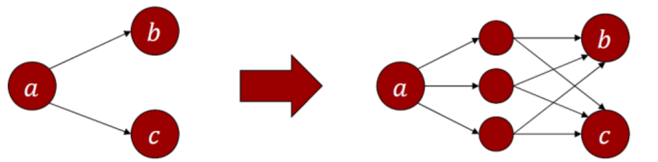
- In a formal sense…
  - Let p_i = stationary probability of visiting i.
  - Let $N(i)$ = set of nodes to which i has an edge
  - Then, $p_i = \sum_{j:i \in N(j)} \frac{p_j}{|N(j)|}$
    - n equations, n variables
- Another way to do this:
  - Matrix A: $A_{i,j} = 1/|N(i)|$ if $(i, j) \in E$ and 0 otherwise
  - We are searching for a solution v such that $Av = v$.
  - Start from any v_0, and compute $\lim_{k \to \infty} A^k v_0$

Axioms

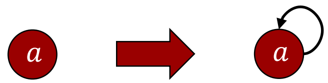
- Axiom 1 (Isomorphism)
  - Permuting node names permutes the final ranking.



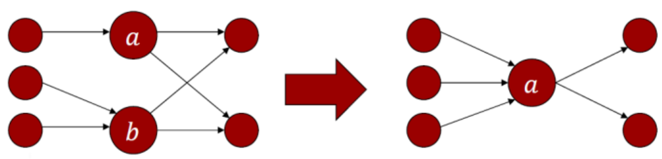
- Axiom 2 (Vote by Committee)
  - Voting through intermediate fake nodes cannot change the ranking.



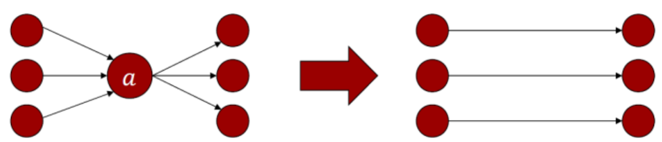
- Axiom 3 (Self Edge)
  - v adding a self edge cannot change the ordering of the *other* nodes.



- Axiom 4 (Collapsing)
  - Merging identically voting nodes cannot change the ordering of the *other* nodes.



- Axiom 5 (Proxy)
  - If k nodes with equal score vote for k other nodes through a proxy, it should be no different than a direct 1-1 voting.



PageRank

- Theorem [Altman and Tennenholtz, 2005]:
  An algorithm satisfies these five axioms if and only if it is PageRank.

Facility Location

Apprx Mechanism Design

1. Define the problem: agents, outcomes, values
2. Fix an objective function (e.g., maximizing sum of values)
3. Check if the objective function is maximized through a strategyproof mechanism
4. If not, find the strategyproof mechanism that provides the best worst-case approximation ratio of the objective function

Facility Location

- Set of agents N
- Each agent i has a true location $x_i \in \mathbb{R}$
- Mechanism f
  - Takes as input reports $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$
  - Returns a location $y \in \mathbb{R}$ for the new facility
- Cost to agent i : $c_i(y) = |y - x_i|$
- Social cost $C(y) = \sum_i c_i(y) = \sum_i |y - x_i|$

Facility Location

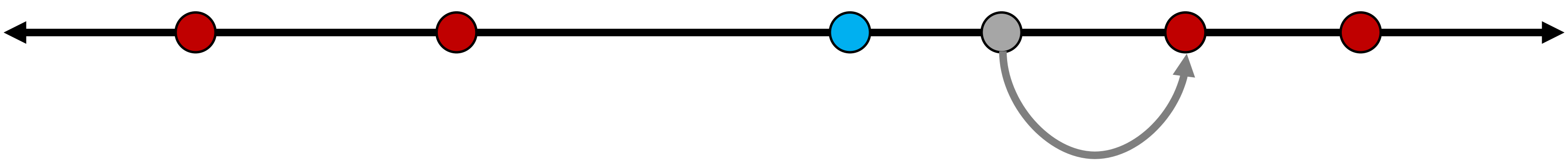
- Social cost $C(y) = \sum_i c_i(y) = \sum_i |y - x_i|$
- Q: Ignoring incentives, what choice of y would minimize the social cost?
- A: The median location $\text{med}(x_1, \dots, x_n)$
  - n is odd → the unique "(n+1)/2"th smallest value
  - n is even → "n/2"th or "(n/2)+1"st smallest value
  - Why?

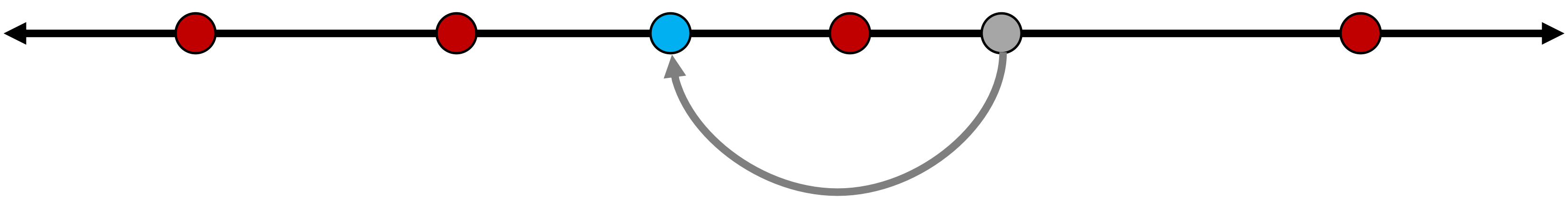
Facility Location

- Social cost $C(y) = \sum_i c_i(y) = \sum_i |y - x_i|$
- Median is optimal (i.e., 1-approximation)
- What about incentives?
  - Median is also strategyproof (SP)!
  - Irrespective of the reports of other agents, agent i is best off reporting x_i

Median is SP

No manipulation can help





Max Cost

- A different objective function $C(y) = \max_i |y - x_i|$
- Q: Again ignoring incentives, what value of y minimizes the maximum cost?
- A: The midpoint of the leftmost $(\min_i x_i)$ and the rightmost $(\max_i x_i)$ locations
- Q: Is this optimal rule strategyproof?
- A: No!

Max Cost

- $C(y) = \max_i |y - x_i|$
- We want to use a strategyproof mechanism.
- Question: What is the approximation ratio of median for maximum cost?
  1. $\in [1,2)$
  2. $\in [2,3)$
  3. $\in [3,4)$
  4. $\in [4, \infty)$

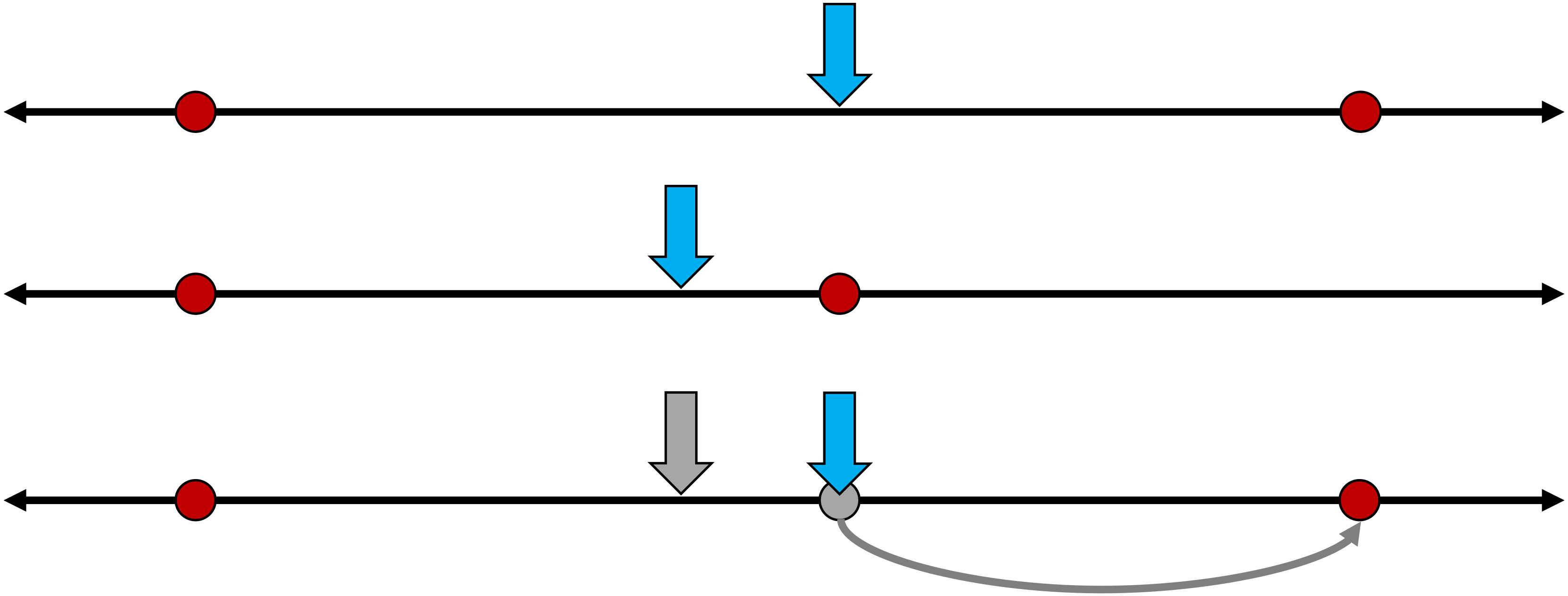
Max Cost

- Answer: 2-approximation

- Other SP mechanisms that are 2-approximation
  - Leftmost: Choose the leftmost reported location
  - Rightmost: Choose the rightmost reported location
  - Dictatorship: Choose the location reported by agent 1
  - …

Max Cost

- Theorem [Procaccia & Tennenholtz, '09]
  No deterministic SP mechanism has approximation ratio < 2 for maximum cost.
- Proof:



Max Cost + Randomized

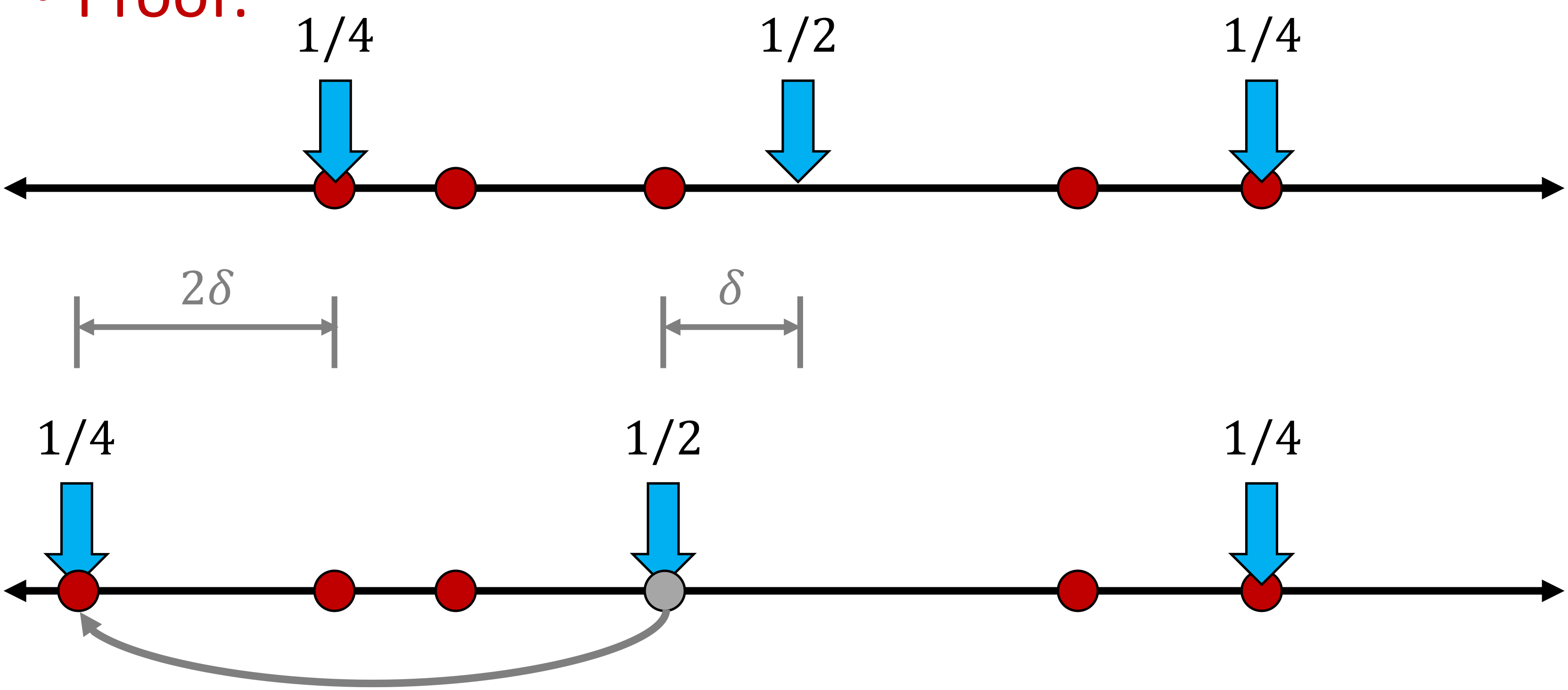
- The Left-Right-Middle (LRM) Mechanism
  - Choose $\min_i x_i$ with probability ¼
  - Choose $\max_i x_i$ with probability ¼
  - Choose $(\min_i x_i + \max_i x_i)/2$ with probability ½

- Question: What is the approximation ratio of LRM for maximum cost?

- At most $\frac{(1/4)*2C+(1/4)*2C+(1/2)*C}{C} = \frac{3}{2}$

Max Cost + Randomized

- Theorem [Procaccia & Tennenholtz, '09]:
  The LRM mechanism is strategyproof.
- Proof:



Max Cost + Randomized

- Exercise for you!
  Try showing that no randomized SP mechanism can achieve approximation ratio $< 3/2$.