Primarily About Primaries

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Abstract

Much of the social choice literature examines *direct* voting systems, in which voters submit their ranked preferences over candidates and a voting rule picks a winner. Real-world elections and decision-making processes are often more complex and involve multiple stages. For instance, one popular voting system filters candidates through *primaries*: first, voters affiliated with each political party vote over candidates of their own party and the voting rule picks a candidate from each party, which then compete in a general election.

We present a model to analyze such multi-stage elections, and conduct the first quantitative comparison (to the best of our knowledge) of the direct and primary voting systems with two political parties in terms of the quality of the elected candidate. Our main result is that every voting rule is guaranteed to perform almost as well (i.e., within a constant factor) under the primary system as under the direct system. Surprisingly, the converse does not hold: we show settings in which there exist voting rules that perform significantly better under the primary system than under the direct system.

1. Introduction

If I could not go to heaven but with a party, I would not go there at all.

– Thomas Jefferson, 1789

Thomas Jefferson, like many of the US constitution's authors, believed that political parties and factions are a bad thing (see also Hamilton et al. [1]). This view stemmed from a long history of British and English political history, in which prison sentences and executions were possible outcomes in the battle between factions for supremacy at the Royal court [2]. However, both in Britain and in the Unites States, once their respective legislative assemblies gained political force, parties turned out to be quite unavoidable. Even Jefferson had

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to start his own party, which ended up quite successful, and was able to vanquish the opposing party from political existence [3].

Fast forwarding to today, political parties have become the bedrock of parliamentary politics throughout the world. In particular, one of political parties' main roles — if not the most important (especially in presidential systems) — is to select the candidates which are voted for by the general public. The mechanisms by which parties make this selection are varied, and they have evolved significantly throughout the past 150 years. But in the past few decades there has been a marked shift by parties throughout the world towards increasing the ability of individual party (or unaffiliated) members to influence the outcome, and in some cases, to be the only element to determine party candidates [4]. In particular, US parties have changed their election methods since the 1970s to focus the selection of presidential, congressional and state-wide candidates on popular support by party members via primaries [5].

Despite this long and established role of parties in whittling down the candidate field in elections, the treatment of a parties' role in elections within the multiagent systems community has been quite limited. While various candidate manipulation attacks have been investigated (e.g., Sybil attacks [6]), and there is recent research into parties as a collection of similar minded candidates (e.g., in gerrymandering, across different districts), the role of parties in removing candidates has not been analyzed.

The focus of this paper is the *primary* voting system, in which each party's electorate selects a winner from among the party's candidates, and among these primary winners, an ultimate election winner is selected by the general public. Our goal is to compare this system to the *direct* voting system, in which all voters directly vote over all candidates.

1.1. Our Results

Our contribution is twofold. First, we formulate a model which allows a quantitative comparison of the two voting systems. Our model is a spatial model of voting in which voters and candidates are located in an underlying multi-dimensional space, and voter preferences are single-peaked. This allows us to compare each candidate's social utility in terms of its total distance to the voters. We make the evaluation objective formal using the notion of *distortion* as advocated by a recent line of research [7, 8]. Our results focus on 2 parties, each selecting a single candidate, with both candidates presented to the general voting public. Specifically, we assume that every candidate is affiliated with one of two parties, but we allow voters to be independent.

Second, we use this model to present a comparison of the direct and primary voting systems. In particular, we show that no voting rule performs much worse (by more than a constant factor) under the primary system than under the direct system. While the converse holds in some cases (that is, there exist voting rules that perform much better under the primary system than under the direct system), we show settings in which it does not, which include many common voting rules.

It is important to note that while we write of parties, voters and elections, this multi-step model applies to a variety of decision-making processes by agents. An organization selecting an "employee of the month" may ask each unit to nominate a single candidate, and then choose from amongst them. A city may ask its regional subdivisions to assess which roads require urgent fixing, and then the city council decides from these options where to invest its efforts. Fundamentally, in many cases where the potential number of options is huge, it is common to use subdivisions to cull the options and present only a few of them for discussion and vote. In such cases, our multi-stage model is appropriate.

2. Related Work

The analysis of regular, direct elections is long and varied, both in the social sciences and in AI [9]. In our particular setting, the voters are located in a metric space, with their preferences related to their distance from candidates. Such settings have been widely investigated in the social science literature since the work of Downs [10], recently summarized by Schofield [11]. In particular, we focus on the concept of distortion, introduced by Procaccia and Rosenschein [7]. While they analyze it in the context of a direct election in which voters have arbitrary utilities for candidates, we consider distortion in a primary setting in which the costs (negative utilities) of voters for candidates are determined by their locations in an underlying metric space, which represents their opinions. The context of voters in metric spaces was investigated in a series of papers [12, 13, 14] for most common voting rules. Feldman et al. [15] explored such a setting for strategyproof mechanisms.

Discussing changes to the set of candidates has mainly focused on two paths of research. Strategic candidates, investigation of which began with the work of Dutta et al. [16] — followed by Dutta et al. [17] discussing strategic candidacy in tournaments, and recently further explored by Brill and Conitzer [18], Polukarov et al. [19] and others — deals mainly with finding equilibria. The other is the addition and removal of candidates, as a form of control manipulation, which was studied by Bartholdi III et al. [20]; see the summaries by Brandt et al. [9] and Rothe [21].

Investigating parties' selection methods and their effect on the election has mostly been done in the social sciences. Kenig [22] details the range of selection methods parties use, and there has been significant focus on more democratic methods for leader selection [23], which seems to be a general trend in many Western countries [4]. There is also significant literature on particular party elections in various countries, such as Britain [24], Belgium [25], Israel [26], and many others. Naturally, the most widely examined country is the US, in which political parties have been a fixture of political life since its early days [3]. The most recent extensive summary of research on it is due to Cohen et al. [5], who try to explain how party power-brokers influence the party membership vote. Norpoth [27] uses primary data to predict election results, and notably Sides et al. [28] show that primary voters are very similar to "regular" voters. In computational fields, recent interest in proxy voting [29, 30], in which voters give other agents the ability to vote for them, may be related to how modern parties are viewed and analyzed.

3. Model

For $k \in \mathbb{N}$, define $[k] = \{1, \ldots, k\}$. Let $V = [n]$ denote a set of n *voters*, and A denote a set of m *candidates*. We assume that voters and candidates lie in an underlying metric space $M = (S, d)$, where S is a set of points and d is a distance function satisfying the triangle inequality and symmetry. More precisely, there exists an *embedding* $\rho : V \cup A \to S$ mapping each voter and candidate to a point in S. For a set $X \subseteq V \cup A$, we slightly abuse the notation and let $\rho(X) = \{\rho(x) : x \in X\}$. Also, for $x, x' \in V \cup A$, we often use $d(x, x')$ instead of $d(\rho(x), \rho(x'))$ for notational convenience.

In this work, we assume that voters and candidates additionally have an affiliation with a political party. Specifically, we study a setting with two parties[1], denoted -1 and 1. Throughout most of this paper, we assume that *every* voter and candidate is affiliated with exactly one of the two parties. In Section 8, we show that our results continue to hold even when some of the voters are independent,unaffiliated with either party. The *party affiliation* function $\pi : V \cup A \to \{-1, 1\}$ maps each voter and candidate to the party they are affiliated with. For $p \in \{-1, 1\}$, let $V_p = \pi^{-1}(p) \cap V$, $A_p = \pi^{-1}(p) \cap A$, $n_p = |V_p|$, and $m_p = |A_p|$. We require $n_p \geq 1$ for each $p \in \{-1, 1\}$.

Collectively, an *instance* is the tuple $I = (V, A, M, \rho, \pi)$. Given I, our goal is to find a candidate $a \in A$ as the winner. The *social cost* of a is its total distance to the voters, denoted $C^I(a) = \sum_{i \in V} d(i, a)$. For party $p \in \{-1, 1\}$, let $C^I_p(a) = \sum_{i \in V_p} d(i, a)$. Hence $C^I(a) = C^I_{-1}(a) + C^I_1(a)$. For $X \subseteq V$, we also use $C^I_X(a) = \sum_{i \in X} d(i, a)$. Given an instance I, we would like to choose a candidate $a_{\text{OPT}} \in \arg\min_{a \in A} C^I(a)$ that minimizes the social cost. We shall drop the instance from superscripts if it is clear from the context.

However, we do not observe the full instance. Specifically, we do not know the underlying metric M or the embedding function ρ. Instead, each voter $i \in N$ submits a *vote*, which is a ranking (strict total order) $\succ_i$ over the candidates in A by their distance to the voter. Specifically, for all $i \in N$ and $a, b \in A$, $a \succ_i b \Rightarrow d(i, a) \leq d(i, b)$. The voter is allowed to break ties between equidistant candidates arbitrarily. The *vote profile* $\overrightarrow{\succ}^I = (\succ_1, \ldots, \succ_n)$ is the collection of votes. Given an instance I, its corresponding *election* $E^I = (V, A, \overrightarrow{\succ}^I, \pi)$ contains all observable information.

In the families of instances that we consider, we fix the number of candidates m and let the number of voters n to be arbitrarily large. This choice is justified because in many typical elections (e.g., political ones), voters significantly outnumber candidates. Let $\mathcal{I}^\alpha_{m,\mathcal{M}}$ be the family of instances satisfying the following conditions.

[1]The model,but not necessarily all the results, can be extended to multiple parties in a reasonably straightforward manner, but we shall focus only on 2 parties.

- Each party has at least an α fraction of the voters affiliated with it, i.e., $n_p \geq \alpha \cdot n$ for each $p \in \{-1, 1\}$. Note that $\alpha \in [0, 0.5]$: $\alpha = 0.5$ is the strictest (exactly half of the voters are affiliated with each party), while $\alpha = 0$ imposes no conditions; in the latter case, we omit the superscript α.

- The number of candidates is at most m.

In particular, we shall focus on a few cases of $\mathcal{M}$:

- $\mathcal{M} = \star$: This allows M to be an arbitrary metric space.

- $\mathcal{M} = \mathbb{R}^k$: The metric space should be $M = (\mathbb{R}^k, d)$, where d is the standard Euclidean distance. In particular, $\mathbb{R} = \mathbb{R}^1$ denotes the real line.

- $\mathcal{M} = \text{sep-}\mathbb{R}^k$: This means the embedding ρ must be such that $\rho(V_{-1} \cup A_{-1})$ and $\rho(V_1 \cup A_1)$ are linearly separable.[2] In this case we also take the metric to be $M = (\mathbb{R}^k, d)$ with d as the standard Euclidean distance. In plain words, the voters and candidates affiliated with each party reside in a certain part of the metric space, separate from those affiliated with the other party. In the single dimension, this means there exists a threshold on the line such that voters and candidates affiliated with one party lie to the left of it, while those affiliated with the other party lie to the right. Note that this is the only choice of $\mathcal{M}$ that restricts the embedding ρ based on party affiliation π.

These families of instances are related by the following relation. For all k,

$$\mathcal{I}^{\alpha}_{m,\text{sep-}\mathbb{R}^k} \begin{array}{ccc} \subset & \mathcal{I}^{\alpha}_{m,\mathbb{R}^k} & \subset \\ \subset & \mathcal{I}^{\alpha}_{m,\text{sep-}\mathbb{R}^{k+1}} & \subset \end{array} \mathcal{I}^{\alpha}_{m,\mathbb{R}^{k+1}} \subset \mathcal{I}^{\alpha}_{m,\star}$$

3.1. Voting Rules and Distortion

A voting rule f takes an election as input, and returns a winning candidate from A. We say that the cost-approximation of f on instance I is

$$\phi(f, I) = \frac{C^I(f(E^I))}{\min_{a \in A} C^I(a)},$$

and given a family of instances $\mathcal{I}$, the distortion of f with respect to $\mathcal{I}$ is

$$\phi_{\mathcal{I}}(f) = \sup_{I \in \mathcal{I}} \phi(f, I).$$

Since distortion is a worst-case notion, we have that when $\mathcal{I} \subseteq \mathcal{I}'$, $\phi_{\mathcal{I}}(f) \leq \phi_{\mathcal{I}'}(f)$ for every voting rule f.

[2] Two sets of points are linearly separable if the interiors of their convex hulls are disjoint, or equivalently, if there exists a hyperplane that contains each set in a distinct closed halfspace.

Standard voting rules choose the winning candidate independently of party affiliations. These include rules such as plurality, Borda count, k-approval, veto, and STV. We refer readers to the book by Brandt et al. [9] for their definitions. We call a voting rule *affiliation-independent* if $f(E) = f(E')$ when elections E and E' differ only in their party affiliation functions. For an affiliation-independent voting rule f that ignores party affiliations, we have $\phi_{\mathcal{I}^{\alpha}_{m,\text{sep-}\mathbb{R}^k}}(f) = \phi_{\mathcal{I}^{\alpha}_{m,\mathbb{R}^k}}(f)$. All of the above-mentioned rules, in addition to being affiliation-independent, share the property of being *unanimous*, i.e., they return candidate a when a is the top choice of all voters.

3.2. Stages and Primaries

Given an affiliation-independent voting rule f, voting systems with primaries employ a specific process to choose the winner, essentially resulting in a different voting rule $\widehat{f}$ that operates on a given election $E = (V, A, \overrightarrow{\succ}, \pi)$ as follows:

1. First, it creates two *primary elections*: for $p \in \{-1, 1\}$, define $E_p = (V_p, A_p, \overrightarrow{\succ}_p, \pi_p)$, where $\overrightarrow{\succ}_p$ denotes the preferences of voters in V_p over candidates in A_p, and $\pi_p : V_p \to \{p\}$ is a constant function.
2. Next, it computes the winning candidate in each primary election (primary winner) using rule f: for $p \in \{-1, 1\}$, let $a^*_p = f(E_p)$.
3. Finally, let $E_g = (V, \left\{a^*_{-1}, a^*_1\right\}, \overrightarrow{\succ}_g, \pi)$ be the *general election*, where $\overrightarrow{\succ}_g$ denotes the preferences of all voters over the two primary winners. The winning candidate is $\widehat{f}(E) = \text{maj}(E_g)$, which is what most voting rules become when dealing with only 2 candidates.[3] Here, maj is the majority rule, which, given two candidates, picks the one that a majority of voters prefer; our results are independent of tie-breaking.

This resembles systems employed by the main US, Canadian and other countries' parties, in which a party's members vote on their party's candidates to select a winner of their primary. In other systems, such selection could be a multi-stage process. While we assume for now that every voter votes in the general election, in Section 8 we show that our results hold as long as at least a constant fraction of the voters participate in the general election.

Given an affiliation-independent voting rule f, the goal of this paper is to compare its performance under the *direct system*, in which f is applied on the given election directly, to its performance under the *primary system*, in which $\widehat{f}$ is applied on the given election instead. Formally, given a family of instances $\mathcal{I}$ and an affiliation-independent voting rule f, we wish to compare $\phi_{\mathcal{I}}(f)$ and $\phi_{\mathcal{I}}(\widehat{f})$ (henceforth, the distortion of f with respect to $\mathcal{I}$ under the direct and the primary systems, respectively).

[3] If one of the parties has no affiliated candidates, then the primary winner of the other party becomes the overall winner. In a setting with more than 2 parties, or where each party nominates several candidates, the general election can use f to determine the outcome (or use some other voting process).

4. Small Primaries Are Terrible

Recall that in a family of instances $\mathcal{I}^{\alpha}_{m,\mathcal{M}}$, we require that at least α fraction of voters be affiliated with each party, i.e., $n_p \geq \alpha n$ for each $p \in \{-1, 1\}$. In other words, each primary election must have at least αn voters.

We first show that when a primary election can have very few voters ($\alpha = 0$), every reasonable voting rule has an unbounded distortion in the primary system, even with respect to our most stringent family of instances $\mathcal{I}_{m,\text{sep-}\mathbb{R}}$.

Theorem 1. *For $m \geq 3$, $\phi_{\mathcal{I}_{m,sep\text{-}\mathbb{R}}}(\widehat{f}) = \infty$ for every affiliation-independent unanimous voting rule f.*

Proof. Consider an instance $I \in \mathcal{I}_{m,\text{sep-}\mathbb{R}}$ in which voter 1 is located at 0 and affiliated with party -1, while the remaining $n-1$ voters are located at 1 and affiliated with party 1. All m candidates are affiliated with party -1: one is at 0, and the rest are at 1.

The candidate a^* at 0 becomes the primary winner of party -1, and trivially becomes the overall winner. Its social cost is $C(a^*) = n - 1$. In contrast, an optimal candidate a_{OPT} at 1 has social cost $C(a_{OPT}) = 1$. Hence, $\frac{C(a^*)}{C(a_{OPT})} = n - 1$. Since the number of voters n is unbounded, $\phi_{\mathcal{I}_{m,\text{sep-}\mathbb{R}}}(\widehat{f}) = \infty$. □

Theorem 1 continues to hold even if we require that at least a constant fraction of candidates be affiliated with each party: we could simply move a constant fraction of the candidates from 1 to 3, and the proof would still go through.

On the other hand, if we require that at least a *constant fraction of voters* be affiliated with each party, the result changes dramatically.

5. Large Primaries Are Never Much Worse than Direct Elections

For every affiliation-independent voting rule f, we bound the distortion of $\widehat{f}$ in terms of the distortion of f *for every instance.* Note that this is stronger than comparing the worst-case distortions of f and $\widehat{f}$ over a family of instances.

Given an instance $I = (V, A, M, \rho, \pi)$ and party $p \in \{-1, 1\}$, we say that $I_p = (V_p, A_p, M, \rho_p, \pi_p)$ is the *primary instance* of party p, where ρ_p and π_p are restrictions of ρ and π to $V_p \cup A_p$. The primary election E_p of party p is precisely the election corresponding to instance I_p.

Theorem 2. *Let $I = (V, A, M, \rho, \pi)$ be an instance. For $p \in \{-1, 1\}$, let I_p be the primary instance of party p, and $n_p = |V_p| \geq \alpha n$. Then,*

$$\phi(\widehat{f}, I) \leq 3 \cdot \frac{1 - \alpha + \max(\phi(f, I_{-1}), \phi(f, I_1))}{\alpha}.$$

Further, for a socially optimal candidate $a_{OPT} \in \arg\min_{a \in A} C^I(a)$, we have a bound depending only on the distortion of the primary election of its party:

$$\phi(\widehat{f}, I) \leq 3 \cdot \frac{1 - \alpha + \phi(f, I_{\pi(a_{OPT})})}{\alpha}.$$

For each family of instances $\mathcal{I}$ that we study, it holds that for every instance $I \in \mathcal{I}$, both its primary instances, if seen as direct elections, are also in $\mathcal{I}$ (since the party division has no effect on the direct election distortion). Hence, we can convert the instance-wise comparison to a worst-case comparison.

Corollary 3. *For every $\alpha > 0$, $k \in \mathbb{N}$, family of instances $\mathcal{I} \in \left\{\mathcal{I}_{m,\star}, \mathcal{I}_{m,\mathbb{R}^k}, \mathcal{I}_{m,sep\text{-}\mathbb{R}^k}\right\}$, and affiliation-independent voting rule f,*

$$\phi_{\mathcal{I}}(\widehat{f}) \leq 3 \cdot \frac{1 - \alpha + \phi_{\mathcal{I}}(f)}{\alpha}.$$

Note that $\phi_{\mathcal{I}}(f) \geq 1$ by definition. Hence, we can write $\phi_{\mathcal{I}}(\widehat{f}) \leq \frac{6}{\alpha} \cdot \phi_{\mathcal{I}}(f)$. In other words, for every affiliation-independent voting rule f, its distortion under the primary system is at most a constant times bigger than its distortion under the direct system, with respect to every family of instances that we consider.

To prove Theorem 2, we need two lemmas. The first lemma shows that if the distortion of a rule f for a primary instance I_p is low, then the corresponding primary winner a_p^* is nearly as good as any candidate in A_p for the overall election as well. The intuition is that when voters in $V \setminus V_p$ drive up the social cost of a_p^* (i.e., when they are far from a^*), they must do so for every candidate in A_p.

Lemma 4. *Let a_p^* denote the primary winner of party p. Then*

$$C(a_p^*) \leq \frac{1 - \alpha + \phi(f, I_p)}{\alpha} \cdot \min_{a \in A_p} C(a).$$

Proof. Let $\theta = \phi(f, I_p)$ (hence $\theta \geq 1$). Fix an arbitrary $a \in A_p$. Then, $C_{V_p}(a_p^*) \leq \theta \cdot C_{V_p}(a)$. Now,

$$\begin{aligned} C(a_p^*) &= C_{V_p}(a_p^*) + C_{V \setminus V_p}(a_p^*) \\ &\leq \theta \cdot C_{V_p}(a) + C_{V \setminus V_p}(a) + (n - n_p) \cdot d(a, a_p^*) \\ &\leq \theta \cdot C(a) + (n - n_p) \cdot d(a_p^*, a), \end{aligned} \tag{1}$$

where the second transition follows due to the triangle inequality. We also have $d(a_p^*, a) \leq d(a_p^*, i) + d(i, a)$ for any $i \in V_p$. Thus,

$$\begin{aligned} d(a_p^*, a) &\leq \frac{C_{V_p}(a_p^*) + C_{V_p}(a)}{n_p} \\ &\leq \frac{1 + \theta}{n_p} \cdot C_{V_p}(a) \leq \frac{1 + \theta}{n_p} \cdot C(a). \end{aligned} \tag{2}$$

Substituting Equation (2) into Equation (1), and using the fact that $\frac{n - n_p}{n_p} \leq \frac{1 - \alpha}{\alpha}$, we get the desired result. □

Our next lemma shows that the primary winner that wins the general election is not much worse than the primary winner that loses the general election. We defer the proof of the lemma until Section 8, where we prove a more general result (Lemma 10).

Lemma 5. *Let a^*_{-1} and a^*_1 be the two primary winners, $a^* \in \{a^*_{-1}, a^*_1\}$ be the winner of the general election, and $\widehat{a} \in \{a^*_{-1}, a^*_1\} \setminus \{a^*\}$. Then,*

$$C(a^*) \leq 3 \cdot C(\widehat{a}).$$

We are now ready to prove our main result.

Proof of Theorem 2. Recall that a^*_{-1} and a^*_1 are the primary winners. Let $p \in \{-1, 1\}$ be such that a^*_p is the winner of the general election. Let $a_{OPT} \in \arg\min_{a \in A} C(a)$ be a socially optimal candidate. We consider three cases.

- *Case 1: $a_{OPT} \in A_p$.* That is, the optimal candidate and the winner are affiliated with the same party. In this case, Lemma 4 yields

$$\phi(\widehat{f}, I) \leq \frac{1 - \alpha + \phi(f, I_p)}{\alpha} = \frac{1 - \alpha + \phi(f, I_{\pi(a_{OPT})})}{\alpha}.$$

- *Case 2: $a_{OPT} \in A_{-p}$, $a_{OPT} = a^*_{-p}$.* That is, the optimal candidate and the winner are affiliated with different parties, and the optimal candidate is a primary winner. In this case, Lemma 5 yields

$$\phi(\widehat{f}, I) \leq 3.$$

- *Case 3: $a_{OPT} \in A_{-p}$, $a_{OPT} \neq a^*_{-p}$.* That is, the optimal candidate is affiliated with a party different than that of the winner, and is not a primary winner. In this case, we use both Lemmas 4 and 5 to derive

$$\begin{aligned}
\phi(\widehat{f}, I) &= \frac{C(a^*_p)}{C(a_{OPT})} = \frac{C(a^*_p)}{C(a^*_{-p})} \cdot \frac{C(a^*_{-p})}{C(a_{OPT})} \\
&\leq 3 \cdot \frac{1 - \alpha + \phi(f, I_{-p})}{\alpha} \\
&= 3 \cdot \frac{1 - \alpha + \phi(f, I_{\pi(a_{OPT})})}{\alpha}.
\end{aligned}$$

Thus, in each case, we have the desired approximation. □

Notice that we do not use the assumption that both parties use the same voting rule f in their primaries. The theorem extends easily to allow the use of different voting rules, with the distortion under the primary system still being bounded in terms of the maximum of the distortions in the two primary elections. Additionally, we show in Section 8 that two other assumptions we made so far — every voter participates in one of the two primaries and every voter participates in the general election — can be relaxed without significantly affecting our results.

6. Large Primaries Are Not Better Without Party Separability

While we showed in the previous section that a voting rule does not perform much worse under the primary system than under the direct system, we now show that it does not perform any better either, at least in the worst case over all instances with at most m candidates. The result continues to hold even if we require each party to have at least a constant fraction of the voters.

Note that this result is weaker than Theorem 2 because it is a worst-case comparison instead of an instance-wise comparison. However, it still applies to all voting rules f. It applies to any metric that does not require separability of parties, in particular to $\mathcal{I}_{m,\star}$ and $\mathcal{I}_{m,\mathbb{R}^k}$.

Theorem 6. *For $\alpha \in [0, 0.5]$, $k \in \mathbb{N}$, $\mathcal{M} \in \{\star, \mathbb{R}^k\}$ (i.e. when the metric space does not require party separability), and affiliation-independent voting rule f, we have $\phi_{\mathcal{I}^\alpha_{m,\mathcal{M}}}(\widehat{f}) \geq \phi_{\mathcal{I}^\alpha_{m,\mathcal{M}}}(f)$.*

Proof. We shall denote $\mathcal{I}^\alpha_{m,\mathcal{M}}$ as $\mathcal{I}$. We want to show that for every instance $I \in \mathcal{I}$, there exists an instance $I' \in \mathcal{I}$ such that $\phi(\widehat{f}, I') \geq \phi(f, I)$. This would imply the desired result.

Fix an instance $I = (V, A, M, \rho, \pi) \in \mathcal{I}$. Let $a_{OPT} \in A$ denote an optimal candidate in I, and $a^* = f(E^I)$. Note that $\phi(f, I) = \frac{C^I(a^*)}{C^I(a_{OPT})}$. Construct instance $I' = (V', A, M, \rho', \pi')$ as follows.

- Let $V' = V \cup \widetilde{V}$, where $\widetilde{V}$ is a new set of voters and $|\widetilde{V}| = |V|$.
- Let $\rho'(x) = \rho(x)$ for all $x \in V \cup A$, and $\rho(x) = \rho(a_{OPT})$ for all $x \in \widetilde{V}$. That is, ρ' matches ρ for existing voters and candidates, and the new voters are co-located with a_{OPT}.
- Let $\pi'(x) = -1$ for all $x \in V \cup A$, and $\pi'(x) = 1$ for all $x \in \widetilde{V}$. That is, all existing voters and candidates are affiliated with party -1, while all new voters are affiliated with party 1.

First, let us check that $I' \in \mathcal{I}$. Since I has m candidates, so does I'. Further, in I', we have $|V'_{-1}| = |V'_1| = |V'|/2$, which satisfies the constraint corresponding to every $\alpha \in [0, 0.5]$. Hence, we have $I' \in \mathcal{I}$.

Let us apply $\widehat{f}$ on I'. One of its primary instances, I'_{-1}, is precisely I. Hence, the primary winner of party -1 is $f(I'_{-1}) = f(I) = a^*$. Because there are no candidates affiliated with party 1, a^* becomes the overall winner.[4]

Next, $C^{I'}(a^*) \geq C^I(a^*)$ because $V \subset V'$. Also, $C^{I'}(a_{OPT}) = C^I(a_{OPT})$ because a_{OPT} has zero distance to all voters in $V' \setminus V$. Together, they yield

$$\phi(\widehat{f}, I') = \frac{C^{I'}(a^*)}{C^{I'}(a_{OPT})} \geq \frac{C^I(a^*)}{C^I(a_{OPT})} = \phi(f, I), \tag{3}$$

[4] Even if we require each party to have at least one affiliated candidate, the proof essentially continues to hold. In this case, we can add one candidate affiliated with party 1 that is located sufficiently far from all the voters, ensuring that a^* still becomes the overall winner. This would show $\phi_{\mathcal{I}^\alpha_{m+1,\mathcal{M}}}(\widehat{f}) \geq \phi_{\mathcal{I}^\alpha_{m,\mathcal{M}}}(f)$ because instance I' may now have $m+1$ candidates.

as desired. $\square$

Our proof actually establishes a slightly stronger result. Instead of showing $\phi_{\mathcal{I}^{\alpha}_{m,\mathcal{M}}}(\widehat{f}) \geq \phi_{\mathcal{I}^{\alpha}_{m,\mathcal{M}}}(f)$, we actually show $\phi_{\mathcal{I}^{0.5}_{m,\mathcal{M}}}(\widehat{f}) \geq \phi_{\mathcal{I}^{\alpha}_{m,\mathcal{M}}}(f)$.

7. The Advantages of Party Separability

The analysis for $\mathcal{I}_{m,\text{sep-}\mathbb{R}^k}$ is not so straightforward. In the proof of Theorem 6, we co-located the new voters affiliated with party 1 and a_{OPT} affiliated with party -1. This was allowed because non-separable metrics like $\star$ and $\mathbb{R}^k$ place no constraints on the embedding.

With $\mathcal{I}^{\alpha}_{m,\text{sep-}\mathbb{R}^k}$, we need the voters and candidates affiliated with one party to be separated from those affiliated with the other. Hence, this operation of putting all of one party's voters at the location of a_{OPT} belonging to another party would be allowed only if, in the original instance I, a_{OPT} is on the boundary of the convex hull of $\rho(V \cup A)$. While this is not the case for all instances, we only need this in at least one *worst-case instance* for f, i.e., for at least one $I \in \mathcal{I}^{\alpha}_{m,\text{sep-}\mathbb{R}^k}$ with $\phi(f, I) = \phi_{\mathcal{I}^{\alpha}_{m,\text{sep-}\mathbb{R}^k}}(f)$. Equation (3) would then yield the desired result. More generally, it is sufficient if, given any $\epsilon > 0$, we can find an instance I such that $\phi(f, I) \geq \phi_{\mathcal{I}^{\alpha}_{m,\text{sep-}\mathbb{R}^k}}(f) - \epsilon$ and a_{OPT} is at distance at most ϵ from the boundary of the convex hull of $\rho(V \cup A)$.

Interestingly, Anshelevich et al. [12] show that this is indeed the case for plurality and Borda count (see the proof of their Theorem 4). Thus, we have the following.

Proposition 7. *Let f be plurality or Borda count. Then, $\phi_{\mathcal{I}^{0.5}_{m,sep\text{-}\mathbb{R}}}(\widehat{f}) \geq \phi_{\mathcal{I}_{m,\star}}(f)$.*

However, known worst cases for the Copeland rule [12] and STV [14] do not satisfy this requirement. It is unknown if these rules admit a different worst case that satisfies it.

This raises an important question. *Does Proposition 7 hold for all affiliation-independent voting rules?* We shall shortly answer this *negatively*.

More precisely, we construct an affiliation-independent voting rule f such that $\phi_{\mathcal{I}^{\alpha}_{m,\text{sep-}\mathbb{R}}}(\widehat{f}) \ll \phi_{\mathcal{I}^{\alpha}_{m,\text{sep-}\mathbb{R}}}(f)$ for every $\alpha > 0$. That is, with large primaries, f performs much better under the primary system than under the direct system, when voters and candidates are embedded on a line and the separability condition is imposed.

Note that instances in $\mathcal{I}_{m,\text{sep-}\mathbb{R}}$ are highly structured. For instance, it is known that when voters and candidates are embedded on a line, there always exists a weak Condorcet winner [31], and selecting such a candidate results in a distortion of at most 3 [12]. Hence, we have $\phi_{\mathcal{I}_{m,\text{sep-}\mathbb{R}}}(f) = 3$ for every Condorcet-consistent, affiliation-independent voting rule f.[5]

[5] Anshelevich et al. [12] also proved that no affiliation-independent (deterministic) voting rule can have distortion better than 3, even with respect to $\mathcal{I}_{m,\text{sep-}\mathbb{R}}$.

Our aim in this section is to construct an affiliation-independent voting rule f_{fail} that with respect to $\mathcal{I}_{m,\text{sep-}\mathbb{R}}$ has an unbounded distortion in the direct system, but at most a constant distortion in the primary system.

Definition 1. Let f_{fail} be an affiliation-independent voting rule that operates on election $E = (V, A, \overrightarrow{\succ})$ as follows. Let $A = \{a_1, \ldots, a_m\}$, and $t = (m+1)/2$.

- *Special Case*: If $m \geq 9$, m is odd, $n \geq m^2$, and $\overrightarrow{\succ}$ has the following structure, then return a_1.
  1. For voter 1, $a_1 \succ_1 \ldots \succ_1 a_m$.
  2. For voter 2, $a_m \succ_2 \ldots \succ_2 a_1$.
  3. For voter 3, a_{t-1} is the most preferred, and $a_{m-2} \succ_3 a_1 \succ_3 a_{m-1} \succ_3 a_m$.
  4. For voter 4, a_{t+1} is the most preferred, and $a_3 \succ_4 a_m \succ_4 a_2 \succ_4 a_1$.
  5. For $j \in [m-2]$, for voter $i = 4 + (2j-1)$, $a_{j+1} \succ_i a_{j+2} \succ_i a_j$, and for voter $i' = 4 + 2j$, $a_{j+1} \succ_{i'} a_j \succ_{i'} a_{j+2}$.
  6. For every other voter v, a_t is the most preferred.
- If E does not fall under the special case, then apply any Condorcet consistent voting rule (e.g., Copeland's rule).

Note that m being odd ensures that t is an integer, and $m \geq 9$ ensures that a_1, a_3, a_{t-1}, a_t, a_{t+1}, a_{m-2}, and a_m are all distinct candidates. The significance of $n \geq m^2$ will be clear later.

We will now establish that a worst-case instance of f_{fail} falls under the special case; for this instance, we need to show that a_t is socially optimal; that f_{fail} returns a_1 on this instance; and most importantly, that the structure of $\overrightarrow{\succ}$ ensures that the optimal candidate a_t is sufficiently far from both the leftmost and the rightmost candidates.

We prove this last fact in the following lemma.

Lemma 8. *Let $I = (V, A, M, \rho, \pi) \in \mathcal{I}_{m,sep\text{-}\mathbb{R}}$ be an instance for which the corresponding election E^I falls under the special case of f_{fail}. Then the following holds.*

1. *Either $\rho(a_1) \leq \ldots \leq \rho(a_m)$, or $\rho(a_1) \geq \ldots \geq \rho(a_m)$, or $|\rho(A)| = 2$.*
2. *If $|\rho(A)| \neq 2$, min $\{d(a_t, a_1), d(a_t, a_m)\} \geq \frac{d(a_1, a_m)}{4}$.*

Proof. Since voter 1 ranks a_m last and preferences are single peaked on the line, a_m is at one edge of the candidate ordering. Similarly, since voter 2 ranks a_1 last, candidate a_1 is also at the edge of the candidate ordering (i.e., $\rho(a_1) = \max_{a \in A} \rho(a)$ or $\rho(a_1) = \min_{a \in A} \rho(a)$ and $\rho(a_m) = \max_{a \in A} \rho(a)$ or $\rho(a_m) = \min_{a \in A} \rho(a)$). If $\rho(a_1) = \rho(a_m)$, this means voters 1 and 2 are located in an equal distance from all candidates (which means all candidates are located in the same location, or some are at some distance from voters 1 and 2, and the rest are at the same distance in the other direction from these voters).

Assume $|\rho(A)| > 2$ (this also means $\rho(a_1) \neq \rho(a_m)$ and $\rho(v_1) \neq \rho(v_2)$), we wish to show the order of candidates is as voter 1 ordered them, i.e., $\rho(a_1) \leq \ldots \leq \rho(a_m)$ or $\rho(a_1) \geq \ldots \geq \rho(a_m)$. If voter 1 is further away from all candidates (i.e, if $\rho(a_1) = \max_{a \in A} \rho(a)$, $\rho(v_1) > \rho(a_1)$, and if $\rho(a_1) = \min_{a \in A} \rho(a)$, $\rho(v_1) < \rho(a_1)$), the ordering of the candidates is as voter 1 orders them. Otherwise, let ℓ be the smallest index such that $\rho(a_1) \neq \rho(a_\ell)$, then $\rho(v_1)$ may be between $\rho(a_1)$ and $\rho(a_\ell)$. If $d(v_1, a_1) < d(v_1, a_\ell)$, once again, the ordering of candidates is as voter 1 ordered them. If $d(v_1, a_1) = d(v_1, a_\ell)$, for any $\ell' > \ell$, $\rho(a_{\ell'}) \neq \rho(a_1)$, as that contradicts voter 2's vote ($a_{\ell'} \succ_2 a_\ell$). Therefore, $\rho(\ell')$ is either at $\rho(\ell)$, or further away from $\rho(a_1)$, meaning that candidates locations are ordered in the order voter 1's ordered them.

For the second condition, we will show that $d(a_t, a_1) \geq d(a_1, a_m)/4$. By symmetry, we also obtain $d(a_t, a_m) \geq d(a_1, a_m)/4$. This is trivial if $|\rho(A)| = 2$ and $\rho(a_1) = \rho(a_m)$. Now let us assume $|\rho(A)| \neq 2$. Without loss of generality, let $\rho(a_1) \leq \ldots \leq \rho(a_m)$ from the first condition. We show that either $\rho(a_1) = \ldots = \rho(a_m)$ or $\rho(a_1) < \ldots < \rho(a_m)$. If not, then we can find three consecutive candidates a_j, a_{j+1}, and a_{j+2} such that either $\rho(a_j) = \rho(a_{j+1}) < \rho(a_{j+2})$ or $\rho(a_j) < \rho(a_{j+1}) = \rho(a_{j+2})$. Both possibilities are impossible in our case due to the existence of voters with preferences $a_{j+1} \succ a_j \succ a_{j+2}$ and $a_{j+1} \succ a_{j+2} \succ a_j$.

Now, if $\rho(a_1) = \ldots = \rho(a_m)$, then the second condition is trivially true. Suppose $\rho(a_1) < \ldots < \rho(a_m)$. Because voter 3 (resp. 4) prefers candidate a_{t-1} (resp. a_{t+1}) the most, we have $\rho(3) \in (\rho(a_{t-2}), \rho(a_t))$ (resp. $\rho(4) \in (\rho(a_t), \rho(a_{t+2}))$. We can now show

$$\begin{aligned}
d(a_1, a_m) &\leq d(a_1, v_4) + d(v_4, a_m) \\
&\leq 2d(a_1, v_4) \qquad (\because a_m \succ_4 a_1) \\
&\leq 2d(a_1, a_{m-2}) \quad (\because \rho(a_t) < \rho(4) < \rho(a_{m-2})) \\
&\leq 2(d(a_1, v_3) + d(v_3, a_{m-2})) \\
&\leq 4d(a_1, v_3) \qquad (\because a_{m-2} \succ_3 a_1) \\
&\leq 4d(a_1, a_t). \qquad (\because \rho(a_1) < \rho(3) < \rho(a_t))
\end{aligned}$$

This concludes the proof. $\square$

Theorem 9. *For $m \geq 9$ and constant $\alpha \in (0, 0.5]$, $\phi_{\mathcal{I}^\alpha_{m,\text{sep-}\mathbb{R}}}(\widehat{f}_{fail})$ is upper bounded by a constant, whereas $\phi_{\mathcal{I}^\alpha_{m,\text{sep-}\mathbb{R}}}(f_{fail})$ is unbounded.*

Proof. First, we show that $\phi_{\mathcal{I}^\alpha_{m,\text{sep-}\mathbb{R}}}(f_{\text{fail}})$ is unbounded. Consider the following instance $I = (V, A, M, \rho, \pi)$. Let $V = \{v_1, \ldots, v_{2n}\}$ (where $n \geq m^2$), $A = \{a_1, \ldots, a_m\}$ (m being odd), and $M = (\mathbb{R}, d)$ with d being the Euclidean distance on the line.

The embedding function ρ is as follows. For $\ell \in [m]$, $\rho(a_\ell) = \frac{\ell-1}{m-1}$; that is, candidates a_1 through a_m are uniformly spaced in $[0, 1]$ with $\rho(a_1) = 0$ and $\rho(a_m) = 1$.

Fix $\epsilon < 1/m^2$. The voters are embedded as follows.

$$\begin{aligned}
\rho(1) &= \rho(a_1) - \epsilon, \\
\rho(2) &= \rho(a_m) + \epsilon, \\
\rho(3) &= \rho(a_t - 1) + \epsilon, \\
\rho(4) &= \rho(a_t + 1) - \epsilon, \\
\rho(4 + (2j-1)) &= \rho(a_{j+1}) + \epsilon && \forall j \in [m-2], \\
\rho(4 + 2j) &= \rho(a_{j+1}) - \epsilon && \forall j \in [m-2], \\
\rho(j) &= \frac{1}{2} && \forall j \geq 2m+1.
\end{aligned}$$

Finally, in the party affiliation π, since we are just showing how bad the distortion of direct elections are and f is affiliation-independent, the party affiliation is not important, and the outcome is independent of π, so we can assign it in any way such that half the voters are of one party and half are of the other. This construction simply ensures (and it is easy to check) that $I \in \mathcal{I}^{0.5}_{m,\text{sep-}\mathbb{R}} \subseteq \mathcal{I}^{\alpha}_{m,\text{sep-}\mathbb{R}}$ for all $\alpha \in (0, 0.5]$.

Next, it is also easy to check that election E^I falls under the special case of f_{fail}. Hence, $f_{\text{fail}}(E^I) = a_1$. Note that $C(a_1) \geq (2n - 2m) \cdot |0 - \frac{1}{2}| > n - m$ because a_1 is at distance $\frac{1}{2}$ from all but $2m$ voters located at $\frac{1}{2}$. In contrast, $C(a_t) \leq 2m \cdot 1$ because a_t is at zero distance from all but $2m$ voters (and its distance from those $2m$ voters is at most 1). Thus, $\phi(f_{\text{fail}}, I) \geq (n-m)/2m$. Since n is unbounded, $\phi_{\mathcal{I}^{\alpha}_{m,\text{sep-}\mathbb{R}}}(f_{\text{fail}})$ is also unbounded.

Finally, we show that $\phi_{\mathcal{I}^{\alpha}_{m,\text{sep-}\mathbb{R}}}(\widehat{f}_{\text{fail}})$ is upper bounded by a constant. Fix an instance $I \in \mathcal{I}^{\alpha}_{m,\text{sep-}\mathbb{R}}$. For notational simplicity, we refer to the number of candidates in I as m, though the proof below works if it is less than m. First, assume $|\rho(A)| \neq 2$. Without loss of generality, assume that $\rho(a_1) \leq \ldots \leq \rho(a_m)$, and that for a fixed $q \in [m]$, candidates $a_1, \ldots, a_q$ are affiliated with party -1 and the rest are affiliated with party 1.

Let I_{-1} and I_1 be the primary instances corresponding to I. Let a_{OPT} be an optimal candidate for I. Without loss of generality, suppose it is affiliated with party -1.

In the proof of Theorem 2, $\phi(\widehat{f}, I)$ depends only on the distortion of f on the primary instance of the party that a_{OPT} is affiliated with. Hence, if primary election $E^{I_{-1}}$ does not fall under the special case of f_{fail}, then f_{fail} applies a Condorcet-consistent rule on I_{-1}, ensuring that $\phi(f, I_{-1})$ is at most 3. In this case, by Theorem 2, $\phi(\widehat{f}, I)$ is also upper bounded by a constant.

Suppose $E^{I_{-1}}$ falls under the special case of f_{fail}. Let $t = (q+1)/2$ and $d_{-1} = d(a_1, a_q)$. Then, by Lemma 8, $\min\{d(a_t, a_1), d(a_t, a_q)\} \geq d_{-1}/4$. From now on, we shall use asymptotic notation liberally for simplicity.

Recall that there is a set of voters $S \subset V_{-1}$ whose top candidate was a_t, and

$$|S| = |V_{-1}| - 2q = \Omega(|V_{-1}|) = \Omega(n),$$

where the second transition holds because in the special case, $|V_{-1}| \geq q^2$, and the final transition holds because $|V_{-1}| \geq \alpha n$.

Note that for every $i \in S$ and $j \in V_1$, $d(i,j) \geq d_{-1}/4$. And $|V_1| \geq \alpha n$. Hence, we have $\Omega(n)$ pairs of voters (i,j) such that $d(i,j) \geq d_{-1}/4$. Further, $d(a_{OPT}, i) + d(a_{OPT}, j) \geq d(i,j)$. Hence, it follows that

$$C(a_{OPT}) = \Omega(n) \cdot d_{-1}. \tag{4}$$

Let $a^* = \widehat{f}(I)$. If $a^* = a^*_{-1} = a_1$, then we have

$$\begin{aligned} C(a^*) &\leq C(a_{OPT}) + n \cdot d(a^*, a_{OPT}) \\ &\leq C(a_{OPT}) + n \cdot d_{-1} = O(C(a_{OPT})), \end{aligned}$$

yielding a constant upper bound on $\phi(\widehat{f}_{\text{fail}}, I) = C(a^*)/C(a_{OPT})$.

On the other hand, if $a^* = a^*_1$, we have

$$\begin{aligned} C(a_{OPT}) &\geq C(a_1) - n \cdot d(a_1, a_{OPT}) \\ &\geq \frac{n}{2} \frac{d(a_1, a^*)}{2} - n \cdot d_{-1} \\ &\geq \frac{n}{2} \frac{d(a_{OPT}, a^*)}{2} - O(C(a_{OPT})). \end{aligned}$$

Here, the second transition follows because in the general election, at least $n/2$ voters vote for a^* over a_1 and $d(a_1, a_{OPT}) \leq d_{-1}$, and the final transition follows from Equation (4). This implies

$$C(a_{OPT}) = \Omega(n \cdot d(a_{OPT}, a^*)). \tag{5}$$

On the other hand, we have

$$C(a^*) \leq C(a_{OPT}) + n \cdot d(a_{OPT}, a^*) = O(C(a_{OPT})),$$

where the last transition follows due to Equation (5). Hence, we again have the desired constant upper bound on $\phi(\widehat{f}_{\text{fail}}, I)$.

Finally, if $\rho(A) = \{x_1, x_2\}$ (w.l.o.g., $x_1 < x_2$), due to separability, we have two options:

1. $\rho(A_{-1}) = x_1$ and $\rho(A_1) = x_2$: Therefore, $a_{OPT} \in \{a*_{-1}, a*_1\}$, and since $a*$ is the majority winner, $a* = a_{OPT}$, and the distortion is 1.
2. $\rho(A_{-1}) = \{x_1, x_2\}$ and $\rho(A_1) = x_2$: If $a_{OPT} \in \{a*_{-1}, a*_1\}$, then it is similar to the previous case. Otherwise, this means $\rho(a_{OPT}) = x_1$ and $\rho(a*_{-1}) = \rho(a*_1) = x_2$. Separability means the voters of party 1 lie in $\{x | x \geq x_2\}$, so $C(a_{OPT}) \geq |V_1| \cdot d(x_1, x_2) \geq \alpha \cdot n \cdot d(x_1, x_2)$. While $C(a*) \leq C(a_{OPT}) + |V_{-1}| \cdot d(x_1, x_2) \leq C(a_{OPT}) + n(1-\alpha) d(x_1, x_2)$. Combining these two equations we get the distortion is $\frac{1+2\alpha}{\alpha}$.

$\square$

8. Relaxing the Assumptions I: Voter Participation

We have analyzed above the primary system under two restrictive assumptions: every voter participates in one of the two primaries, and every voter participates in the general election. In reality, only a fraction of eligible voters participate in the primaries or in the general election. *What happens to the distortion under the primary system when not all eligible voters vote, but we still measure the social cost of the winning candidate with respect to all eligible voters?* We show that under such a relaxation, the distortion of any voting rule under the primary system changes by at most a constant factor. Hence, all of our results continue to hold.

To relax the assumptions, we extend our formal framework as follows. Recall that so far, we denoted an instance by $I = (V, A, M, \rho, \pi)$, where V is a set of n voters, A is a set of candidates, and party affiliation $\pi : V \cup A \to \{-1, 1\}$ maps every voter and candidate to one of the two parties. We further assumed that each voter v participates in the primary of party $\pi(v)$ as well as in the general election.

In the extended framework, we denote an instance by $I = (V, A, M, \rho, \pi, \tau)$. Here, the party affiliation function $\pi : V \cup A \to \{-1, 0, 1\}$ is allowed to map a voter to 0, which indicates that the voter does not participate in either party's primary election. This incorporates not only independent voters, but also voters affiliated with a party that do not participate in the party's primary election. We still require that $\pi(a) \in \{-1, 1\}$ for every $a \in A$, i.e., that every candidate is affiliated with a party and participates in that party's primary election. We also have the additional function $\tau : V \to \{0, 1\}$, which maps each voter to 1 if the voter participates in the general election, and to 0 otherwise. This relaxes the assumption that all voters participate in the general election.

Finally, recall that for $\alpha \in [0, 0.5]$, we defined a family of instances $\mathcal{I}^{\alpha}_{m,\mathcal{M}}$ such that in every instance $I \in \mathcal{I}^{\alpha}_{m,\mathcal{M}}$ with n voters, at least αn voters are affiliated with each party (and participate in that party's primary election). In the relaxed framework, for $\alpha \in [0, 0.5]$, $\beta \in [0, 1]$, and $\gamma \in [0, 1]$, we define a family of instances $\mathcal{I}^{\alpha,\beta,\gamma}_{m,\mathcal{M}}$ such that in every instance $I \in \mathcal{I}^{\alpha,\beta,\gamma}_{m,\mathcal{M}}$ with n voters, at least αn voters participate in each primary election, at least βn of voters participated in the primaries (obviously, $\beta \geq 2\alpha$), and at least γn voters participate in the general election. Formally, we require $|\pi^{-1}(p)| \geq \alpha n$ for each $p \in \{-1, 1\}$, $|\pi^{-1}(-1) \cup \pi^{-1}(1)| \geq \beta n$ and $|\tau^{-1}(1)| \geq \gamma n$. Note that the relaxed family $\mathcal{I}^{\alpha,\beta,\gamma}_{m,\mathcal{M}}$ includes all instances from the restricted family $\mathcal{I}^{\alpha}_{m,\mathcal{M}}$, which are obtained when $\beta = \gamma = 1$.

While our restricted framework required that $\pi(v) \in \{-1, 1\}$ and $\tau(v) = 1$ for every voter v, the extended framework allows the following possibilities.

1. $\pi(v) = 0$ and $\tau(v) = 1$: the voter does not participate in either primary but participates in the general election.
2. $\pi(v) \in \{-1, 1\}$ and $\tau(v) = 0$: the voter participates in a primary election

but does not participate in the general election.[6]

3. $\pi(v) \in 0$ and $\tau(v) = 0$: the voter does not participate in any election, primary or general.

Our goal is to show that for any voting rule f and constants $\alpha, \beta, \gamma > 0$, the distortion of f under the primary system in the relaxed framework ($\phi_{\mathcal{I}^{\alpha,\beta,\gamma}_{m,\mathcal{M}}}(\widehat{f})$) is no more than a constant times higher than in the distortion in the restricted framework ($\phi_{\mathcal{I}^{\alpha}_{m,\mathcal{M}}}(\widehat{f})$); the fact that $\phi_{\mathcal{I}^{\alpha,\beta,\gamma}_{m,\mathcal{M}}}(\widehat{f}) \geq \phi_{\mathcal{I}^{\alpha}_{m,\mathcal{M}}}(\widehat{f})$ follows trivially from the fact that the relaxed framework is strictly more general than the restricted framework.

We begin by proving a generalization of Lemma 5 in the relaxed framework, in which we establish that the social costs of the two primary winners must not differ by more than a constant factor.

Lemma 10. *Consider an instance $I \in \mathcal{I}^{\alpha,\beta,\gamma}_{m,\mathcal{M}}$ where $\alpha, \beta, \gamma > 0$. In the primary system, let a^*_{-1} and a^*_1 be the two primary winners, $a^* \in \left\{a^*_{-1}, a^*_1\right\}$ be the winner of the general election, and $\widehat{a} \in \left\{a^*_{-1}, a^*_1\right\} \setminus \{a^*\}$. Then,*

$$C(a^*) \leq \left(\frac{4}{\gamma} - 1\right) \cdot C(\widehat{a}).$$

Proof. Because at least γn voters participate in the general election and a^* wins by a majority vote, there must exist $X \subseteq V$ with $|X| \geq \frac{\gamma n}{2}$ such that $d(i, a^*) \leq d(i, \widehat{a})$ for every $i \in X$. Combining with the triangle inequality $d(a^*, \widehat{a}) \leq d(a^*, i) + d(i, \widehat{a})$, we get $d(i, \widehat{a}) \geq \frac{d(a^*, \widehat{a})}{2}$ for every $i \in X$. Hence,

$$C(\widehat{a}) \geq \frac{\gamma n}{2} \cdot \frac{d(a^*, \widehat{a})}{2} \Rightarrow d(a^*, \widehat{a}) \leq \frac{4}{\gamma n} \cdot C(\widehat{a}). \tag{6}$$

Now, we have

$$\begin{aligned} C(a^*) &= \sum_{i \in X} d(i, a^*) + \sum_{i \in V \setminus X} d(i, a^*) \\ &\leq \sum_{i \in X} d(i, \widehat{a}) + \sum_{i \in V \setminus X} (d(i, \widehat{a}) + d(a^*, \widehat{a})) \\ &\leq C(\widehat{a}) + \left(n - \frac{\gamma n}{2}\right) \cdot d(a^*, \widehat{a}), \end{aligned} \tag{7}$$

where the second transition follows because $d(i, a^*) \leq d(i, \widehat{a})$ for $i \in X$, and $d(i, a^*) \leq d(i, \widehat{a}) + d(a^*, \widehat{a})$ due to the triangle inequality. Substituting Equation (6) into Equation (7), we get the desired result. ☐

We are now ready to prove the main result of this section.

[6] While data indicates that almost all voters who participate in the primary elections also participate in the general election [28], we consider this possibility for the sake of completeness.

Theorem 11. *For every choice of $\mathcal{M}$, $m \in \mathbb{N}$, affiliation-independent voting rule f, and constants $\alpha, \beta > 0$,*

$$\phi_{\mathcal{I}^{\alpha,\beta,\gamma}_{m,\mathcal{M}}}(\widehat{f}) \leq \left(\frac{4}{\gamma} - 1\right)\left(1 + \frac{4}{\beta}\right) \cdot \phi_{\mathcal{I}^{\alpha}_{m,\mathcal{M}}}(\widehat{f}).$$

Proof. Consider an instance $I = (V, A, M, \rho, \pi, \tau) \in \mathcal{I}^{\alpha,\beta,\gamma}_{m,\mathcal{M}}$ with n voters in the relaxed framework. Recall that function $\pi : V \cup A \to \{-1, 0, 1\}$ dictates whether each voter participates in the primary of party -1, in the primary of party 1, or in neither primary. Similarly, function $\tau : V \to \{0, 1\}$ dictates whether each voter participates in the general election.

Let us now construct an instance $\overline{I} = (\overline{V}, A, M, \rho, \overline{\pi}) \in \mathcal{I}^{\alpha}_{m,\mathcal{M}}$ in the restricted framework as follows: We set $\overline{V} = \left(\pi^{-1}(-1) \cup \pi^{-1}(1)\right) \cap V$; that is, we keep the voters who participate in the primaries in $\overline{I}$, and delete the voters who do not. Note that $|\overline{V}| \geq \beta n$. Let $\overline{\pi} : \overline{V} \cup A \to \{-1, 1\}$ be the restriction of π to $\overline{V} \cup A$ (note that π does not map any voter in $\overline{V}$ to 0, and cannot map any candidate in A to 0). Note that at least $\alpha n \geq \alpha|\overline{V}|$ voters participate in each primary election; hence, this is a valid instance of $\mathcal{I}^{\alpha}_{m,\mathcal{M}}$.

Crucially, note that instances I and $\overline{I}$ match in the set of voters that participate in each primary election. They only differ in the set of voters that participate in the general election.[7] Hence, the primary winners in instances I and $\overline{I}$ must be identical.

Let a_{-1} and a_1 denote the primary winners of parties -1 and 1, respectively, in both I and $\overline{I}$. Let $a_{OPT} \in \arg\min_{a \in A} C^I(a)$ and $\overline{a_{OPT}} \in \arg\min_{a \in A} C^{\overline{I}}(a)$ denote the socially optimal candidates in I and $\overline{I}$, respectively. Let a^* and $\overline{a^*}$ denote the winners of the general elections in I and $\overline{I}$, respectively. Hence, $\phi(\widehat{f}, I) = C^I(a^*)/C^I(a_{OPT})$ and $\phi(\widehat{f}, \overline{I}) = C^{\overline{I}}(\overline{a^*})/C^{\overline{I}}(\overline{a_{OPT}})$.

For any candidate a, we have

$$\begin{aligned}
\beta n \cdot d(\overline{a_{OPT}}, a) \leq |\overline{V}| \cdot d(\overline{a_{OPT}}, a) &\leq \sum_{i \in \overline{V}} d(i, \overline{a_{OPT}}) + d(i, a) \\
&= C^{\overline{I}}(\overline{a_{OPT}}) + C^{\overline{I}}(a) \\
&\leq 2C^{\overline{I}}(a),
\end{aligned}$$

where the first transition follows because $|\overline{V}| \geq \beta n$, the second transition follows from the triangle inequality, and the final transition holds because $\overline{a_{OPT}}$ is the socially optimal candidate in $\overline{I}$. Hence, for every candidate a, we have

$$d(\overline{a_{OPT}}, a) \leq \frac{2}{\beta n} C^{\overline{I}}(a). \tag{8}$$

[7] In the general election of $\overline{I}$, we removed voters who participated in the general election but not in the primaries in I, and added voters who participated in the primaries but not in the general election in I.

Now, we have

$$
\begin{aligned}
C^I(a^*) &\le \left(\frac{4}{\gamma}-1\right) C^I(\overline{a^*}) = \left(\frac{4}{\gamma}-1\right) \sum_{i \in V} d(i, \overline{a^*}) \\
&\le \left(\frac{4}{\gamma}-1\right) \cdot \sum_{i \in V} d(i, a_{OPT}) + d(a_{OPT}, \overline{a^*}) \\
&\le \left(\frac{4}{\gamma}-1\right) \cdot \left(C^I(a_{OPT}) + n \cdot \left(d(a_{OPT}, \overline{a_{OPT}}) + d(\overline{a_{OPT}}, \overline{a^*})\right)\right) \\
&\le \left(\frac{4}{\gamma}-1\right) \cdot \left(C^I(a_{OPT}) + n \cdot \left(\frac{2}{\beta n} C^{\overline{I}}(a_{OPT}) + \frac{2}{\beta n} C^{\overline{I}}(\overline{a^*})\right)\right) \\
&\le \left(\frac{4}{\gamma}-1\right) \cdot \left(\left(1+\frac{2}{\beta}\right) C^I(a_{OPT}) + \frac{2}{\beta} C^{\overline{I}}(\overline{a^*})\right), \qquad (9)
\end{aligned}
$$

where the first transition follows from the fact that both a^* and $\overline{a^*}$ are primary winners in I and from Lemma 10, the third and the fourth transitions follow from the triangle inequality, the fifth transition uses Equation (8), and the final transition uses the fact that $C^{\overline{I}}(a) \le C^I(a)$ for any candidate a since the set of voters in I is a superset of the set of voters in $\overline{I}$. Finally, we have

$$
\begin{aligned}
\phi(\widehat{f}, I) &= \frac{C^I(a^*)}{C^I(a_{OPT})} \\
&\le \left(\frac{4}{\gamma}-1\right) \cdot \frac{1}{C^I(a_{OPT})} \left(\left(1+\frac{2}{\beta}\right) \cdot C^I(a_{OPT}) + \frac{2}{\beta} \cdot C^{\overline{I}}(\overline{a^*})\right) \\
&\le \left(\frac{4}{\gamma}-1\right) \cdot \left(\left(1+\frac{2}{\beta}\right) + \frac{2}{\beta} \cdot \frac{C^{\overline{I}}(\overline{a^*})}{C^I(a_{OPT})}\right) \\
&\le \left(\frac{4}{\gamma}-1\right) \cdot \left(\left(1+\frac{2}{\beta}\right) + \frac{2}{\beta} \cdot \frac{C^{\overline{I}}(\overline{a^*})}{C^{\overline{I}}(\overline{a_{OPT}})}\right) \\
&= \left(\frac{4}{\gamma}-1\right) \cdot \left(\left(1+\frac{2}{\beta}\right) + \frac{2}{\beta} \cdot \phi(\widehat{f}, \overline{I})\right) \\
&\le \left(\frac{4}{\gamma}-1\right) \left(1+\frac{4}{\beta}\right) \phi(\widehat{f}, \overline{I}),
\end{aligned}
$$

where the first transition uses Equation (9), and the second transition uses the fact that $C^I(a_{OPT}) \ge C^{\overline{I}}(a_{OPT}) \ge C^{\overline{I}}(\overline{a_{OPT}})$. Thus, we have the desired result. □

We proved that having independent voters in the general election and having voters who only participate in the primaries cannot improve the distortion of a voting rule under the primary system. Hence, our earlier results which establish that primaries are no better than direct elections (Theorems 1 and 6) continue to hold in this extended framework. Additionally, we proved that the existence of such voters can make the distortion under the primary system worse by at most a constant factor (which depends on β and γ). Hence, our earlier results which establish that primaries cannot be much worse than direct elections (Theorem 2) or that primaries can be significantly better than direct elections (Theorem 9) also continue to hold in this extended framework.

9. Relaxing the Assumptions II: Candidate Elimination Structure

10. Using Simulations to Go Beyond Worst Case

So far we compared the distortion of a voting rule under the direct and primary systems, taken in the *worst case* over a family of instances. In practice, such worst case instances may not arise naturally. In this section, we compare the distortion of a voting rule under the direct and primary systems, in the *average case* over simulated instances. To control for the effect of vastly different numbers of voters or candidates affiliated with the two parties, we place the restriction that half of the voters and half of the candidates must be affiliated with each party. Furthermore, we wish to examine instances in which voters and candidates affiliated with one party are separated from those affiliated with the other party.

We fix the metric space to be $[0,1]^k$ for $k \in \{1,3,5,7,9\}$ with the Euclidean distance. We generate 1000 instances satisfying the following restrictions. First, we place a set V of $n = 1000$ voters at uniformly random locations in $[0,1]^k$. Next, we find a hyperplane dividing voters into two equal halves. Due to symmetry, we simply find a threshold t on the k^{th} coordinate such that the locations of half of the voters (call this set V_{-1}) have a smaller k^{th} coordinate, while the locations of the rest (call this set V_1) have a greater k^{th} coordinate. We do not affiliate the voters yet. Next, we place $\frac{m}{2} = 10$ candidates (call this set A_{-1}) uniformly at random on one side of the hyperplane, and $\frac{m}{2} = 10$ candidates (call this set A_1) uniformly at random on the other side.

Once the locations of the voters and candidates are fixed, we create two instances. In one instance ("split"), we assign $V_{-1} \cup A_{-1}$ to party -1, and assign $V_1 \cup A_1$ to party 1. This instance belongs to $\mathcal{I}^{0.5}_{m,\text{sep-}\mathbb{R}^k}$. In the other instance ("random"), we assign half of the voters and half of the candidates chosen uniformly at random to party -1, and the rest to party 1. This instance belongs to $\mathcal{I}^{0.5}_{m,\mathbb{R}^k}$, but not necessarily to $\mathcal{I}^{0.5}_{m,\text{sep-}\mathbb{R}^k}$. This allows us to directly compare the effect that separability has on the distortion. Finally, we run five voting rules — plurality, Borda, STV, Copeland and maximin — on both instances under the direct and primary systems, and measure the distortion. Note that their distortion under the direct system would be identical for both instances because the two instances only differ in party affiliations. Thus, for each rule, we obtain three numbers: Direct, Primary-split, and Primary-random. We average the distortion numbers across the 1000 instances.

See Figure 1 for selected simulation results. The results for $k > 3$ resemble those for $k = 3$ in their comparison of the direct versus the primary system; the only difference is that the overall distortions are lower for higher k. To compare the three distortion numbers in each case, we ran a repeated measures ANOVA

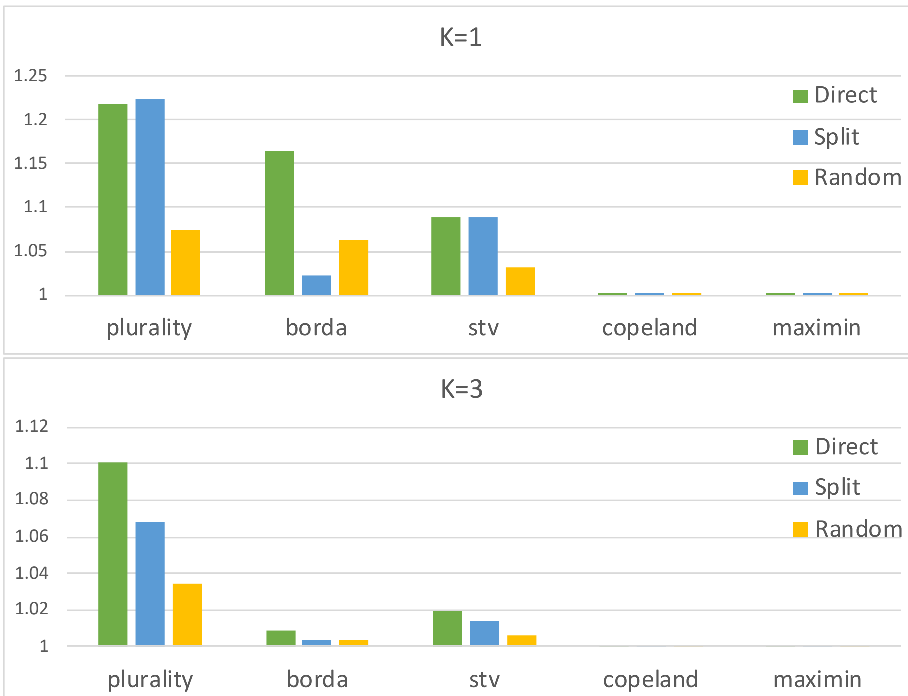

Figure 1: The distortion values for various settings (Direct, Primary-split and Primary-random) and voting rules for $k \in \{1, 3\}$. Each bar represents the average distortion over 1000 instances. Within each figure, all of the bars represent average distortion over the same set of voter and candidate locations, but not necessarily party affiliations. ANOVA on all elections returned confidence $p < 10^{-9}$ when $k = 1$ and confidence $p < 10^{-5}$ when $k = 3$.

comparing the distortion values, and in all but 2 of 25 cases, had a p value under 0.05 (the two cases had $k = 9$).

Perhaps the most important observation is that our simulation results stand in direct contrast to our worst-case bounds (Theorems 2 and 6 and Proposition 7). In almost all of our settings, the distortion under the primary system (split and random) is better than that the distortion under its direct counterpart, and often shows a significant improvement. This is especially noticeable in the non-Condorcet consistent rules (plurality, Borda and STV), as in all but 3 of 15 cases the distortion significantly improves the primary system in both cases. This effect is most pronounced with plurality. With Condorcet consistent rules, the distortion values are very low, regardless of whether the direct system is used or the primary system. In general, as we increase the dimension k, the groups become more homogenous and the p values grow, while distortions approach 1.

Overall, the simulation results show a distortion that is far below our theoretical worse-case results. We suspect that the reason for this difference might have to do with our choice of uniform voter and candidate distributions, and distortion numbers might differ under different distributions.

11. Discussion

Our paper initiates the novel quantitative study of multi-stage elections (and their comparison to single-stage elections), but leaves plenty to explore. Some directions are fairly straightforward extensions of our results. The most straightforward question is to tighten our bounds, and beyond that, to examine if our results on the improvement of primaries over direct elections in a synthetic voting rule hold for more common ones (e.g., STV) or not. Moreover, our results contrasting separable and non-separable metric spaces may be extended to spaces which are nearly-separable. There is also the question of explaining the trends we observe in the average case, which sometimes differ from our worst-case results. A next step would be to study realistic distributional models of voter preferences and candidate positions in the political spectrum, and analyze their effect on distortion.

Other extensions are seemingly more involved. Extending our framework to more than two parties requires the use of a ranked voting rule in the general election, which may significantly affect the analysis. Interestingly, such an extension would also incorporate independent candidates because one can imagine an independent candidate to be a party of their own. Examining the use of multiple and different voting rules as Narodytska and Walsh [32] do for two-step voting (though without candidate elimination between stages) is an enticing direction. For example, plurality might be used in the general election, while STV might be used by the parties in their primaries. It is also reasonable to consider that each party has its own voting rule. It would be interesting as well to examine manipulations by parties, by candidates, and by voters in primary systems. In particular, it is reasonable to believe that candidates may strategically shift, to some extent, their location following the primaries, to make themselves more appealing to the general electorate. Another topic where more research is needed is investigating multi-winner elections in party elections (e.g., in party lists, in countries where this is common).

We believe that the study of multi-stage elections and party mechanisms can not only contribute novel theoretical challenges to tackle, but can also bring research on computational social choice closer to reality and increase its impact.

12. Acknowledgements

This work was partially supported by NSERC under the Discovery Grants program.

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