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Author(s): Allan Gibbard
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MANIPULATION OF SCHEMES THAT MIX VOTING WITH CHANCE[1]

BY ALLAN GIBBARD

A *decision scheme* makes the probabilities of alternatives depend on individual strong orderings of them. It is *strategy-proof* if it logically precludes anyone's advantageously misrepresenting his preferences. It is *unilateral* if only one individual can affect the outcome, and *duple* if it restricts the final outcome to a fixed pair of alternatives. Any strategy-proof decision scheme, it is shown, is a probability mixture of schemes each of which is unilateral or duple. If it guarantees Pareto optimal outcomes, it is a probability mixture of dictatorial schemes. If it guarantees *ex ante* Pareto optimal lotteries, it is dictatorial.

1. INTRODUCTION

AN INDIVIDUAL *manipulates* a system of voting if, by misrepresenting his preferences, he secures a result he prefers to the result that would obtain if he expressed his true preferences. For systems of pure voting, where chance plays no role in settling which alternative is adopted, the following result is known: such a scheme, if it is to preclude individual strategic manipulation, must either make someone dictator, or restrict the possible outcome to a fixed pair of alternatives. (See Gibbard [**6**] and Satterthwaite [**13**]). This paper deals with systems of voting of a more general kind: systems by which a social decision is made through a combination of voting and chance. It will be shown that any such scheme, if it is to preclude individual strategic manipulation, must be a probability mixture of schemes, each of which either (i) accords a monopoly of influence to a single voter, or (ii) restricts the final outcome to a fixed pair of alternatives. Schemes of the first kind I shall call *unilateral*; of the second kind, *duple*.

What is meant here by a combination of voting with chance? Suppose a decision is made in the following way: first, voting of some kind is used to pick out a set of one or more winning alternatives; then, in case there is more than one such winner, one of them is chosen by lot. Such a scheme, in effect, uses the way people vote to determine the probability each alternative has of being adopted. This I shall take as the defining feature of a scheme which combines voting with chance: on the basis of the way people vote, it assigns to each alternative a probability of being adopted.

This paper deals only with voting by *rank order ballot*: in the schemes to be considered here, voting consists in each voter's ranking the alternatives in a professed order of preference. An individual is not allowed to express indifference between alternatives. The theorem in this paper applies to all systems of the kind I have characterized: to all systems by which voters' rank order ballots—no

[1] I have been helped in revising this paper by conversations with Mark Satterthwaite, Thomas Schwartz, and Hugo Sonnenschein, and by letters from Peter Fishburn and Richard Zeckhauser. I am grateful to the referee for remarkably detailed suggestions for shortening the proof of the main theorem.

indifference allowed—determine the probability of each alternative's being adopted.

Systems of this kind will be called *decision schemes*, and they are defined, more precisely, as follows. Let there be a finite set of mutually exclusive alternatives, from which the community must select exactly one. Each voter ranks the alternatives on his ballot in professed order of preference. On the basis of these orderings, a probability of being adopted is assigned to each alternative, and the final choice is made by a suitable chance device. A *decision scheme*, then, is a function of the following kind. Let there be n voters, and let V be the set of mutually incompatible alternatives open to the community. Call an ordering of V with no tries a *ranking* and call an n-tuple of rankings a *ranking n-tuple*. Finally, let a *lottery* be an assignment of a probability to each alternative, with the probabilities adding up to one. A *decision scheme* is a function d whose domain is the set of all preference n-tuples, and whose values are lotteries.

How can manipulability be defined for decision schemes? A decision scheme is manipulable if there is a logically possible situation in which someone manipulates it, and an individual manipulates a decision scheme if, by misrepresenting his preferences, he secures a lottery he prefers to the lottery that would have obtained if he had expressed his true preferences. Whether he manipulates the scheme, then, depends on his preferences among lotteries. Now if an ordering of lotteries satisfies rationality conditions such as those of von Neumann and Morgenstern [**8**, p. 26], then it can most conveniently be given by an assignment of cardinal utilities to the alternatives. Whether individual k manipulates the scheme to his advantage, then, depends not only on the way everyone else votes, the way k votes, and the way k genuinely orders the alternatives; it depends further on the way k genuinely orders lotteries—on k's cardinal utilities.

Manipulability, then, can be characterized as follows. In the first place, k *manipulates* decision scheme d if (i) where the actual votes are given by ranking n-tuple $\langle P_1, \ldots, P_n \rangle$ and k's true utility scale is U, k's avowed ranking P_k is not the ranking of the alternatives given by scale U, and (ii) if k had voted the ranking given by scale U, he would have secured a lottery of lower expected utility, as reckoned by U, than the lottery he actually secures. A decision scheme d is *manipulable*, then, if for some ranking n-tuple $\langle P_1, \ldots, P_n \rangle$, for some person k, and for some utility scale U, k manipulates d. If it is not manipulable, it will be called *strategy-proof*. These definitions are given explicitly in Section 4.

Unattractive examples of strategy-proof decision schemes are not hard to find. Here are three:

SCHEME 1: Put everyone's ballot in a hat, draw one at random, and choose the alternative which is ranked first on that ballot. (For a discussion of this scheme, see Gibbard, [**6**, pp. 592–593], and Zeckhauser, [**18**, pp. 938–940].)

SCHEME 2: First collect the ballots. Next, put the names of the alternatives in a hat and select two at random. Then use the collected ballots to decide between those two alternatives by majority vote. This amounts to a decision scheme, since under it, the ballots cast determine the probability of each alternative's being adopted. Now if a voter misrepresents his preferences under this scheme, it can affect the outcome only to his disadvantage. His misrepresentation can affect the outcome only if the following holds: for some pair of alternatives x and y, he prefers x to y but ranks y above x on his ballot, the names of x and y are drawn from the hat, and he swings the outcome from x to y by his vote—thus getting an outcome he likes less than the honest outcome.[2]

SCHEME 3: A coin is flipped, and Scheme 1 is used if the coin lands heads; Scheme 2 if the coin lands tails.

It might have been hoped that there were strategy-proof decision schemes more attractive than these: schemes, for instance, which select one or more optimal alternatives in a reasonable way on the basis of the way people vote, and then, in case there is more than one optimal alternative, choose the alternative actually to be adopted from among them by chance. The theorem in this paper shows, however, that all strategy-proof decision schemes are much like the unattractive schemes I have given as examples: all involve, in effect, selecting a ballot or a pair of alternatives by chance, and either ignoring all ballots but the one selected, or choosing somehow between the two selected alternatives. All, in other words, are probability mixtures of schemes, each of which is either unilateral or duple.

The precise statement and proof of this theorem are given in Section 4. Three corollaries are stated and proved in Section 5. The first is this: suppose a decision scheme guarantees Pareto optimal outcomes. Suppose, in other words, that no matter how people vote, if one alternative is unanimously outranked by another, then it gets a probability of zero. Suppose also that there are at least three alternatives, and that the decision scheme is strategy-proof. Then the decision scheme is a probability mixture of dictatorial schemes.[3]

The second corollary is this. Suppose a decision scheme gives lotteries which are Pareto optimal *ex ante*, where a lottery is *Pareto optimal ex ante* if there is no other lottery which is unanimously preferred to it. Suppose, in other words, that no matter what each person's utility scale is, if each person votes the ranking of alternatives given by his utility scale, then the resulting lottery ρ has this property: there is no other lottery ρ' which ranks higher than ρ on everyone's utility scale. Suppose, as before, that there are at least three alternatives, and that the decision scheme is strategy-proof. Then the decision scheme is dictatorial—it is not, that is to say, merely a probability mixture of dictatorial schemes; it is itself dictatorial. This corollary extends to schemes which allow the expression of individual indifference.

The third corollary is simply the earlier theorem on non-chance voting schemes [**6**]. The proof in this paper, then, constitutes a new proof of that earlier theorem.

[2] Zeckhauser [**18**, p. 939] describes an extension of the "random dictator system" as follows: "Provide each voter with q ballots for his first choice, r for his second, s for his third, etc., with $q > r > s$. The selection procedure is random as before." He goes on to say, "Thus we find that only variants of the random dictator system will elicit ballots unique with respect to individuals' . . . ordinal preferences." (Being 'unique' in Zeckhauser's terminology is roughly the same as being "strategy-proof" in mine). If by "variants of the random dictator system" he means systems of the form specified in the passage I have quoted, then Scheme 2 is a counterexample to this claim.

[3] I owe this corollary to Hugo Sonnenschein.

2. BACKGROUND

The notion of manipulability used in this paper is a variant of the one formulated by Dummett and Farquharson [**2**]. Manipulability and closely related matters are discussed by Arrow [**1**, p. 7], Vickrey [**16**], Murakami [**7**, pp. 74–81], Farquharson [**3**], Sen [**15**, pp. 192–196], and Pattanaik [**9**, **10**, **11**, and **12**]. The theorem cited at the outset of this paper is proved independently, in quite different ways, by Gibbard [**6**] and Satterthwaite [**13**]. A precise statement of the theorem is this: any scheme which uses rank order balloting in a nonchance way to select a single alternative is either manipulable, dictatorial (in that someone is guaranteed his first choice from among the possible outcomes), or restricted to no more than two possible outcomes. This rèsult holds both for schemes which allow individual indifference to be expressed and for schemes which do not. A streamlined proof of the theorem is given by Schmeidler and Sonnenschein [**14**]. This earlier theorem does not apply to systems of voting which allow tied outcomes. In my discussion of that theorem [**6**, pp. 592–593], I argued that it makes no sense to study the manipulability of schemes which allow ties unless one considers the system by which ties are to be broken. If ties are to be broken by chance, I argued, then the full system to be studied in effect yields outcomes which are lotteries among alternatives.

It was Zeckhauser [**17**] who broached the study of voting with lotteries as alternatives. Fishburn [**4** and **5**] studies the subject further. A subsequent paper by Zeckhauser [**18**] is on virtually the topic of this paper. There Zeckhauser studies systems of voting which rely on individuals' self-interested balloting and may have lotteries as outcomes. He concludes [**18**, Theorem V, p. 945] that no such system can guarantee an outcome which is both *ex ante* Pareto optimal and, in a special sense, "nondictatorial". (For the case of two voters, an outcome is "dictatorial" in Zeckhauser's sense if it is the first choice of one voter and the last choice of the other). Zeckhauser's result is logically independent of the one in this paper. It is stronger in one respect: the results here are confined to systems with rank order balloting, whereas Zeckhauser's is not. Zeckhauser, on the other hand, requires that the lotteries that serve as outcomes of the schemes he considers be Pareto optimal *ex ante*, and, in his special sense, "nondictatorial". The main theorem in this paper does not invoke Pareto optimality, and whereas the second corollary here is suggested by Zeckhauser's result, Zeckhauser's nondictatorship condition is much stronger than the nondictatorship condition in the corollary. Zeckhauser's result, then, neither directly entails the results in this paper nor is directly entailed by them.

3. INDIVIDUAL INDIFFERENCE

The main result in this paper fails when extended to systems that permit a voter to express indifference between alternatives. No doubt the easiest example of this failure is a serial dictatorship. Let a fixed "dictator" always get his first choice, and if more than one alternative ties as his first choice, let a "prime henchman" get his first choice from among the alternatives the dictator likes best. Let further ties be broken arbitrarily, say by selecting the tying alternative which is first on some predetermined list. Now a serial dictatorship is clearly strategy-proof: neither the dictator, the prime henchman, nor anyone else can ever gain by misrepresenting his preferences. If there are more than two alternatives, however, then the serial dictatorship is not equivalent to any probability mixture of unilateral or duple schemes, as the following considerations show.

Note at the outset that a serial dictatorship is not unilateral. A scheme is unilateral, in the sense the term has been given here, only if it accords a single voter—call him the *ruler*—a monopoly of influence, so that no matter how anyone votes, the ballots of all voters other than the ruler are ignored. Under a serial dictatorship, the ballot of the prime henchman is not invariably ignored, and hence a serial dictatorship is not itself unilateral.

Now a serial dictatorship is not a probability mixture which has any duple scheme as a part. If it were, then for at least one fixed pair of alternatives, the probability of the adopted alternative's being in that pair would have to be nonzero independently of how anyone voted. Under a serial dictatorship, both alternatives in any pair have probability zero of being adopted whenever neither is a first choice of the dictator. Thus if a serial dictatorship is a probability mixture of schemes each of which is unilateral or duple, then it is a probability mixture of unilateral schemes alone.

The only unilateral schemes that could be part of this mixture, though, are ones for which the dictator is ruler. Otherwise, there would be a nonzero chance that no matter what the dictator's ballot said, it would be ignored, so that for some way the dictator and others might vote, an alternative which was not the first choice of the dictator would be adopted with nonzero probability. Under a serial dictatorship, on the other hand, the probability that the dictator will fail to get his first choice (or one of his first choices in case he has no unique first choice) is always zero. Thus if a serial dictatorship were a probability mixture of unilateral and duple schemes, it would have to consist of a single unilateral scheme with the dictator as ruler—and we have already seen that it does not.

A serial dictatorship, then, is not a probability mixture of schemes which are unilateral or duple, and hence the theorem in this paper does not in general extend to systems which allow individuals to express indifference.

What, then, can be said about systems with ballots which do allow individual indifference to be expressed? What the theorem here tells us is this: if such a system s is strategy-proof, then there is a probability mixture m of unilateral and duple schemes which coincides with s whenever no one is indifferent between any pair of alternatives. For any ranking n-tuple $\boldsymbol{P}$ with no ties, in other words, m assigns the same prospect to $\boldsymbol{P}$ as does s.

The force of the theorem, then, extends to systems which allow individuals to express indifference. For the force of the theorem lies in the judgment that any probability mixture of unilateral and duple schemes is grossly defective as a way of making community decisions. What the theorem says about systems with no individual indifference is, in effect, that nonmanipulability can be had only in systems which are otherwise grossly defective. Now take a system which is nonmanipulable and allows individual indifference. For all cases in which no one is indifferent between alternatives, the system is a fixed mixture of unilateral and duple schemes. That in itself is a gross defect: for a significant class of combinations of individual preferences, the system stands ready to make the community choice in an unacceptable way. The theorem in this paper shows, then, that even in systems which permit a voter to express indifference, nonmanipulability may be had only at an exhorbitant price.

4. DEFINITIONS AND PROOFS

Let V be a finite set, called the set of *alternatives*. Variables w, x, y, and z will have V as their range of values. A *strict ordering* of V is a binary relation P which, for all x, y, and z, satisfies:

Asymmetry: $xPy \rightarrow \sim yPx$.

Negative transitivity: $xPy \rightarrow (xPz \vee zPy)$.

Such a relation is transitive, and may allow indifference between alternatives.
A *ranking* of V is a strict ordering of V which, for all x and y, satisfies:

Connectedness: $x \neq y \rightarrow (xPy \vee yPx)$.

A ranking *n-tuple* over V is an n-tuple $\langle P_1, \ldots, P_n \rangle$ of rankings of V. Ranking n-tuples will be represented by bold type on the pattern: $\boldsymbol{P} = \langle P_1, \ldots, P_n \rangle$, $\boldsymbol{P}^* = \langle P_1^*, \ldots, P_n^* \rangle$, and the like. $\boldsymbol{P}$ and $\boldsymbol{P}'$ *agree off k* iff for all $i \neq k$, $P_i' = P_i$. $\boldsymbol{P}/_k P$ is the preference n-tuple $\boldsymbol{P}'$ such that $P_k' = P$ and $\boldsymbol{P}'$ agrees with $\boldsymbol{P}$ off k.

We now define "proto-scheme", "scheme", and "decision scheme". A *measure* over V is a function ρ which assigns a nonnegative real number, $\rho(x)$, to each member x of V. The sum $\Sigma_x \rho(x)$ of these numbers is called the *weight* of the measure. A *lottery* is a measure of weight one.

A proto-scheme is a function d such that, for some positive integer n, called the *number of voters* of d, and for some finite set V, whose members are called *alternatives* of d, the domain of d is the set of all ranking n-tuples over V, and the values of d are measures over V. The value of d at $\boldsymbol{P}$ will be written $d\boldsymbol{P}$, and the probability $d\boldsymbol{P}$ assigns to an alternative x will be written $d(x, \boldsymbol{P})$. A *scheme* is a proto-scheme all of whose values have the same weight; this will be called the *weight* of the scheme. A *decision scheme* is a scheme of weight one. It thus assigns to each ranking n-tuple $\boldsymbol{P}$ a lottery over V.

We consider, then, a fixed set V of alternatives and number n of voters. The variables will range as follows: w, x, y, and z are alternatives in V; X, Y, and Z are sets of alternatives, i.e., subsets of V; P and Q are rankings of V; $\boldsymbol{P}$ and $\boldsymbol{Q}$ are ranking n-tuples $\langle P_1, \ldots, P_n \rangle$ over V; i, j, and k are integers from 1 to n which stand for voters; b, c, and d are schemes for n voters and set V of alternatives. Subscripts, superscripts, primes, and the like do not affect the range of variables.

A *utility scale U* over V is an assignment of real numbers to the members of V. Where U is a utility scale over V and ρ is a lottery over V, we define the *expected utility $U(\rho)$ of ρ on* scale U in this way: $U(\rho) = \Sigma_x U(x)\rho(x)$. Utility scale U *fits* a strict ordering P iff for all x and y, $U(x) > U(y) \leftrightarrow xPy$.

A decision scheme d is *potentially manipulable* by k at $\boldsymbol{P}$ iff there are a utility scale U which fits P_k and a ranking P'_k of V such that where $\boldsymbol{P}' = \boldsymbol{P}/_k P'_k$, $U(d\boldsymbol{P}') > U(d\boldsymbol{P})$. A decision scheme d is *manipulable* iff there are a voter k and a ranking n-tuple $\boldsymbol{P}$ such that d is potentially manipulable by k at $\boldsymbol{P}$. Otherwise, d is *strategy-proof.*

We now give a number of definitions which will allow the theorem on strategy-proof decision schemes to be stated in a preliminary, weak version.

DEFINITION 1: Scheme d is *unilateral* iff there is a k such that for all $\boldsymbol{P}$ and $\boldsymbol{P}'$, if $P'_k = P_k$, then $d\boldsymbol{P}' = d\boldsymbol{P}$.

DEFINITION 2: Scheme d is *duple* iff there are alternatives x and y such that for every other alternative z, $d(z, \boldsymbol{P}) = 0$ for all $\boldsymbol{P}$.

DEFINITION 3: Scheme d is a *probability mixture* of schemes $d_1, \ldots, d_m$ iff there is a sequence $\alpha_1, \ldots, \alpha_m$, with $0 < \alpha_\iota \leqslant 1$ for each $\iota \in \{1, \ldots, m\}$ and $\Sigma_{\iota=1}^m \alpha_\iota = 1$, such that for each $\boldsymbol{P}$ and x,

$$d(x, \boldsymbol{P}) = \alpha_1 d_1(x, \boldsymbol{P}) + \ldots + \alpha_m d_m(x, \boldsymbol{P}).$$

Where d is such a probability mixture, we shall write

$$d = \alpha_1 d_1 + \ldots + \alpha_m d_m,$$

and where $d = b + c$, we shall write $b = d - c$.

THEOREM (*weak version*): *If d is a strategy-proof decision scheme, then d is a probability mixture of decision schemes each of which is either unilateral or duple.*

This theorem can be strengthened to give conditions which are sufficient as well as necessary for a decision scheme's being strategy-proof. For any set X of alternatives and scheme d, we shall write $d(X, \boldsymbol{P})$ for $\Sigma_{x \in X} d(x, \boldsymbol{P})$, the total probability assigned by measure $d\boldsymbol{P}$ to members of X. X *heads* ranking P_k iff for any $x \in X$ and $y \notin X$, $xP_k y$.

DEFINITION 4: Proto scheme d is *localized* iff for every k, $\boldsymbol{P}$, P'_k, and X such that X heads both P_k and P'_k, $d(X, \boldsymbol{P}/_k P'_k) = d(X, \boldsymbol{P})$.

A *switch* is a reversal of two adjacent alternatives in a ranking. A scheme is *nonperverse* if switching an alternative upward never decreases its probability.

DEFINITION 5: $xP_k!y$ means that $xP_k y$ and $\sim(\exists z)(xP_k z$ and $zP_k y)$. Where $xP_k!y$, P_k^y is the ranking which switches xy in P_k and permutes no other alternative, $\boldsymbol{P}^{ky} = \boldsymbol{P}/_k P_k^y$, and $\varepsilon_k^y(d, \boldsymbol{P})$, the *effect* under d of k's switching y upward, is $d(y, \boldsymbol{P}^{ky}) - d(y, \boldsymbol{P})$. Scheme d is *nonperverse* iff for every $\boldsymbol{P}$, k, *and* y such that $\{y\}$ does not head P_k, $\varepsilon_k^y(d, \boldsymbol{P}) \geqslant 0$.

THEOREM: *A decision scheme d is strategy-proof if and only if it is a probability mixture of decision schemes, each of which is localized, non-perverse, and either unilateral or duple.*

The proof of the Theorem consists chiefly of five lemmas.

DEFINITION 6: A proto-scheme d is *pairwise responsive* iff for every $\boldsymbol{P}$, k, x, y, and z, if $xP_k!y$ and $z \notin \{x, y\}$, then $d(z, \boldsymbol{P}^{ky}) = d(z, \boldsymbol{P})$.

LEMMA 1: *The following are equivalent*: (*i*) *d is a localized proto-scheme*; (*ii*) *d is a pairwise responsive scheme*; (*iii*) *d is a pairwise responsive proto-scheme, and for all x, y,* $\boldsymbol{P}$, *and k such that* $xP_k!y$, $d(\{x, y\}, \boldsymbol{P}^{ky}) = d(\{x, y\}, \boldsymbol{P})$.

PROOF. Suppose d is a localized proto-scheme. Then since V heads any $\boldsymbol{P}$ and $\boldsymbol{P}^*$, $d(V, \boldsymbol{P}) = d(V, \boldsymbol{P}^*)$, and d is a scheme. Now suppose that $xP_k!y$, $z \notin \{x, y\}$, and W is the set of alternatives ranked above z in P^k. Then both W and $W \cup \{z\}$ head both P_k and P_k^y. Thus since d is localized, k's switching y upward changes neither the total probability of W nor the total probability of $W \cup \{z\}$. Thus it leaves the probability of z unchanged, and d is pairwise responsive. Thus (i) entails (ii). For any pairwise responsive scheme, a switch of xy changes neither the total probability of $V - \{x, y\}$ nor that of V; thus it leaves that of $\{x, y\}$ unchanged, and (ii) entails (iii). Now suppose (iii); it follows that if $xP_k!y$ and $\{x, y\} \subseteq Z$, then $d(Z, \boldsymbol{P}^{ky}) = d(Z, \boldsymbol{P})$. If Z heads both P_k and P'_k, then P'_k can be formed from P_k by switches between members of Z and switches between nonmembers of Z, neither of which, we have seen, change the total probability of Z. Thus d is localized, and (iii) entails (i).

LEMMA 2: *A decision scheme d is strategy-proof iff d is localized and nonperverse.*[4]

PROOF: Suppose that d is not localized, so that for some k, some $\boldsymbol{P}$ and $\boldsymbol{P}'$ that agree off k, and some X which heads both P_k and P'_k, $d(X, \boldsymbol{P}') - d(X, \boldsymbol{P}) = \varepsilon > 0$. Let U fit P_k and be such that for all $x \in X$, $1 \leqslant U(x) < 1 + \varepsilon$, and for all $y \notin X$, $0 \leqslant U(y) < \varepsilon$. Then

$$U(d\boldsymbol{P}) < (1+\varepsilon)d(X, \boldsymbol{P}) + \varepsilon[1 - d(X, \boldsymbol{P})] = d(X, \boldsymbol{P}) + \varepsilon.$$

$$U(d\boldsymbol{P}') \geqslant 1 \cdot d(X, \boldsymbol{P}') + 0 \cdot [1 - d(X, \boldsymbol{P}')] = d(X, \boldsymbol{P}') = d(X, \boldsymbol{P}) + \varepsilon.$$

Therefore $U(d\boldsymbol{P}') > U(d\boldsymbol{P})$, and so d is potentially manipulable by k at $\boldsymbol{P}$.

If d is localized but perverse, then for some x, y, and k, $xP_k!y$ and k's switching y upwards lowers the probability of y by some amount $\varepsilon > 0$. By (iii) of Lemma 1, the switch raises the probability of x by ε, and changes no other probabilities. Hence, if U fits P_k, so that $U(x) > U(y)$, then $U(d\boldsymbol{P}^{ky}) - U(d\boldsymbol{P}) = \varepsilon U(x) - \varepsilon U(y) > 0$, and so d is potentially manipulable by k at $\boldsymbol{P}$.

Now suppose d is localized and nonperverse, and consider any k, $\boldsymbol{P}$, P'_k, and U which fits P_k. Where $\boldsymbol{P}' = \boldsymbol{P}/_k P'_k$, we shall show that $U(d\boldsymbol{P}') \leqslant U(d\boldsymbol{P})$. Form P'_k from P_k by successive switches as follows: take the top alternative in P'_k and switch

[4] Zeckhauser [**18**, pp. 938–939] proves a similar result for systems that solicit individuals' first place preferences only.

it from its position in P_k successively to the top, then take the second alternative in P'_k and switch it successively up from its position in P_k to its position in P'_k, and so forth. At each step, an alternative y is switched with an alternative which is above it in P_k. Since U fits P_k, $U(x) > U(y)$, and so by (iii) of Lemma 1 and the nonperversity of d, utility on scale U cannot be increased by such steps. Hence $U(d\boldsymbol{P}') \leq U(d\boldsymbol{P})$. That proves the Lemma.

DEFINITION 7: $P_i \uparrow \{x, y\}$ is P_i restricted to $\{x, y\}$, and $\boldsymbol{P} \uparrow \{x, y\} = \langle P_1 \uparrow \{x, y\}, \ldots, P_n \uparrow \{x, y\}\rangle$. A scheme d is *pairwise isolated* iff for any k, $\boldsymbol{P}$, $\boldsymbol{P}^*$, x, and y, if $xP_k!y$, $P^*_k = P_k$, and $\boldsymbol{P}^* \uparrow \{x, y\} = \boldsymbol{P} \uparrow \{x, y\}$, then $\varepsilon^y_k(d, \boldsymbol{P}^*) = \varepsilon^y_k(d, \boldsymbol{P})$. d is *decomposable* iff for any fixed k, x, and y with $x \neq y$, there are functions γ and δ such that for all $\boldsymbol{P}$ with $xP_k!y$, $\varepsilon^y_k(d, \boldsymbol{P}) = \gamma(\boldsymbol{P}\uparrow\{x, y\}) + \delta(P_k)$.

DEFINITION 8: P_{kxy} is P_k with x and y moved to the bottom, their ordering with respect to each other preserved, and the ordering of all other alternatives with respect to each other preserved. $\boldsymbol{P}_{xy}$ is $\langle P_{1xy}, \ldots, P_{nxy}\rangle$, P_{kx} is P_{kxx}, and $\boldsymbol{P}_x$ is $\boldsymbol{P}_{xx}$.

LEMMA 3: *Let scheme d be localized. Then d is pairwise isolated and decomposable.*

PROOF THAT d IS PAIRWISE ISOLATED: We first show that the switch of a pair by one person does not alter the effect of the switch of another pair by another person. Suppose $j \neq k$, $wP_j!z$, $xP_k!y$, and $\{w, z\} \neq \{x, y\}$.

Case 1 ($y \notin \{w, z\}$): d is pairwise responsive and $\boldsymbol{P}^{jz}$ differs from $\boldsymbol{P}$ only in j's switching wz; thus $d(y, \boldsymbol{P}^{jz}) = d(y, \boldsymbol{P})$. Likewise, $\boldsymbol{P}^{jzky}$ differs from $\boldsymbol{P}^{ky}$ only in j's switching wz; thus $d(y, \boldsymbol{P}^{jzky}) = d(y, \boldsymbol{P}^{ky})$. Hence,

$$d(y, \boldsymbol{P}^{jzky}) - d(y, \boldsymbol{P}^{jz}) = d(y, \boldsymbol{P}^{ky}) - d(y, \boldsymbol{P}),$$

which is to say $\varepsilon^y_k(d, \boldsymbol{P}^{jz}) = \varepsilon^y_k(d, \boldsymbol{P})$.

Case 2 ($x \notin \{w, z\}$): By an argument like that in Case 1, $\varepsilon^x_k(d, \boldsymbol{P}^{jzky}) = \varepsilon^x_k(d, \boldsymbol{P}^{ky})$. It follows from this and (iii) of Lemma 1 that $\varepsilon^y_k(d, \boldsymbol{P}^{jz}) = \varepsilon^y_k(d, \boldsymbol{P})$. Now let $xP_k!y$, $P^*_k = P_k$, and $\boldsymbol{P}^* \uparrow \{x, y\} = \boldsymbol{P} \uparrow \{x, y\}$. Then $\boldsymbol{P}^*$ can be formed from $\boldsymbol{P}$ by a sequence of switches by voters other than k, none of which switches x with y. We have just seen that none of these changes the value of ε^y_k, and thus $\varepsilon^y_k(d, \boldsymbol{P}^*) = \varepsilon^y_k(d, \boldsymbol{P})$. Thus d is pairwise isolated.

PROOF THAT d IS DECOMPOSABLE: Take k, x, and y with $x \neq y$. For any $\boldsymbol{P}$ with $xP_k!y$, define $\gamma(\boldsymbol{P} \uparrow \{x, y\}) = \varepsilon^y_k(d, \boldsymbol{P}_{kxy})$. Now let $\boldsymbol{P}$ and $\boldsymbol{P}^*$ be such that $xP_k!y$ and $P^*_k = P_k$; we shall show

(1) $$\varepsilon^y_k(d, \boldsymbol{P}^*) - \gamma(\boldsymbol{P}^*\uparrow\{x, y\} = \varepsilon^y_k(d, \boldsymbol{P}) - \gamma(\boldsymbol{P}\uparrow\{x, y\}).$$

Since d is pairwise isolated, $\varepsilon^y_k(d, \boldsymbol{P})$ depends only on P_k and $\boldsymbol{P} \uparrow \{x, y\}$; thus we may suppose without loss of generality that everyone other than k ranks x and y last. Now form P^k_y from P_k by the following sequence of switches.

(a) Progressively switch y to bottom.
(b) Progressively switch x down to just above y.
(c) Switch y with x.
(d) Progressively switch y up to its original position.
(e) Progressively switch x up to just below y.

Call this sequence $P_k^0, \ldots, P_k^\mu$, and consider the difference

(2) $$d(y, \boldsymbol{P}/_kP_k^\iota) - d(y, \boldsymbol{P}^*/_kP_k^\iota)$$

as ι goes from 0 to μ. This difference changes only in step (c). For the steps in (a) and (d) consist of switching y with various alternatives $z \notin \{x, y\}$. Everyone except k ranks z above y in both $\boldsymbol{P}$ and $\boldsymbol{P}^*$, and so since d is pairwise isolated, both terms of (2) change by the same amount, and (2) is unchanged. The steps in (b) and (e) consist of switching x with alternatives other than y; since d is pairwise responsive, this changes neither term of (2). Now at step (c), x and y are switched in P_{kxy}. The change in (2) at step (c), then, is

(3) $$\varepsilon_k^y(d, \boldsymbol{P}_{xy}) - \varepsilon_k^y(d, \boldsymbol{P}_{xy}^*).$$

This, then, is the change in (2) from $\iota = 0$ to $\iota = \mu$, that is,

$$[d(y, \boldsymbol{P}^{ky}) - d(y, \boldsymbol{P}^{*ky})] - [d(y, \boldsymbol{P}) - d(y, \boldsymbol{P}^*)],$$

which is $\varepsilon_k^y(d, \boldsymbol{P}) - \varepsilon_k^y(d, \boldsymbol{P}^*)$. From the equality of this with (3), (1) follows. Since the quantity in (1) depends only on P_k, let $\delta(P_k)$ be this quantity; then $\varepsilon_k^y(d, \boldsymbol{P}) = \gamma(\boldsymbol{P} \upharpoonright \{x, y\}) + \delta(P_k)$, and d is decomposable.

Definition 9: k's *unilateral component* of decision scheme d is the function d_k such that for all x and $\boldsymbol{P}$,

$$d_k(x, \boldsymbol{P}) = \min_{\boldsymbol{Q}} \{d(x, \boldsymbol{Q}/_kP_k) - d(x, \boldsymbol{Q}/_kP_{kx})\}.$$

Since the value of $d_k(x, \boldsymbol{P})$ depends only on x and P_k, this will be written $d_k(x, P_k)$.

Lemma 4: *Let d be a strategy-proof decision scheme, and let d_k be k's unilateral component of d. Then (i) if $xP_k!y$, then*

$$\varepsilon_k^y(d_k, \boldsymbol{P}) = \min_{\boldsymbol{Q}} \varepsilon_k^y(d, \boldsymbol{Q}/_kP_k).$$

(ii) d_k is a scheme which is unilateral, localized, and nonperverse.

Proof of (i): Let $xP_k!y$, let $\alpha = d_k(y, P_k)$, and let $\beta = d_k(y, P_k^y)$. Then $\varepsilon_k^y(d_k, \boldsymbol{P}) = \beta - \alpha$. Let $\varepsilon = \min_{\boldsymbol{Q}} \varepsilon_k^y(d, \boldsymbol{Q}/_kP_k)$; we are to prove that $\varepsilon = \beta - \alpha$.

By the definition of β, for some $\boldsymbol{Q}$,

$$\beta = d(y, \boldsymbol{Q}/_kP_k^y) - d(y, \boldsymbol{Q}/_kP_{ky}),$$

and by the definitions of α and ε,

$$\alpha \leqslant d(y, \boldsymbol{Q}/_kP_k) - d(y, \boldsymbol{Q}/_kP_{kx});$$

$$\varepsilon \leqslant d(y, \boldsymbol{Q}/_kP_k^y) - d(y, \boldsymbol{Q}/_kP_k).$$

Therefore, by addition, $\alpha+\varepsilon \leqslant \beta$.

We will have $\alpha+\varepsilon \geqslant \beta$ if there is a $\boldsymbol{Q}$ such that

$$d(y, \boldsymbol{Q}/_k P_k^y) - d(y, \boldsymbol{Q}/_k P_{ky}) = \alpha+\varepsilon, \tag{4}$$

since β is the minimal value of this difference. Construct such a $\boldsymbol{Q}$ as follows. By the definition of α, for some $\boldsymbol{P}^*$ with $P_k^* = P_k$, $\alpha = d(y, \boldsymbol{P}^*) - d(y, \boldsymbol{P}^*/_k P_{ky})$. This difference is the sum of the effects, in context $\boldsymbol{P}^*$, of k's successively switching y from bottom to just below x in P_k. Since d is pairwise isolated and y is not switched with x, each of these effects is independent of where others besides k place x in their rankings. On the other hand, since d is pairwise isolated, the effect under d of k's switching y with x is independent of the way others vote *except* for their ranking of x with respect to y. By the definition of ε, for some $\boldsymbol{P}'$, $P'_k = P_k$ and $\varepsilon = \varepsilon_k^y(d, \boldsymbol{P}')$. Form $\boldsymbol{Q}$ from $\boldsymbol{P}^*$ by moving x, in the ranking of each voter $i \neq k$, to just above y in P_i^* or just below y in P_i^* according as $xP'_i y$ or $yP'_i x$. Then, we have seen,

$$\alpha = d(y, \boldsymbol{Q}) - d(y, \boldsymbol{Q}/_k P_{ky}),$$

$$\varepsilon = \varepsilon_k^y(d, \boldsymbol{Q}) = d(y, \boldsymbol{Q}/_k P_k^y) - d(y, \boldsymbol{Q}),$$

and hence by addition, (4) holds.

PROOF OF (ii): Since d is nonperverse, $d(x, \boldsymbol{Q}/_k P_k) - d(x, \boldsymbol{Q}/_k P_{kx}) \geqslant 0$ for all $\boldsymbol{Q}$. Hence its minimal value $d_k(x, P_k)$ is nonnegative, and d_k is a proto-scheme. Now let $xP_k!y$. By (i),

$$d_k(x, P_k) - d_k(x, P_k^y) = \min_{\boldsymbol{Q}} \varepsilon_k^x(d, \boldsymbol{Q}/_k P_k^y);$$

$$d_k(y, P_k^y) - d_k(y, P_k) = \min_{\boldsymbol{Q}} \varepsilon_k^y(d, \boldsymbol{Q}/_k P_k).$$

By (iii) of Lemma 1, these two minima must be equal, and hence

$$d_k(\{x, y\}, P_k^y) = d_k(\{x, y\}, P_k).$$

Now let $z \notin \{x, y\}$, and take any $\boldsymbol{Q}$. Since d is pairwise responsive, $d(z, \boldsymbol{Q}/_k P_k^y) = d(z, \boldsymbol{Q}/_k P_k)$. By (iii) of Lemma 1, $d(z, \boldsymbol{Q}/_k P_{kz}^y) = d(z, \boldsymbol{Q}/_k P_{kz})$. Therefore,

$$d(z, \boldsymbol{Q}/_k P_k^y) - d(z, \boldsymbol{Q}/_k P_{kz}^y) = d(z, \boldsymbol{Q}/_k P_k) - d(z, \boldsymbol{Q}/_k P_{kz}),$$

and so the minima are equal: $d_k(z, P_k^y) = d_k(z, P_k)$. Thus (iii) in Lemma 1 is satisfied for d_k, and so d_k is a localized scheme. Finally, by (i), if $xP_k!y$, then for some $\boldsymbol{Q}$, $\varepsilon_k^y(d_k, P_k) = \varepsilon_k^y(d, \boldsymbol{Q})$, and since d is nonperverse, this is nonnegative; therefore d_k is nonperverse. d_k is clearly unilateral, and the lemma is proved.

DEFINITION 10: A scheme d *ignores external comparisons* iff for any x, y, k, $\boldsymbol{P}$, and $\boldsymbol{P}^*$, if $xP_k!y$ and $\boldsymbol{P}^* \uparrow \{x, y\} = \boldsymbol{P} \uparrow \{x, y\}$, then $\varepsilon_k^y(d, \boldsymbol{P}^*) = \varepsilon_k^y(d, \boldsymbol{P})$.

LEMMA 5: *Suppose d is a strategy-proof decision scheme. For each k, let d_k be k's unilateral component of d, and define functions d_0, c, and c_{yz} for each $\{y, z\}$ such* *that $y \neq z$ as follows.*

$$d_0(x, \boldsymbol{P}) = d(x, \boldsymbol{P}_x);$$

$$c(x, \boldsymbol{P}) = d(x, \boldsymbol{P}) - d_0(x, \boldsymbol{P}) - \ldots - d_n(x, \boldsymbol{P});$$

$$c_{yz}(x, \boldsymbol{P}) = c(x, \boldsymbol{P}_{yz}) \quad \text{for} \quad x \in \{y, z\} \quad \text{and} \quad 0 \quad \text{for} \quad x \notin \{y, z\}.$$

Then c ignores external comparisons. Each function $d_0, \ldots, d_n$, c, and each c_{xy} is a localized, nonperverse scheme, $d_0, \ldots, d_n$ are unilateral, each c_{xy} is duple, and for all x and $\boldsymbol{P}$,

$$(5) \qquad d(x, \boldsymbol{P}) = d_0(x, \boldsymbol{P}) + \textstyle\sum_k d_k(x, \boldsymbol{P}) + \sum_{yz} c_{yz}(x, \boldsymbol{P}),$$

where Σ_{yz} sums over all pairs $\{y, z\}$ with $y \neq z$.

PROOF: d_0 is constant for each x, and thus, like each d_k, a unilateral, localized, nonperverse scheme. Now consider c, and let $xP_k!y$.

$$\varepsilon_k^y(d_i, \boldsymbol{P}) = 0 \quad \text{for} \quad i \neq k,$$

and thus

$$\varepsilon_k^y(c, \boldsymbol{P}) = \varepsilon_k^y(d, \boldsymbol{P}) - \varepsilon_k^y(d_k, \boldsymbol{P}).$$

By Lemma 4(i),

$$\varepsilon_k^y(d_k, \boldsymbol{P}) \leqslant \varepsilon_k^y(d, \boldsymbol{P}); \quad \text{hence,} \quad \varepsilon_k^y(c, \boldsymbol{P}) \geqslant 0$$

and c is nonperverse. Since $d_0(x, \boldsymbol{P}_x) = d(x, \boldsymbol{P}_x)$ and $d_i(x, \boldsymbol{P}_x) = 0$ for $i \neq 0$, $c(x, \boldsymbol{P}_x) = 0$ for all x and $\boldsymbol{P}$. Thus since c is nonperverse, $c(x, \boldsymbol{P}) \geqslant 0$ for all x and $\boldsymbol{P}$, and c is a proto-scheme. Since $d, d_0, \ldots, d_n$ are all localized, c is localized and, hence, a scheme.

The function c ignores external comparisons. For, let $xP_k!y$. Since d is decomposable, there are functions γ and δ such that for any $\boldsymbol{Q}$ with $xQ_k!y$, $\varepsilon_k^y(d, \boldsymbol{Q}) = \gamma(\boldsymbol{Q} \uparrow \{x, y\}) + \delta(Q_k)$. Thus $\varepsilon_k^y(d, \boldsymbol{P}) = \gamma(\boldsymbol{P} \uparrow \{x, y\}) + \delta(P_k)$. Let $\boldsymbol{Q}$ minimize $\varepsilon_k^y(d, \boldsymbol{Q})$ for $Q_k = P_k$; then by Lemma 4(i), $\varepsilon_k^y(d_k, \boldsymbol{P}) = \varepsilon_k^y(d, \boldsymbol{Q}) = \gamma(\boldsymbol{Q} \uparrow \{x, y\}) + \delta(P_k)$. Therefore $\varepsilon_k^y(c, \boldsymbol{P}) = \gamma(\boldsymbol{P} \uparrow \{x, y\}) - \gamma(\boldsymbol{Q} \uparrow \{x, y\})$.

Now let $\boldsymbol{P}^* \uparrow \{x, y\} = \boldsymbol{P} \uparrow \{x, y\}$; by a like argument, where $\boldsymbol{Q}^*$ minimizes $\varepsilon_k^y(d, \boldsymbol{Q}^*)$ for $Q_k^* = P_k^*$, $\varepsilon_k^y(c, \boldsymbol{P}^*) = \gamma(\boldsymbol{P}^* \uparrow \{x, y\}) - \gamma(\boldsymbol{Q}^* \uparrow \{x, y\})$. Since $\boldsymbol{Q}$ minimizes $\varepsilon_k^y(d, \mathbf{Q}) = \gamma(\boldsymbol{Q} \uparrow \{x, y\}) + \delta(P_k)$ for $Q_k = P_k$, $\boldsymbol{Q}$ minimizes $\gamma(\boldsymbol{Q} \uparrow \{x, y\})$ for xQ_ky, and since xP_k^*y, $\boldsymbol{Q}^* = \boldsymbol{Q}/_k P_k^*$ minimizes $\gamma(\boldsymbol{Q}^* \uparrow \{x, y\})$ for $Q_k^* = P_k^*$. For this $\boldsymbol{Q}^*$, $\boldsymbol{Q}^* \uparrow \{x, y\} = \boldsymbol{Q} \uparrow \{x, y\}$, and so $\varepsilon_k^y(c, \boldsymbol{P}^*) = \gamma(\boldsymbol{P}^* \uparrow \{x, y\}) - \gamma(\boldsymbol{Q}^* \uparrow \{x, y\}) = \gamma(\boldsymbol{P} \uparrow \{x, y\}) - \gamma(\boldsymbol{Q} \uparrow \{x, y\}) = \varepsilon_k^y(c, \boldsymbol{P})$. Thus c ignores external comparisons.

Since $c(z, \boldsymbol{P}) \geqslant 0$ for all z and $\boldsymbol{P}$, from the way c_{xy} is defined, $c_{xy}(z, \boldsymbol{P}) \geqslant 0$. Moreover, $c_{xy}(V, \boldsymbol{P}) = c(\{x, y\}, \boldsymbol{P}_{xy})$, and since c is localized, this is constant for all $\boldsymbol{P}$, and c_{xy} is a scheme. c_{xy} is pairwise responsive and, hence, localized: an xy switch leaves all other probabilities zero, and any other switch leaves all probabilities unchanged. c_{xy} is nonperverse: if $wP_k!z$ and $\{w, z\} \neq \{x, y\}$, $\varepsilon_k^z(c_{xy}, \boldsymbol{P}) = 0$, and if $xP_k!y$, $\varepsilon_k^y(c_{xy}, \boldsymbol{P}) = \varepsilon_k^y(c, \boldsymbol{P}_{xy}) \geqslant 0$, since c is nonperverse.

Finally, $c(x, \boldsymbol{P}) = \Sigma_{yz} c_{yz}(x, \boldsymbol{P})$ for $y \neq z$. We noted earlier that $c(x, \boldsymbol{P}_x) = 0$ for all x and $\boldsymbol{P}$; thus $c_{yz}(x, \boldsymbol{P}_x) = 0$. For if $x \notin \{y, z\}$, then $c_{yz}(x, \boldsymbol{P}) = 0$ for all $\boldsymbol{P}$, and $c_{yz}(y, \boldsymbol{P}_y) = c(y, (\boldsymbol{P}_y)_{yz}) = 0$, since $(\boldsymbol{P}_y)_{yz}$ has y uniformly on the bottom. Therefore, $c(x, \boldsymbol{P}_x) = 0 = \Sigma_{yz} c_{yz}(x, \boldsymbol{P}_x)$. Now form $\boldsymbol{P}$ from $\boldsymbol{P}_x$ by successively shifting x upward in each ranking; call the resulting sequence $\boldsymbol{P}^0, \ldots, \boldsymbol{P}^\mu$. At each step from $\boldsymbol{P}^\iota$ to $\boldsymbol{P}^{\iota+1}$, some k switches x with a w such that $wP_k^\iota!x$. Now $c_{wx}(x, \boldsymbol{P}^{\iota+1}) - c_{wx}(x, \boldsymbol{P}^\iota) = c(x, \boldsymbol{P}_{wx}^{\iota+1}) - c(x, \boldsymbol{P}_{wx}^\iota) = \varepsilon_k^x(c, \boldsymbol{P}_{wx}^\iota) = \varepsilon_k^x(c, \boldsymbol{P}^\iota)$ since c ignores external comparisons; thus $c(x, \boldsymbol{P}^{\iota+1}) - c(x, \boldsymbol{P}^\iota) = c_{wx}(x, \boldsymbol{P}^{\iota+1}) - c_{wx}(x, \boldsymbol{P}^\iota)$. For $\{y, z\} \neq \{x, w\}$, $P_{kyz}^{\iota+1} = P_{kyz}^\iota$, and so $c_{yz}(x, \boldsymbol{P}^{\iota+1}) = c_{yz}(x, \boldsymbol{P}^\iota)$. Therefore, $c(x, \boldsymbol{P}^{\iota+1}) - c(x, \boldsymbol{P}^\iota) = \Sigma_{yz} c_{yz}(x, \boldsymbol{P}^{\iota+1}) - \Sigma_{yz} c_{yz}(x, \boldsymbol{P}^\iota)$. Since $c(x, \boldsymbol{P}^0) = \Sigma_{yz} c_{yz}(x, \boldsymbol{P}^0)$, by induction $c(x, \boldsymbol{P}^\iota) = \Sigma_{yz} c_{yz}(X, \boldsymbol{P}^\iota)$ for all $\boldsymbol{P}^\iota$, and thus $c(x, \boldsymbol{P}) = \Sigma_{yz} c_{yz}(x, \boldsymbol{P})$. Equation (5) follows immediately.

PROOF OF THEOREM: In (5), drop schemes of weight zero, so that d is a sum $b_1^* + \ldots + b_m^*$ of schemes of positive weight $\alpha_1, \ldots, \alpha_m$, respectively. Let each decision scheme $b_\iota = (1/\alpha_\iota) b_\iota^*$; then $d = \alpha_1 b_1 + \ldots + \alpha_m b_m$, and each b_ι is localized, nonperverse, and either unilateral or duple. Conversely, if decision scheme d is a probability mixture of this kind, then d is clearly localized and nonperverse; hence, by Lemma 2, d is strategy-proof. That completes the proof of the theorem.

5. COROLLARIES

DEFINITION 11: k is *dictator* for decision scheme d iff for every $\boldsymbol{P}$, x and y, if $xP_k y$ then $d(y, \boldsymbol{P}) = 0$. d is *dictatorial* iff there is a dictator for d.

DEFINITION 12: Lottery ρ is *Pareto optimal ex post* for ranking n-tuple $\boldsymbol{P}$ iff for any x, if there is a y such that $yP_i x$ for all i, then $\rho(x) = 0$. Decision scheme d is *Pareto optimific ex post* iff for every $\boldsymbol{P}$, lottery $d\boldsymbol{P}$ is Pareto optimal *ex post* for $\boldsymbol{P}$.

COROLLARY 1 (Sonnenschein): *Let decision scheme d be strategy-proof and Pareto optimific ex post. Let the set V of alternatives for d have at least three members. Then d is a probability mixture of dictatorial decision schemes.*

PROOF: Since d is strategy-proof, d is a probability mixture of decision schemes, each of which is unilateral or duple. Let

$$d = \alpha_1 d_1 + \ldots + \alpha_m d_m,$$

where for each $\iota \in \{1, \ldots, m\}$, $\alpha_\iota > 0$ and d_ι is nonnull and either unilateral or duple. Then no d_ι is duple. For since d is Pareto optimific *ex post*, the alternatives in any pair $\{x, y\}$ get a probability of zero whenever some alternative z is unanimously preferred to them. Thus for any pair $\{x, y\}$, $d_\iota(\{x, y\}, \boldsymbol{P}) = 0$ whenever $zP_i x$ and $zP_i y$ for each i; therefore d_ι is not an xy duple scheme. Hence, each d_ι is unilateral.

Now let d_ι be unilateral on the part of k and let xP_ky. Then, we shall show, $d_\iota(y, \boldsymbol{P})=0$. For let $\boldsymbol{P}'$ be such that $P'_k=P_k$ and for all i, xP_iy. Then since d is Pareto optimific *ex post*, $d(y, \boldsymbol{P}')=0$, and thus $d_\iota(y, \boldsymbol{P}')=0$. Since d_ι is unilateral on the part of k and $P'_k=P_k$, $d_\iota(y, \boldsymbol{P})=0$. We have shown that for any $\boldsymbol{P}$, x, and y, if xP_ky then $d_\iota(y, \boldsymbol{P})=0$. Thus k is dictator for d_ι. We have shown that each d_ι is dictatorial, and the corollary is proved.

DEFINITION 13: Lottery ρ is *Pareto optimal ex ante* for utility scales $U_1, \ldots, U_n$ iff there is no lottery ρ' such that for each i, $U_i(\rho')>U_i(\rho)$. Decision scheme d is *Pareto optimific ex ante* iff for every ranking n-tuple $\boldsymbol{P}$ and every n-tuple $\langle U_1, \ldots, U_n\rangle$ of utility scales such that for each i, P_i fits U_i, lottery $d\boldsymbol{P}$ is Pareto optimal *ex ante* for $U_1, \ldots, U_n$.

COROLLARY 2: *Let decision scheme d be strategy-proof and Pareto optimific ex ante. Let the set V of alternatives for d have at least three members. Then d is dictatorial.*

Proof: Note first that if d is Pareto optimific *ex ante*, then d is Pareto optimific *ex post*. For, let preference rankings $P_1, \ldots, P_n$ fit $U_1, \ldots, U_n$, respectively, and suppose lottery ρ is not Pareto optimal *ex post* for $P_1, \ldots, P_n$. Then for some pair of alternatives x and y, yP_ix for all i, but $\rho(x)\neq 0$. Now let ρ' give x's probability to y, so that $\rho'(x)=0$, $\rho'(y)=\rho(x)+\rho(y)$, and $\rho'(z)=\rho(z)$ for all $z\notin\{x, y\}$. Then for each i, since P_i fits U_i, $U_i(y)>U_i(x)$, and so $U_i(\rho')>U_i(\rho)$. Therefore ρ is not Pareto optimal *ex ante* for $U_1, \ldots, U_n$, and d is not Pareto optimific *ex ante*.

Suppose now that d is Pareto optimific *ex ante*, and therefore Pareto optimific *ex post*. Then d is a probability mixture of dictatorial decision schemes. Let

$$d=\alpha_1 d_1+\ldots+\alpha_n d_n,$$

where for each i, $\alpha_i\geqslant 0$ and i is dictator for d_i.

Suppose that d is not itself dictatorial, so that $\alpha_i>0$ for more than one i. Let $\alpha_k>0$, let x, y, and z be distinct alternatives, and let the utility scales $U_1, \ldots, U_n$ be as follows.

$$\begin{aligned}&U_k(x)=1,\quad 1>U_k(y)>\alpha_k,\quad U_k(z)=0,\\ &\text{and for all } w\notin\{x, y, z\},\quad U_k(w)<0.\end{aligned}$$

For all $i\neq k$,

$$\begin{aligned}&U_i(z)=1,\quad 1>U_i(y)>1-\alpha_k,\quad U_i(x)=0,\\ &\text{and for all } w\notin\{x, y, z\},\quad U_i(w)<0.\end{aligned}$$

For each i, let P_i fit U_i, so that $\{x\}$ heads P_k and $\{z\}$ heads P_i for all $i\neq k$. Then $d(x, \boldsymbol{P})=\alpha_k$, $d(z, \boldsymbol{P})=1-\alpha_k$, and $d(w, \boldsymbol{P})=0$ for all w distinct from x and y. Therefore $U_k(d\boldsymbol{P})=\alpha_k$, and for $i\neq k$, $U_i(d, \boldsymbol{P})=1-\alpha_k$.

Now let $\hat{y}$ be the lottery that gives y as a sure thing. Then $U_k(\hat{y})>\alpha_k$ and for $i\neq k$, $U_i(\hat{y})>1-\alpha_k$. Therefore for all i, $U_i(\hat{y})>U_i(d\boldsymbol{P})$, and so $d\boldsymbol{P}$ is not Pareto optimal *ex ante* for $U_1, \ldots, U_n$. On the supposition that d is not dictatorial, we have shown that d is not Pareto optimific *ex ante*. Therefore d is dictatorial, and the corollary is proved.[5]

Corollary 2 can be extended to schemes that allow individual indifference. Let a *preference n-tuple* over V be an n-tuple of strict orderings of V. Let an *unrestricted decision scheme* (UDS) be a function which, for some finite set V of alternatives and number n, takes as arguments all preference n-tuples over V, and takes as values lotteries over V. Manipulability is defined as before, with the term "ranking" replaced by "strict ordering".

The following Lemma allows us both to extend Corollary 2 to UDS's, and to derive the old theorem on non-chance voting schemes. Where d is a decision scheme or UDS, a possible outcome for d is an alternative x such that for some $\boldsymbol{P}$ in the domain of d, $d(x, \boldsymbol{P})>0$. A *weak dictator* for d is a voter k such that for every $\boldsymbol{P}$, where X is the set of possible outcomes which are first among possible outcomes in P_k, $d(X, \boldsymbol{P})=1$. d is *weakly dictatorial* iff there is a weak dictator for d.

LEMMA 6: *Let d be a strategy-proof UDS, and let d' be the decision scheme which is d with its domain restricted to ranking n-tuples. Then* (*i*) *any possible outcome of d is a possible outcome of d', and* (*ii*) *a weak dictator for d' is weak dictator for d.*

PROOF OF (i): Suppose x is a possible outcome of d but not of d'. Let $\boldsymbol{P}$ be a ranking n-tuple such that x ranks first in every P_i, and let $\boldsymbol{P}^*$ be a preference n-tuple such that $d(x, \boldsymbol{P}^*)>0$. Form a sequence of preference n-tuples $\boldsymbol{P}^0, \ldots, \boldsymbol{P}^n$ as follows: let $\boldsymbol{P}^0=\boldsymbol{P}$, and for each i, let $\boldsymbol{P}^i=\boldsymbol{P}^{i-1}/_i P_i^*$, so that $\boldsymbol{P}^n=\boldsymbol{P}^*$. Then since $\boldsymbol{P}^0$ is a ranking n-tuple and x is not a possible outcome of d', we have that $d(x, \boldsymbol{P}^0)=0$, whereas $d(x, \boldsymbol{P}^n)>0$. Take the least j such that $d(x, \boldsymbol{P}^j)>0$, and let $d(x, \boldsymbol{P}^j)=\varepsilon$. Then $d(x, \boldsymbol{P}^{j-1})=0$, and $P_j^{j-1}=P_j$. Since P_j ranks x first, there is a utility scale U which fits P_j, such that $U(x)=1$ and for all $y \neq x$, $0 \leqslant U(y)<\varepsilon$. Since $d(x, \boldsymbol{P}^j)=\varepsilon$, $U(x)=1$, and for all $y \neq x$, $U(y) \geqslant 0$, we have that $U(d\boldsymbol{P}^j) \geqslant \varepsilon$. Since $d(x, \boldsymbol{P}^{j-1})=0$ and for all $y \neq x$, $U(y)<\varepsilon$, we have that $U(d\boldsymbol{P}^{j-1})<\varepsilon$. Thus $U(d\boldsymbol{P}^j)>U(d\boldsymbol{P}^{j-1})$, and since U fits P_j and $P_j^{j-1}=P_j$, d is potentially manipulable by j at $\boldsymbol{P}^{j-1}$.

PROOF OF (ii): From (i), the possible outcomes of d and d' are the same. Now suppose k is weak dictator for d' but not for d. Then for some preference n-tuple $\boldsymbol{P}$, where X is the set of possible outcomes ranked first in P_k, $d(X, P) \neq 1$. For some $x \in X$, let P_k^* rank x first, and for every $i \neq k$, let P_i^* rank x last. Let $\boldsymbol{P}^0=\boldsymbol{P}/_k P_k^*$. Then $d(x, \boldsymbol{P}^0) \neq 1$, for otherwise, on any utility scale U which fits P_k, we would have that $U(d\boldsymbol{P}^0)=U(x)$ and $U(d\boldsymbol{P})<U(x)$, so that $U(d\boldsymbol{P}^0)>U(d\boldsymbol{P})$. Thus d would be potentially manipulable by k at $\boldsymbol{P}$. Now form sequence $\boldsymbol{P}^0, \ldots, \boldsymbol{P}^n$ by letting $\boldsymbol{P}^i=\boldsymbol{P}^{i-1}/P_i^*$ for each i, so that $\boldsymbol{P}^n=\boldsymbol{P}^*$. Then since $\boldsymbol{P}^n$ is a ranking n-tuple with k, who is weak dictator for d', ranking possible outcome x first, we have that $d(x, \boldsymbol{P}^n)=1$. Take the least j such that $d(x, \boldsymbol{P}^j)=1$; then $d(x, \boldsymbol{P}^{j-1})<1$. We know that $j \neq k$, since from the way $\boldsymbol{P}^j$ is defined, $\boldsymbol{P}^k=\boldsymbol{P}^{k-1}$. Therefore x is at

[5] Aspects of this proof are suggested by arguments in Zeckhauser [**18**].

the bottom of P_j^*, and so on any utility scale U which fits P_j^*, $U(d\boldsymbol{P}^{j-1}) > U(x)$, whereas $U(d\boldsymbol{P}^j) = U(x)$. Thus $U(d\boldsymbol{P}^{j-1}) > U(d\boldsymbol{P}^j)$, and since $P_j^j = P_j^*$ and U fits P_j^*, d is potentially manipulable by j at $\boldsymbol{P}^j$. That proves the lemma.

COROLLARY 2′: *Let UDS d be strategy-proof and Pareto optimific ex ante, and let d cover at least three alternatives. Then d is dictatorial.*

PROOF: Since d is Pareto optimific *ex ante*, all alternatives are possible outcomes, and so a weak dictator is dictator. By Corollary 2, d with its domain restricted to ranking n-tuples is dictatorial, and thus by Lemma 6, d is dictatorial.

DEFINITION 14: A *voting scheme* is a UDS v such that for every x and preference n-tuple $\boldsymbol{P}$, either $v(x, \boldsymbol{P}) = 1$ or $v(x, \boldsymbol{P}) = 0$.

COROLLARY 3: *If a voting scheme is strategy-proof, then it is either duple or weakly dictatorial.*

PROOF: Let voting scheme v be strategy-proof, and let v' be v with its domain restricted to ranking n-tuples. Then by the Theorem, v' is a probability mixture $\alpha_1 d_1 + \ldots + \alpha_m d_m$, where each α_ι is positive, $\Sigma\alpha_\iota = 1$, and each d_ι is unilateral or duple. Now if $v'(x, \boldsymbol{P}) = 1$, then for each ι, $d_\iota(x, \boldsymbol{P}) = 1$, and if $v'(x, \boldsymbol{P}) = 0$, then for each ι, $d_\iota(x, \boldsymbol{P}) = 0$. Hence, each d_ι is identical with v', and v' is either unilateral or duple.

If v' is duple—that is, has at most two possible outcomes—then by (i) of Lemma 6, v is duple.

Let v' be unilateral, with k as ruler. Then v' is weakly dictatorial. For let x be a possible outcome. Then for some ranking n-tuple $\boldsymbol{P}^*$, $v'(x, \boldsymbol{P}^*) > 0$, and, hence, $v'(x, \boldsymbol{P}^*) = 1$. Thus since v' is unilateral, $v'(x, \boldsymbol{P}) = 1$ wherever $P_k = P_k^*$. Suppose, then, that v' is not weakly dictatorial, so that for some alternative x and ranking n-tuple $\boldsymbol{P}'$, x is first in P_k' but $v'(x, \boldsymbol{P}') \neq 1$. Where $\boldsymbol{P} = \boldsymbol{P}'/_k P_k^*$, we know that $v'(x, \boldsymbol{P}) = 1$. Thus for any utility scale U which fits P_k', $U(v\boldsymbol{P}) > U(v\boldsymbol{P}')$, and v' is potentially manipulable by k at $\boldsymbol{P}$. The supposition that v' is not weakly dictatorial has been shown false. It follows from (ii) of Lemma 6 that v itself is weakly dictatorial. That proves the corollary.

University of Pittsburgh

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